Math 416

Worksheet 12. Graphs, Chapter 0

We briefly review graphs, though many of you have probably seen them before.

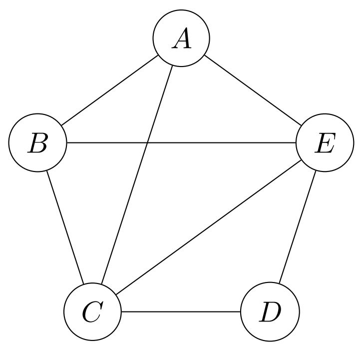
Definition. A **graph** is a set $V = V(G)$ of **vertices** (typically something like $V = \{1, 2, \ldots, n\}$) together with a set $E = E(G) \subseteq [V]^2$ of **edges**. (Edges are unordered pairs.)

Aside. The graphs we're considering now are **simple graphs**, that is graphs with at most 1 edge between two vertices and no loops. There are (at least) three natural ways to generalize them.

(i) You might allow loops, so that (x, x) can be an edge.
(ii) In a **directed graph** the edges are ordered pairs, so that (x, y) might be an edge while (y, x) is not.
(iii) You might also allow parallel edges, so that an edge (x, y) comes with a multiplicity that tells you how many copies of it appear in the graph.

Problem 1. Suppose that G is a graph with $n = |V|$ vertices. Is there a maximum number of edges that G can have? If so, what is that maximum?

Here is a picture of a graph with vertex set $\{A, B, C, D, E\}$.



(1)

In computing, a graph is often coded by its **adjacency matrix**. If G has n vertices $x_1, \ldots, x_n$, then the adjacency matrix of G (associated to this ordering of the vertices) is the $n \times n$ matrix M whose entries are either 0 or 1, as the following formula explains.

$$M_{ij} = \begin{cases} 1 & \text{if } (x_i, x_j) \in E \\ 0 & \text{if } (x_i, x_j) \notin E \end{cases}$$

Problem 2. Give the adjacency matrix of the graph pictured in (1).

Problem 3. How can you look at an $n \times n$ matrix and tell whether it is the adjacency matrix of some simple graph?

Another data structure associated to graph is the **adjacency list**, which stores in its i^{th} entry the list of vertices to which the i^{th} is adjacent. But let's not worry about that for now.

Definition. In a graph G, two vertices x and y are said to be **adjacent** if $\{x, y\}$ is an edge in G. We also say that x and y are **neighbors**.

The **degree** of a vertex x in a graph G is the number of its neighbors. We write this as

$$\deg_G(x) = |\{y \in V(G) : y \text{ is adjacent to } x\}| .$$

(Or just $\deg(x)$ if the graph G is clear from context.)

On the Problem Set you will prove...

Lemma (Handshake lemma)**.** The sum of the degrees is twice the number of edges. That is,

$$\sum_{v \in V(G)} \deg(v) = 2 \cdot |E|\,.$$

Definition (many standard definitions)**.** For a graph $G = (V, E)$, we define the following terms.

(a) A **walk** of **length** $n - 1$ is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices in which v_k and v_{k+1} are adjacent for every $k < n$. (Notice that vertices can be repeated.)
(b) A walk $(v_0, \ldots, v_n)$ is called a (simple) **path** of length n if the vertices $v_0, \ldots, v_n$ visited by the walk are all distinct: $j \neq k$ implies $v_j \neq v_k$.
(c) A walk $(v_0, \ldots, v_n)$ is a **cycle** if $(v_0, \ldots, v_{n-1})$ is a path of length $n - 1$, and v_{n-1} and v_n are adjacent, and $v_0 = v_n$.
(d) Vertices x and y are **connected** (in G) if there is a path from x to y. Connectedness defines an equivalence relation; the equivalence classes are called the **connected components** of G. If any two vertices in G are connected, then G is said to be a **connected** graph.
(e) A connected graph with no cycles (**acyclic**) is a **tree**.

Problem 4.

(a) Give examples of connected and disconnected graphs.
(b) Give two examples of trees. In each one, count the vertices and the edges. What do you notice?

Problem 5. Suppose that $G = (V, E)$ is a graph.

(a) Let v, w be vertices of G. Prove that if there is a walk from v to w in G, then there is a path from v to w in G.
(b) Sketch a proof that connectedness is an equivalence relation on the vertices of G. (Caution: there is something that needs to be proved!)

Remark. We say that a graph algorithm is **linear** if its running time is $O(m + n)$, where $m = |E|$ and $n = |V|$. It is a consequence of the Handshake Lemma that $n + m \in O(n^2)$.

Definition. A **complete graph** is a simple graph where every pair of vertices is connected by an edge. We denote the complete graph on n vertices by K_n.

Problem 6. (For fun)

(1) Prove that K_n is the simple graph with the most possible edges.
(2) Show that K_2, K_3, and K_4 can be *embedded in* $\mathbb{R}^2$ – meaning you can draw them in $\mathbb{R}^2$ without crossing any edges.
(3) Prove that K_5 is not planar, i.e. it cannot be embedded in $\mathbb{R}^2$.