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LOCAL PAILEY WIENER THEOREMS

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Purpose

The purpose of this paper is to survey new and simplified statements of local Paley Wiener theorems, due to the authors, on the $n-1$ dimensional unit sphere realized as a subset of $n=2,3$ Euclidean space. More precisely, given a function $f:\mathbb{C}^n\to\mathbb{C}$, $n=2,3$, whose restriction to an $n-1$ sphere is analytic, we establish necessary and sufficient conditions determining whether f is the Fourier transform of a compactly supported, bounded function $F:\mathbb{R}^n\to\mathbb{C}$. The essence of this investigation is that because of the local nature of the problem, the mapping $f\to F$ is not in general invertible and so the problem cannot be studied via a Fourier integral. Our proofs are new.

Summary

A problem of interest in radiation and scattering problems is that of determining the support of a compactly supported, square integrable scattering potential, $F:\mathbb{R}^n\to\mathbb{C}$, $n=2,3$ from far field data given by a function $f:\mathbb{C}^n\to\mathbb{C}$.

Suppose, approri, that we know that for every vector $\mathbf{z}\in S^{n-1}$, the $n-1$ dimensional unit sphere, realized as a proper subset of $\mathbb{R}^n$, the far field data f is given **locally** by an integral such as

$$f(\mathbf{z})=\int_\tau F(\mathbf{x})e^{-ik(\mathbf{z}.\mathbf{x})}d^n(x) \tag{0.1}$$

for some bounded set $\tau\in\mathbb{R}^n$ and bounded, compactly supported $F:\mathbb{R}^n\to\mathbb{C}$ with support in τ. In practice, $\mathbf{z}$ is the unit vector in the direction where f is measured and k is an absolute real constant. The inverse support problem, as treated here, studies the problem of determining bounds for the support of the set τ, assuming the model (0.1). For many applications, assumption (0.1) on the given f is strong and often not obvious from the given data.

The purpose of this paper is to survey new and simplified statements of **local Paley Wiener** theorems on S^{n-1}. More precisely, given a function $f:\mathbb{C}^n\to\mathbb{C}$, $n=2,3$, whose restriction to S^{n-1} is analytic, we establish necessary and sufficient conditions determining whether f is the Fourier transform of a compactly supported, bounded function $F:\mathbb{R}^n\to\mathbb{C}$. The essence of this investigation, is that because of the **local** nature of the problem, the mapping $f\to F$ is not in general invertible and so the problem cannot be studied via a Fourier integral. Our proofs are new and use a beautiful interplay between plane wave expansions and Euclidean geometrical arguments. We will show that provided the restriction of f to S^{n-1} is analytic and f satisfies a growth condition of exponential type at infinity, then (0.1) holds for some compact set $\tau\in\mathbb{R}^n$ and bounded $F:\mathbb{R}^n\to\mathbb{C}$ with compact support in τ. We also describe the smallest supports sets for which our results are best possible. Results of this type are typically known in the literature as Paley Wiener theorems [PW]. In this paper, we seek generalizations of [PW] in that we do not assume that f is entire nor that its restriction to $\mathbb{R}^n$ is square integrable. The later assumptions are basic in [PW]. Indeed, we show that (a) a growth condition of f of exponential type at infinity and (b) an assumption that the restriction of f to S^{n-1} is analytic, are enough to deduce (0.1). In [PW], the function F obtained is square integrable on $\mathbb{R}^n$, compactly supported but not necessarily bounded.

Structure of paper

The remainder of this paper is as follows:

(a) Notation
(b) The Paley Wiener Theorems in $\mathbb{R}^n$
(c) Main Results: Local Paley Weiner theorems on spheres, comparisons and problems with local inversion

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Notation

In order to motivate what follows, we need to introduce some needed notation. Throughout, for any non zero real sequences a_n and b_n, we shall write $a_n = O(b_n)$ if the ratio a_n/b_n is uniformly bounded in n and $a_n \sim b_n$ if $a_n/b_n \to 1$, $n \to \infty$. Similar notation will be used for functions and sequences of functions. Throughout, we shall say that a function $f : \mathbb{C}^n \to \mathbb{C}$ is of exponential type $a > 0$ at infinity, if

$$|f(z)| = O\left(e^{a|z|}\right), \, |z| \to \infty.$$

Finally, given $f : \mathbb{C}^n \to \mathbb{C}$, by f_A, we will always mean the restriction of f to a proper subset A of $\mathbb{C}^n$. Given $x \in \mathbb{R}^n$, by the vector $\mathbf{x}$, we will mean the point x with a given direction from the origin. Similar notation will be used for complex vectors. Associated with the euclidean metric on $\mathbb{R}^n$ and $\mathbb{C}^n$, we have the usual inner product and in what follows, $\mathbf{v}_1.\mathbf{v}_2$ will denote the usual inner product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. Throughout, $L_2(\mathbb{R}^n)$ will denote the class of all square integrable functions $F : \mathbb{R}^n \to \mathbb{C}$.

The Paley Wiener Theorems in $\mathbb{R}^n$

The relationship of the growth of an entire function with the properties of its Fourier transform are embodied in the well known Paley-Wiener theorems. For functions of complex variable, we have:

Theorem 1 (PW1). *A function $f : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type with its restriction $f_{\mathbb{R}} \in L_2(\mathbb{R})$ iff*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_a^b F(x) e^{-ixz} dx, \, z \in \mathbb{C}.$$

for some $F \in L_2(\mathbb{R})$ with support in $[a,b]$ Moreover, $[a,b]$ is the smallest set containing the support of F and

$$a = -\limsup_{y \to \infty} \frac{\log|f(-iy)|}{y}, \, b = \limsup_{y \to \infty} \frac{\log|f(iy)|}{y}.$$

Analogues of [PW1], which we denote for simplicity by [PW2], exist for $n \geq 1$ complex variables. See for example [6, pg 181, Theorem 7.3.1] for a clear exposition of these results.

In what follows, we will need the notion of a *support function* of the smallest convex set outside of which which a compactly supported function $F : \mathbb{R}^n \to \mathbb{C}$ vanishes. This function is defined by

$$\rho(\mathbf{u}) := \sup_{\mathbf{x} \in \tau_F} (\mathbf{u}.\mathbf{x})$$

where τ_F is the support of F and $\mathbf{u} \in \mathbb{R}^n$ is a given unit vector. The function $\rho(\mathbf{u})$ is used to define a convex region $\tau_{c,F} \supset \tau_F$ which is formed from tangent planes to τ_F having normal vectors $\mathbf{u}$ and located at a distance $\rho(\mathbf{u}, \tau_F)$ from the origin.

It is clear that the function ρ depends on F via its support τ_F but since F is always fixed, for ease of notation, we will drop this dependence henceforth.

Main Results: Local Paley Weiner theorems, comparisons and problems with local inversion

In this section, we state our main results which constitute generalizations of [PW] for $n = 2, 3$. Following is our first result:

Theorem 2 (LPW1). *Let $n = 2, 3$ and let $f : \mathbb{C}^n \to \mathbb{C}$ be a function whose restiction to S^{n-1} is analytic. Let k, a be positive numbers. Suppose that for any fixed real vector $\mathbf{u} \in S^{n-1}$.*

$$|f(\mathbf{z})| = O\left(e^{ka(\mathbf{u}.\Im(\mathbf{z}))}\right), \, \mathbf{u}.\Im(\mathbf{z}) \to \infty, \, \mathbf{z} \in \mathbb{C}^n.$$

Then,

$$f(\mathbf{z}) = \int_{B_a} F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} d^n x, \, \mathbf{z} \in S^{n-1}$$

where F is a bounded function supported in the closed ball $B_a \in \mathbb{R}^n$ with centre 0 and radius a.

Remark 1

(a) Note that in the statement of [LPW1], compared to that of [PW1], we use smoothness properties of f only on S^{n-1} to establish our result and square integrability and analyticity of the restriction of f off the given sphere is not required nor used in our results. The function F obtained is both bounded and compactly supported in $\mathbb{R}^n$. As $\mathbf{z} \in \mathbb{C}^n$ and $\mathbf{u} \in \mathbb{R}^n$, the inner product $\Im\mathbf{z}.\mathbf{u}$ is well defined.

(b) [LPW1] is similar to [7, Theorems 9, 12, Section 4] which were established earlier. In these later results, the authors consider a wide class of distribution functions and study bounds on their coefficients under similar smoothness assumptions to ours. The method of proof in [7], uses a combination of clever and sophisticated machinary of Bessel and spherical harmonics and is of independent interest. In particular, we mention that Bessel and spherical harmonics allow for extensions from $n = 2$ to $n = 3$. We are able to establish [LPW1] using different techniques which involve an interplay between plane wave expansions and geometrical arguments. These later techniques also provide a natural but different method to move from $n = 2$ to $n = 3$.

(c) It is easy to establish a partial converse of [LPW1] which is the following: Suppose

$$f(\mathbf{z}) = \int_{B_a} F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} d^n x, \mathbf{z} \in \mathbb{C}^n$$

where F is a bounded function supported in the closed ball $B_a \in \mathbb{R}^n$ with centre 0 and radius a. Then

$$|f(\mathbf{z})| = O\left(e^{ka(\mathbf{u}.\Im(\mathbf{z}))}\right), \mathbf{u}.\Im(\mathbf{z}) \to \infty.$$

(c) [LPW1] stated above does not, in general, yield the smallest support volume for the function F. However, the smallest *convex* support volume for this function can be obtained in analogy with [PW2]. This is contained in Theorem 3 below.

Theorem 3 (LPW II). *Let $n = 2, 3$, k a positive constant and $f : \mathbb{C}^n \to \mathbb{C}$ be a function whose restiction $f_{S^{n-1}}$ is analytic. Suppose that for any fixed real vector $\mathbf{u} \in S^{n-1}$,*

$$|f(\mathbf{z})| = O\left(e^{k\rho(\mathbf{u})(\mathbf{u}.\Im(\mathbf{z}))}\right), \mathbf{u}.\Im(\mathbf{z}) \to \infty, \mathbf{z} \in \mathbb{C}^n.$$

Then

$$f(\mathbf{z}) = \int_{\tau_c} F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} d^n x, \mathbf{z} \in S^{n-1}$$

where F is a bounded function supported in the convex region τ_c having support function $\rho(\mathbf{u}, \tau_c)$ for any vector $\mathbf{u} \in S^{n-1}$.

Remark 2 [LPW2] is similar to [7, Corollary 4.7]. The proof in this former paper uses much in the spirit of [7, Theorems 9, 12], Bessel and spherical harmonics whereas our proof uses a different method of proof which relies on geometry and wave expansions. We believe both methods of proof to be of independent interest. It is easy to see also that much as in [LPW1], we have the following: Suppose that

$$f(\mathbf{z}) = \int_{\tau_c} F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} d^n x, \mathbf{z} \in \mathbb{C}^n$$

where F is a bounded function supported in the convex region τ_c having support function $\rho(\mathbf{u}, \tau_c)$ for any vector $\mathbf{u} \in S^{n-1}$. Then

$$|f(\mathbf{z})| = O\left(e^{k\rho(\mathbf{u})(\mathbf{u}.\Im(\mathbf{z}))}\right), \mathbf{u}.\Im(\mathbf{z}) \to \infty.$$

Problems with local inversion

The new theorems [LPW1] and [LPW2] stated here are inherently different from the conventional Paley Wiener theorems [PW1] and [PW2]. For example, consider the case where the function $f(\mathbf{z})$ is the boundary value of an entire function $G(\omega)$ which satisfies the conditions of the conventional Paley Wiener theorem; i.e.,

$$f(\mathbf{z}) = G(\omega)|_{\mathbf{z}=k\omega}.$$

While each theorem guarantees that the associated function is the transform of a compactly supported function in R^n, the supports for these two functions will, in general, be different. Indeed, $f(\mathbf{z})$ is totally independent of components of $G(\omega)$ which vanish on the sphere $\mathbf{z} = k\omega$ so that while these components contribute to the overall support a ssociated with $G(\omega)$, they do not contribute to the support associated with $f(\mathbf{z})$. Such components, which are known as *non-radiating sources* and are well known to play an important role in inverse source and scattering problems.

The difference between the conventional and generalized Paley Wiener theorems is also apparent from the fact that the two functions f and F are reciprocally related via a Fourier transform pair in the conventional Paley Wiener theorem. On the other hand, in the new theorems f can be computed from F via the boundary value of a Fourier transform but a unique inverse mapping does not exist. Indeed, any non-radiating source supported within the support of F can be added to F without changing f.

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