A DIRECT PROOF THAT TORIC RANK 2 BUNDLES ON PROJECTIVE SPACE SPLIT

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Abstract

In this paper we give a short, direct proof that rank 2 toric vector bundles on n-dimensional projective space split once n is at least 3. This result is originally due to Bertin and Elencwajg, and there is also related work by Kaneyama, Klyachko, and Ilten-Süss. The idea is that, after possibly twisting the vector bundle, there is a section which is a complete intersection.

Let k be an algebraically closed field and let $\mathbb{G}_m$ act on $\mathbb{P}^n_k$ by scaling the last n coordinates. In this paper we present a direct proof that $\mathbb{G}_m$-equivariant rank 2 vector bundles on $\mathbb{P}^n_k$ split when $n \geq 3$.

A conjecture of Hartshorne ([3, Conj. 6.3]) states that every rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^n_k$ splits as a sum of line bundles as soon as $n \geq 7$. We know of two interesting pieces of evidence for this conjecture. First, let $Z \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth complex subvariety of codimension 2 which is the zero locus of a rank 2 vector bundle. It is natural to try to show that $\mathcal{E}$ is not split by studying the topology of Z, i.e. to show that it is not homeomorphic to a complete intersection, which is the zero locus of a split bundle. For example, Horrocks and Mumford ([4]) constructed a rank 2 vector bundle on $\mathbb{P}^4$ with a section whose zero locus is an abelian surface. It follows that their bundle is not split, as any smooth complete intersection surface has trivial fundamental group. On the other hand, when n is large then Barth proved that many of the cohomology groups of Z are isomorphic to the cohomology groups of a complete intersection. More precisely, Barth proved ([1]) that for a nonsingular complex subvariety $X \subset \mathbb{P}^n_{\mathbb{C}}$ of codimension e, the restriction maps $H^i(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$ are isomorphisms for $i \leq n - 2e$.

The second piece of evidence comes when one considers equivariant vector bundles. Considering $\mathbb{P} = \mathbb{P}^n_k$ as a toric variety, then a *toric vector bundle* is a vector bundle $\mathcal{E}$ on $\mathbb{P}$ such that the total space $\mathbb{E}$ is equipped with a $(\mathbb{G}_m)^n$-action which commutes with the action on $\mathbb{P}$. Bertin and Elencwajg ([2, Thm. 3.4]) first proved that every rank 2 toric vector bundle on $\mathbb{P}^n$ is split once $n \geq 3$. Kaneyama ([6, Cor. 3.5]) proved that every toric vector bundle on $\mathbb{P}$ of rank $r < n$ splits. Klyachko ([7]) gave a description of the category of toric vector bundles and used this description to give a different proof of Kaneyama's result. Recently Ilten and Süss ([5]) studied smaller dimensional tori $(\mathbb{G}_m)^\ell$ acting on $\mathbb{P}$ by scaling and gave a description of the category of $(\mathbb{G}_m)^\ell$-equivariant vector bundles. They proved that if $\mathcal{E}$ is a rank r, $(\mathbb{G}_m)^\ell$-vector bundle on $\mathbb{P}$ and $r \leq \min\{n, \ell+3\}$ then $\mathcal{E}$ splits.

The goal of this note is to give a direct proof of the splitting of rank 2 $\mathbb{G}_m$-bundles on $\mathbb{P}$ for $n \geq 3$. Consider the following scaling $\mathbb{G}_m$-action:

$$\mathbb{G}_m \times \mathbb{P} \to \mathbb{P}$$
$$t \times [a_0 : a_1 : \cdots : a_n] \mapsto [a_0 : ta_1 : \cdots : ta_n].$$

This acts by scaling on the affine space $\mathbb{A}^n_k = \mathbb{P} \setminus (x_0 = 0)$ and fixes the point $[1 : 0 : \cdots : 0]$ as well as the hyperplane $H = (x_0 = 0) \subset \mathbb{P}$. In this setting we prove:

THEOREM 0.1. Let $n \geq 3$. If $\mathcal{E}$ is a rank 2, $\mathbb{G}_m$-equivariant vector bundle on $\mathbb{P}$, then $\mathcal{E}$ splits as a direct sum of line bundles.

REMARK 0.2. As the above action is the restriction of the standard $(\mathbb{G}_m)^n$-action on $\mathbb{P}$ to a subgroup, this proves that every toric vector bundle on $\mathbb{P}$ splits.

PROOF. Clearly $\mathcal{E}$ is split $\iff$ $\mathcal{E}(NH)$ is split. The goal is to show that some twist $\mathcal{E}(NH)$ has a section $s \in H^0(\mathbb{P}, \mathcal{E}(NH))$ such that either the zero locus $Z = (s = 0) \subset \mathbb{P}$ is empty, or it is a scheme-theoretic complete intersection of codimension 2 in $\mathbb{P}$. This is sufficient for the following reasons. First, if Z is empty then there is a short exact sequence:

$$0 \to \mathcal{O} \to \mathcal{E}(NH) \to \det(\mathcal{E}(NH)) \to 0.$$

This sequence is necessarily split as $\mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\det(\mathcal{E}(NH)), \mathcal{O}_\mathbb{P}) = H^1(\det(\mathcal{E}(NH))^\vee) = 0$ (using $\dim(\mathbb{P}) \geq 3$). Second, if $Z \neq \emptyset$, then it is cut out by equations of degree a and b. In this case we have:

$$\mathcal{E}(NH)^\vee|_Z \cong I_Z/I_Z^2 \cong \mathcal{O}_Z(-a) \oplus \mathcal{O}_Z(-b)$$

By taking first Chern classes and pushing forward to $\mathbb{P}$ we see that $\det(\mathcal{E}(NH)^\vee) = \mathcal{O}(-a-b)$. Thus we have two Koszul resolutions:

$$0 \to \mathcal{O}_\mathbb{P}(-a-b) \to \mathcal{O}_\mathbb{P}(-a) \oplus \mathcal{O}_\mathbb{P}(-b) \to I_Z \to 0, \quad \text{and}$$
$$0 \to \mathcal{O}_\mathbb{P}(-a-b) \to \mathcal{E}(NH)^\vee \to I_Z \to 0.$$

Applying $\mathrm{Hom}(-, \mathcal{O}_\mathbb{P}(-a-b))$ to the first Koszul resolution shows that

$$\mathrm{Hom}(\mathcal{O}_\mathbb{P}(-a-b), \mathcal{O}_\mathbb{P}(-a-b)) \cong \mathrm{Ext}^1(I_Z, \mathcal{O}_\mathbb{P}(-a-b)).$$

is 1-dimensional. Therefore, by the Yoneda interpretation of $\mathrm{Ext}^1(I_Z, \mathcal{O}_\mathbb{P}(-a-b))$ it follows that $\mathcal{E}(NH)^\vee \cong \mathcal{O}(-a) \oplus \mathcal{O}(-b)$.

As in the introduction, let $\mathbb{A}^n_k = (x_0 \neq 0)$ and let H denote the hyperplane $(x_0 = 0)$. Then $\mathcal{E}|_{\mathbb{A}^n}$ is isomorphic to the trivial rank 2 bundle. (Note, this does not require the Quillen-Suslin theorem as the equivariant structure makes $\mathcal{E}(\mathbb{A}^n_k)$

a graded projective module on the graded local ring $k[x_1, \cdots, x_n]$, so it is free.) Choose a section $s \in \mathcal{E}(\mathbb{A}^n_k)$ which does not vanish on $\mathbb{A}^n_k$ and is an eigenvector for the $\mathbb{G}_m$-action. This gives rise to a meromorphic section of $\mathcal{E}$. After twisting by some multiple of H (a $\mathbb{G}_m$-invariant divisor) we can assume s extends to a global section of $\mathcal{E}(NH)$ (also denoted by s) which does not vanish in codimension 1 and which is a $\mathbb{G}_m$-eigenvector. Let $Z = (s = 0)$ be the scheme-theoretic vanishing locus of s. If $Z = \emptyset$ we are done, so assume $Z \neq \emptyset$.

We want to understand the ideal I_Z. The support of Z is contained in H. Consider a point $p \in Z$ and an affine $\mathbb{G}_m$-neighborhood U containing p where $\mathcal{E}$ is trivialized. Such a neighborhood U is an $\mathbb{A}^1$-bundle over $W = U \cap H$. Then we have an isomorphism of $\mathbb{G}_m$-algebras:

$$k[U] \cong k[W][x_0],$$

where $k[W]$ is trivial as a $\mathbb{G}_m$-representation and $\mathbb{G}_m$ acts on $k \cdot x_0$ with weight 1. The trivialization of $\mathcal{E}$ on U allows us to write $I_Z(U) = (f, g) \subset k[U]$ where $f, g \in k[U]$ are eigenvectors for the $\mathbb{G}_m$-action (in particular, $Z \cap U$ is a complete intersection in U). By the above isomorphism of $\mathbb{G}_m$-algebras we have $f = f_W x_0^\alpha$ and $g = g_W x_0^\beta$ where $f_W, g_W \in k[W]$. Without loss of generality (using that Z has codimension 2) we can assume $\alpha = 0$. Because Z is supported on H, it follows that g_W is a unit in $k[W]/(f_W)$. Thus, we can write:

$$I_Z(U) = (f_W, x_0^\beta)$$

for some $f_W \in k[W]$ and $\beta > 0$.

Now this local description can be extended to global homogeneous equations for Z. First, the ideal $(f_W) \subset k[W]$ can be intrinsically defined by

$$(f_W) = I_Z(U) \cap k[W] \subset k[W].$$

Geometrically this means the following: if we consider the map $Z \rightarrow H$ given by linear projection from the point $[1 : 0 : \cdots : 0]$, the scheme-theoretic image of the map is an effective Cartier divisor $D \subset H$. Let $F \in H^0(H, \mathcal{O}_H(a))$ be the homogeneous equation for D. Consider F as a section of $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a))$ (i.e. as a homogeneous equation which does not depend on x_0). Locally on U we have the equality of ideals $(F|_U) = (f_W)$. Second, it suffices to show that the exponent β does not depend on the choice of point $p \in Z$. This is clear for any two points in an irreducible component of Z. Because $n \geq 3$, any two irreducible components of Z must intersect as they are set theoretically hypersurfaces in H. Therefore β is independent of $p \in Z$, and Z is a complete intersection defined by the equations $(F = x_0^\beta = 0) \subset \mathbb{P}$.

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REFERENCES

1. Barth, W., *Transplanting cohomology classes in complex-projective space*, Amer. J. Math. 92 (1970), 951–967.
2. Bertin, J. and Elencwajg, G., *Symétries des fibrés vectoriels sur* $\mathbf{P}^n$ *et nombre d'Euler*, Duke Math. J. 49 (1982), no. 4, 807–831.
3. Hartshorne, R., *Varieties of small codimension in projective space*, Bull. Amer. Math. Soc. 80 (1974), 1017–1032.
4. Horrocks, G. and Mumford, D., *A rank* 2 *vector bundle on* $\mathbf{P}^4$ *with* $15,000$ *symmetries*, Topology 12 (1973), 63–81.
5. Ilten, N. and Süss, H., *Equivariant vector bundles on* T*-varieties*, Transform. Groups 20 (2015), no. 4, 1043–1073.
6. Kaneyama, T., *Torus-equivariant vector bundles on projective spaces*, Nagoya Math. J. 111 (1988), 25–40.
7. Klyachko, A. A., *Equivariant bundles over toric varieties*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 5, 1001–1039, 1135.

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