# Review of the Proj construction.

$S$ a graded ring.

$$S = \bigoplus_{d \geq 0} S_d.$$

$\mathrm{Proj}(S)$ is a scheme with

- points = homogeneous prime ideals $\mathfrak{p}$ s.t. $\mathfrak{p} \not\supset \bigoplus_{d>0} S_d$.

- closed sets = $V(\mathfrak{a}) = \{ \mathfrak{p} \subset S \mid \mathfrak{p} \supseteq \mathfrak{a} \}$
  $\uparrow$
  homog. ideal.

If $f \in S_d$ $(d > 0)$, define:

- $\mathcal{U}_f = \mathrm{Proj}\, S \setminus V(f)$.

- $\mathcal{O}_{\mathrm{Proj}(S)}(\mathcal{U}_f) := (S_f)_0$
  $$= \left\{ \frac{a_i}{f^j} \,\middle|\, a_i, f^j \in S_{d_j} \right\}.$$
  "degree-0 part"

**Ex.** $A[\underbrace{x_0,\dots,x_n}_{\text{degree}=1}]$

$$\mathrm{Proj}(A[x_0,\dots,x_n]) = \mathbb{P}^n_A.$$

**Ex:** If $S \twoheadrightarrow S'$ is a surjection of graded rings then

$$\mathrm{Proj}(S') \subseteq \mathrm{Proj}(S)$$

is a closed embedding.

**Ex:** If $f_1,\dots,f_k \in A[x_0,\dots,x_n]$ are homog. polynomials then

$$\mathrm{Proj}\left(A[x_0,\dots,x_n]/(f_1,\dots,f_e)\right) \subseteq \mathbb{P}^n_A.$$

**Remark:** There is a map:

$$\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$$

Graded S-MODULES $M = \bigoplus_{d \geq d_0} M_d$.
give rise to q. coherent sheaves $\tilde{M}$ on Proj S.

$$\left(\text{For } f \in S_d : d > 0,\ (M_f)_0 = \tilde{M}(\mathcal{U}_f)\right).$$

## EXTENDED EXAMPLE

$S = \mathbb{C}[\underbrace{x_0, x_1}_{\text{degree}=1}]$ $\quad$ $\text{Proj } S = \mathbb{CP}^1$

$M = S(m)$

$$:= 0 \cdots 0 \oplus \underset{-m}{\mathbb{C}} \oplus \underset{-m+1}{\mathbb{C}\{x, y\}} \oplus \underset{-m+2}{\mathbb{C}\{x^2, xy, y^2\}} \oplus \cdots$$

degree

$$\cdots \oplus \underset{0}{\mathbb{C}\{x^m, x^{m-1}y, \ldots, y^m\}} \oplus \cdots$$

$\tilde{M} = \mathcal{O}_{\mathbb{CP}^1}(m)$.

Proj $S$ has an affine cover by:

($1/x$-chart) $S_x = \mathbb{C}[x^{\pm 1}, y]_0 = \mathbb{C}[\frac{y}{x}]$.

$\operatorname{Spec} \mathbb{C}[\frac{y}{x}] = \mathbb{A}^1_{\mathbb{C}}$

($1/y$-chart) $S_y = \mathbb{C}[x, y^{\pm 1}]_0 = \mathbb{C}[\frac{x}{y}]$.

$\operatorname{Spec} \mathbb{C}[x/y] = \mathbb{A}^1_{\mathbb{C}}$.

- $S(m)_x = \mathbb{C}[x^{\pm 1}, y]$

$$(S(m)_x)_0 = \bigoplus_{d \geq -m} \mathbb{C}\, y^{d+m}/x^d.$$

(generated by $x^m$ : rank 1 free $\mathbb{C}[x/y]$-mod)

- $S(m)_y = \mathbb{C}[x, y^{\pm 1}]$.

$$(S(m)_y)_0 = \bigoplus_{d \geq -m} x^{d+m}/y^d.$$

(generated by $y^m$ —— " —— )

# GLOBAL SECTIONS.

$\mathcal{O}_{\mathbb{CP}^1}(m)(\mathbb{CP}^1)$

can be obtained by gluing sections on $\mathbb{A}^1_{x/y}$ & $\mathbb{A}^1_{y/x}$.

$$\mathcal{O}_{\mathbb{CP}^1}(m)(\mathbb{A}^1_{y/x}) = \mathbb{C}x^m \oplus \mathbb{C}x^{m-1}y \oplus \cdots \oplus \mathbb{C}y^m \oplus \mathbb{C}\frac{y^{m+1}}{x} \oplus \cdots$$

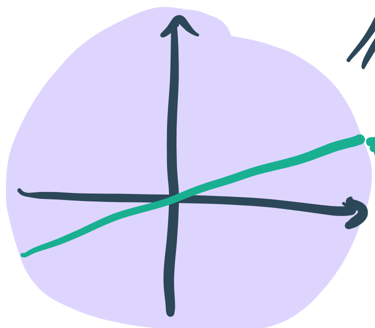
$$\mathcal{O}_{\mathbb{CP}^1}(m)(\mathbb{A}^1_{x/y}) =$$
$$\cdots \oplus \mathbb{C}\frac{x^{m+1}}{y} \oplus \mathbb{C}x^m \oplus \mathbb{C}x^{m-1}y \oplus \cdots \oplus \mathbb{C}y^m$$

$$\mathcal{O}_{\mathbb{CP}^1}(m)(\mathbb{CP}^1) = \mathbb{C}x^m \oplus \mathbb{C}x^{m-1}y \oplus \cdots \oplus \mathbb{C}y^m.$$

# Universal Line

$\mathrm{Spec}(\mathbb{C}[u,v])$ $\boxed{\begin{array}{l} u \text{ "=" } x \\ v \text{ "=" } y. \end{array}}$ $\mathrm{Proj}(\mathbb{C}[x,y])$

$\mathbb{A}^2$




$\mathbb{CP}^1$

closed pt $\leftarrow$ corresponds to a homog. prime ideal.

$\parallel$

a line.

$$\mathbb{A}^2 \times \mathbb{CP}^1 \supseteq \{(a,b)\times \ell \mid (a,b) \in \ell\}.$$

$$\parallel$$

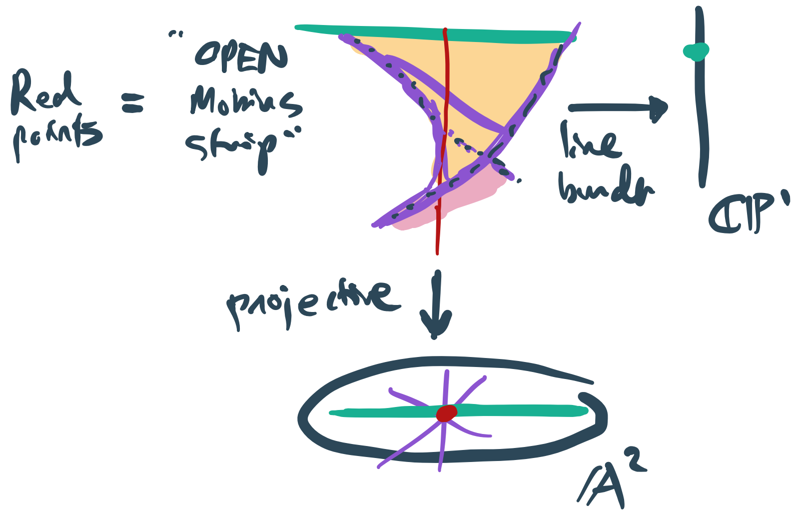
$$V(uy = vx)$$

<u>NOTE</u>: $\mathbb{A}^2 \times \mathbb{CP}^1 \cong \mathrm{Proj}(\underbrace{\mathbb{C}[u,v]}_{\text{degree } 0}[\underbrace{x,y}_{\text{degree } 1}])$

$uy - vx$ is homogeneous

"Universal line" := $\mathrm{Proj}(\mathbb{C}[u,v][x,y]/uy-vx)$.

## PICTURE




"the blow-up of $\mathbb{A}^2$ at a point."

# MAPS to Projective SPACE

$$\mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \dots, x_n])$$

has the invertible sheaf.

$$\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n_A}(1) := \widetilde{A[x_0, \dots, x_n](1)}.$$

and global sections:

$$x_0, \dots, x_n \in \mathcal{O}(1)(\mathbb{P}^n_A).$$

**MOTTO:** A map from a scheme $X/A$ to $\mathbb{P}^n_A$ is determined by

1. A line bundle (or invertible sheaf $\mathcal{L}$ on $X$)

2. $(n+1)$ global sections

$$s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$$

satisfying (*) (global generation)

What is this?

**Theorem** (Hartshorne 7.1).

(a) If $\varphi : X \longrightarrow \mathbb{P}^n_A$ is an $A$-morphism, then $\varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$ is an invertible sheaf globally generated by the sections

$$\varphi^*(x_i) \in \Gamma(X, \varphi^*\mathcal{O}(1)).$$

(b) If $\mathcal{L}$ is an invertible sheaf on $X$ and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ are sections that globally generate $\mathcal{L}$, then $\exists!$ $A$-morphism:

$$\varphi : X \longrightarrow \mathbb{P}^n_A$$

such that $\mathcal{L} \cong \varphi^*\mathcal{O}(1)$ & $s_i = \varphi^*(x_i)$.

**Proof:** (a) $s_i$ globally generates $\mathcal{L}$

$$\Longleftrightarrow \quad \mathcal{O}_X^{\oplus n+1} \xrightarrow[(s_0, \dots, s_n)]{} \mathcal{L}. \text{ surjective.}$$

Know: $\mathcal{O}_{\mathbb{P}^n_A} \xrightarrow[(x_0, \dots, x_n)]{} \mathcal{O}(1)$ surjective

and pullback is right exact.

(b) IDEA: Define the map:

$$X \longrightarrow \mathbb{P}^n_A$$

in pieces. Let:

$$X_i = X \setminus (s_i = 0)$$

$$\mathcal{U}_i = \mathbb{P}^n_A \setminus (x_i = 0)$$

<u>First</u>: $\bigcup X_i = X$.

(global generation)

<u>Second</u>: Define:

$$\varphi_i : X_i \longrightarrow \mathcal{U}_i = \operatorname{Spec}\left(A\left[\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right]\right) \cong \mathbb{A}^n.$$

by: $\varphi_i^*\left({}^{x_j}/_{x_i}\right) := {}^{s_j}/_{s_i}$.

NOTE: $\mathcal{O}_X \xrightarrow{s_i} \mathcal{L}$

restricts to $\mathcal{O}_X(X_i) \underset{\cong}{\xrightarrow{s_i}} \mathcal{L}(X_i)$
an iso
on $X_i$.

${}^1/_{s_i}$ = inverse. So: ${}^{s_j}/_{s_i} \in \mathcal{O}_X(X_i)$

Third: Need to check compatibility.

(exercise/office hours/if time available) ∎

IF TIME...

If $s_i \neq 0$ & $s_\ell \neq 0$, then:

$$\begin{tikzcd}
 & X_i \arrow[r, "\varphi_i"] & \mathbb{A}^n_i & \\
U_{i\ell} \arrow[ur] \arrow[r, "\varphi_{i\ell}"] \arrow[dr, "\varphi_{\ell i}"'] \arrow[ddr] & \mathbb{A}^n_i \setminus \left(\frac{x_\ell}{x_i} = 0\right) \arrow[u, hook] \arrow[d, "\wr"] & = \operatorname{Spec} A\left[\frac{x_1}{x_i}, \dots, \left(\frac{x_\ell}{x_i}\right)^{\pm 1}, \dots, \frac{x_n}{x_i}\right]. \\
 & \mathbb{A}^n_\ell \setminus \left(\frac{x_i}{x_\ell} = 0\right) \arrow[d, hook] & = \operatorname{Spec} A\left[\frac{x_1}{x_\ell}, \dots, \left(\frac{x_i}{x_\ell}\right)^{\pm 1}, \dots, \frac{x_n}{x_\ell}\right] \\
 & X_\ell \arrow[r] & \mathbb{A}^n_\ell &
\end{tikzcd}$$

$$\varphi_i^*\left(\frac{x_j}{x_i}\right) = s_j / s_i$$

$$\varphi_{i\ell}^*\left(\frac{x_j}{x_i}\right) = \varphi_{\ell i}\left(\frac{x_j}{x_\ell} \cdot \frac{x_\ell}{x_i}\right) = \frac{s_j}{s_\ell} \cdot \left(\frac{s_i}{s_\ell}\right)^{-1}$$

Q: Given a map:

$$\varphi : X \longrightarrow \mathbb{P}^n_A$$

corresponding to

$$(s_0, \dots, s_n, \mathcal{L})$$

as above. When is $\varphi$ a closed immersion?

<u>Proposition</u>. $\varphi$ is a closed immersion iff

(A) each $X_i$ is affine, and

(B) The maps:

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i})$$

above are surj.

<u>Proof</u>. Almost tautological. $\square$.

**Theorem** Let $k = \bar{k}$. $X$ a projective $k$-scheme.

$$\varphi : X \longrightarrow \mathbb{P}^n_k.$$

corresponding to $s_0, \dots, s_n$, $\mathcal{L}$. Let

$$\langle s_0, \dots, s_n \rangle = V \subset \Gamma(X, \mathcal{L})$$

$\varphi$ is a closed immersion $\Longleftrightarrow$

① Elements of $V$ separate points

i.e. for any 2 closed pts $P, Q \in X$ $\exists$ $s \in V$ s.t. $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$.

② Elements of $V$ separate tangent vectors.

$\forall$ closed points $P \in X$ the set $\{ s \in V \mid s_P \in m_P \mathcal{L}_P \}$ spans the $k$-vector space

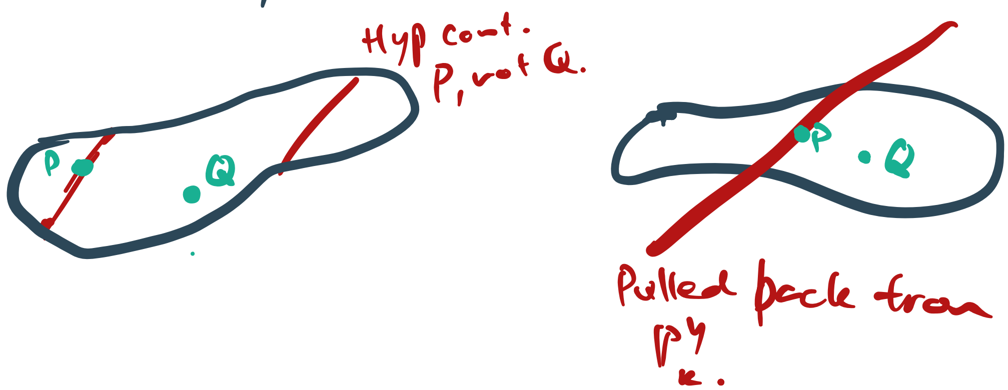
$$m_P \mathcal{L}_P / m_P^2 \mathcal{L}_P.$$

**Proof:** ($\Rightarrow$): is less interesting.

($\Leftarrow$): Sections in $V$ are pulled back from $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(1))$. So

1st: $\varphi : X \to \mathbb{P}^n_k$ is injective at the level of $k$-points.

Why?




2nd: Now: $\varphi$ is proper $\Rightarrow$ closed.

$\Rightarrow$ continuous closed injective $\Rightarrow$ $\varphi$ a homeo. onto $\varphi(X) \subseteq \mathbb{P}^n_k$

closed subscheme.

It suffices to show:

$$\varphi : X \to \varphi(X)$$

satisfies $\mathcal{O}_{\varphi(X)} \xrightarrow{\cong} \varphi_* \mathcal{O}_X$:

suffice to show:

$$\mathcal{O}_{\mathbb{P}^n} \longrightarrow \varphi_* \mathcal{O}_X \text{ is surjective.}$$

Can be checked locally at closed points.

$$\mathcal{O}_{\mathbb{P}^n, P} \longrightarrow \mathcal{O}_{X,P}$$

1. SAME RESIDUE FIELD.
2. $\varphi^* : \mathfrak{m}_{\mathbb{P}^n, P} \longrightarrow \mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$ is surjective.
3. $\varphi_* \mathcal{O}_X$ is a coherent $\mathcal{O}_{\mathbb{P}^n}$-mod.

**Lemma/Exercise/Hartshorne II.7.4/ APPLICATION of Nakayama's Lemma**

$f : A \to B$ local hom. of local Noetherian rings. w/

- $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ an isom.
- $\mathfrak{m}_A \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ surj.
- $B$ a f.g. $A$-mod.

$\Rightarrow f$ surjective.

$\blacksquare$

# Linear Systems

A map $X \to \mathbb{P}^n$ is determined by a line bundle $\mathcal{L}$ & sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$.

sections of $\mathcal{L}$ $\longleftrightarrow$ divisors on $X$.

Assume:

- $X$ a nonsingular projective variety $/k = \bar{k}$.
- $\mathcal{L}$ an invertible sheaf on $X$.
- $s \in \Gamma(X, \mathcal{L}) \setminus 0$.

There is a divisor (Weil = Cartier) "defined by" $s = 0$.

$s : \mathcal{O}_X \to \mathcal{L}$. determines $s^\vee : \mathcal{L}^{\vee} \to \mathcal{O}_X$

1. $s^\vee : \mathcal{L}^{\vee} \to \mathcal{O}_X$ is injective, so defines an ideal sheaf.
2. As $\mathcal{L}^{\vee}$ is locally free of rank $= 1$, the ideal is principal.

**Propn:** $X$ as above. $D_0 \subset X$ a divisor.
$\mathcal{L} \cong \mathcal{L}(D_0)$.

(a) $\forall\ s \in \Gamma(X, \mathcal{L}) \setminus 0$, the divisor
$$D = (s=0) \subset X$$
is effective and linearly equivalent to $D_0$.

(b) Every divisor $D \subset X$ s.t. $D \equiv_{\text{lin}} D_0$
is $(s=0) \subset X$ for some $s \in \Gamma(X, \mathcal{L}) \setminus 0$.

(c) 2 sections ; $s, s' \in \Gamma(X, \mathcal{L}) \setminus 0$.
define the same divisor $\Longleftrightarrow$ $s = \lambda s'$.
for some $\lambda \in k^{\times}$.

**ASIDE before proof of (b).**

**Q1.** How do we define $\mathcal{L}(D_0)$?

$k(X) \leftarrow$ field of fractions of $X$: is a q-coh. sheaf actually.
$\eta = \operatorname{Spec} k(X) \xrightarrow{i} X$.
i.e. $\mathcal{O}_\eta(\mathcal{U}) = k(X)$.

- For any open set $\mathcal{U} \subset X$ where
$$I_{D_0 \cap \mathcal{U}} = (f_{\mathcal{U}}) \subset \mathcal{O}_X(\mathcal{U}).$$
is principal, consider:
$$\mathcal{O}_X(\mathcal{U}) \cdot \frac{1}{f_{\mathcal{U}}} \subset k(X).$$

- These glue & $\left(\mathcal{U}, f_{\mathcal{U}} \cdot \frac{1}{f_{\mathcal{U}}}\right) = s$ glues to a global section that vanishes on $D_0$.

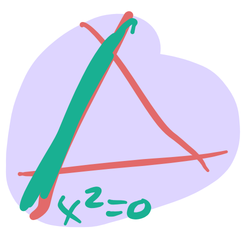
The other global sections of $\mathcal{L}(D_0)$ are the rational functions

$$h \in k(X)$$

such that for a trivializing cover $\{\mathcal{U}_i\}$ of $\mathcal{L}(D_0)$,

$$h \in \mathcal{O}_X(\mathcal{U}_i)\cdot \frac{1}{f_{\mathcal{U}_i}} \subset k(X).$$

**Ex.** $(x^2=0) \subset \mathbb{P}^2 = \mathrm{Proj}(k[x,y,z])$.




$\frac{1}{x}$-chart: $\mathrm{Spec}(k[\frac{y}{x}, \frac{z}{x}])$
generator is $1$.
Look at $y/x$, $(y/x)^2$, $(y/x)^3$.

$\frac{1}{y}$-chart: $\mathrm{Spec}(k[\frac{x}{y}, \frac{z}{y}])$
generator is $(x/y)^2$.

$\frac{y}{x} = \left(\frac{x}{y}\right)\cdot\left(\frac{x}{y}\right)^{-2}$ ✓ $\quad k[\frac{x}{y}, \frac{z}{y}]\cdot\left(\frac{x}{y}\right)^{-2}$

$\left(\frac{y}{x}\right)^2 = 1\cdot\left(\frac{x}{y}\right)^{-2}$ ✓ $\quad \left(\frac{y}{x}\right)^3 = \left(\frac{x}{y}\right)^{-1}\cdot\left(\frac{x}{y}\right)^{-2}$ ✗

CAN CHECK: GLOBAL SECTIONS of $\mathcal{O}((x^2=0))$ restricted to $\frac{1}{x}$-chart are

$$\begin{array}{ccc} 1 & \frac{y}{x} & \left(\frac{y}{x}\right)^2 \\ \frac{z}{x} & \frac{zy}{x^2} & \\ \frac{z^2}{x^2} & & \end{array}$$

**Lemma / Exercise.** (Done already?)
Let $\mathcal{L} = \mathcal{L}(D_0)$. Global sections of $\mathcal{L}$ correspond to $h \in k(X)$ such that

$(h) + D_0$ is effective.

**Proof of (b)** Want to show, if $D$ effective & $D \equiv_{lin} D_0$
$\exists\ s \in \Gamma(X, \mathcal{L}(D_0))$ such that

$$(s=0) = D \subset X.$$

$D \equiv_{lin} D_0 \iff D - D_0 = (h)$
for some $h \in k(X)$.

But then $(h) + D_0$ is effective

$\Rightarrow$ $h$ gives a global section of $\mathcal{L}(D_0)$

& locally $h = \dfrac{f_D}{f_{D_0}}$ eqns defining the ideal of $D, D_0$ resp:

$\Rightarrow$ locally the section $h$ is $f_D$ times the generator of $\mathcal{L}(D_0)$.

$\square$

$$\left\{ \begin{matrix} \text{eff. divisors} \\ D \equiv_{\text{lin}} D_0 \end{matrix} \right\} = \mathbb{P}(\Gamma(X, \mathcal{L}))$$

a projective space!

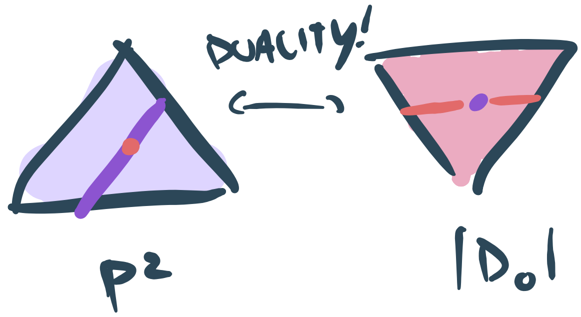
<u>Defn:</u> A **complete linear system** on a nonsingular projective variety $X$ is the projective space of effective divisors linearly equiv. to some $D_0$.

<u>EXAMPLE</u>: $\mathbb{P}^2 = \mathbb{P}^2_k$; $D_0 = (x_0 = 0)$.

$\mathcal{O}(D_0) = \mathcal{O}_{\mathbb{P}^2}(1)$.

$\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$.

$|D_0| \cong \mathbb{P}^2$



**Defn:** A linear system $\Lambda$ on $X$ is a subset of a complete linear series $|D_0|$ corresponding to a vector subspace:

$$V \subset \Gamma(X, \mathcal{L}).$$

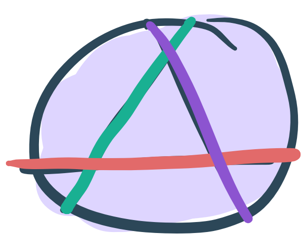
A base point $p$ of $\Lambda$ is a point $p \in X$ s.t. $p \in D$ for all $D \in \Lambda$.

The dimension of the linear system is $\dim_k V - 1$.

## MORAL

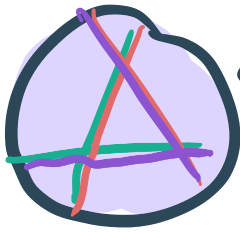
$$\left\{ \begin{array}{c} \text{Non-degenerate maps} \\ X \to \mathbb{P}^n_k \\ \text{up to } PGL_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Base point free linear} \\ \text{systems on } X \\ \text{of dimension } n \end{array} \right\}.$$

**Ex.**



$V = \langle x^2, y^2, z^2 \rangle \in \Gamma(\mathbb{P}^2, \mathcal{O}(2))$ is a base-point free, nondegenerate 2D linear system.

BUT <u>not</u> complete.

Ex.  $\langle xy, yz, xz \rangle \subset \Gamma(\mathbb{P}^2, \mathcal{O}(2))$

is 2-dim'l but <u>not</u> basepoint-free & not complete.

It defines a map on the complement of

$$[1:0:0],\ [0:1:0],\ [0:0:1].$$

(this gives an example of a birational automorphism that is <u>not</u> an automorphism.

## Relative Proj

Proj: Input: graded algebra $S = \bigoplus_{d \geq 0} S_d$.

Output: Proj $S$ scheme w/ a map to $\operatorname{Spec}(S_0)$.

Proj: Input: sheaf of graded $\mathcal{O}_X$-algebras.

$$\mathcal{O}_X \to S_\cdot = \bigoplus_{d\geq 0} S_d$$

Output: $\mathrm{Proj}(S_\cdot)$ a scheme with a map to $X$.

Assumptions:

- $S_\cdot = \bigoplus_{d\geq 0} S_d$ is a q-coherent $\mathcal{O}_X$-algebra.
- $S_0 = \mathcal{O}_X$, $S_1$ is coherent, $S_\cdot$ is generated as an algebra by $S_1$.

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Construction:

For each affine open set $\mathcal{U} \subset X$ let:

- $$\mathrm{Proj}(S_\cdot)_{\mathcal{U}} := \mathrm{Proj}(\mathcal{O}_X(\mathcal{U}) \oplus S_1 \oplus \cdots)$$
$$\downarrow \pi_{\mathcal{U}}.$$
$$\mathcal{U} = \mathrm{Spec}(\mathcal{O}_X(\mathcal{U})).$$

- For any affine open sets $V, \mathcal{U}$ with $V \overset{i}{\subset} \mathcal{U}$.

$$\begin{tikzcd} \mathrm{Proj}(S.)_V \arrow[r] \arrow[d, "\pi_V"'] & \mathrm{Proj}(S.)_{\mathcal{U}} \arrow[d, "\pi_{\mathcal{U}}"] \\ V \arrow[r, "i"] & \mathcal{U} \end{tikzcd}$$

These glue to a scheme called

$$\mathrm{Proj}(S.) \xrightarrow{\pi} X.$$

- The invertible sheaf $\mathcal{O}(1)$ defined on each

$$\mathrm{Proj}(S.)_{\mathcal{U}} \leftarrow \text{affine}$$

extends to an invertible sheaf on all $\mathrm{Proj}(S.)_{\mathcal{U}}$.

(**Remark:** under very mild assumptions: $\pi: \mathrm{Proj}(S.) \longrightarrow X$ is projective & it is always proper.)

# IMPORTANT EXAMPLES.

1. Let $X$ be Noetherian & let $Z \subset X$ be a closed subscheme w/ ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$.

   Then

$$\mathrm{Proj}\left(\bigoplus \mathcal{I}_Z^k\right) \longrightarrow X \qquad (\mathcal{I}_Z^0 := \mathcal{O}_X).$$

   is the blow-up of $X$ at $Z$.

2. Let $\mathcal{E}$ be a locally-free sheaf of rank $= r$.

   Consider the sheaf

$$\mathrm{Sym}^\bullet \mathcal{E} = \bigoplus_{k \geq 0} \mathrm{Sym}^k \mathcal{E}.$$

   1. $\mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{E}) := \mathbb{E}^\vee \longrightarrow X$

      the rank $r$ vector-bundle associated to $\mathcal{E}^\vee$.

   2. $\mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E}) := \mathbb{P}(\mathcal{E}) \longrightarrow X.$

      · the projective bundle associated to $\mathcal{E}$.

**Proposition:** Let $X, \mathcal{E}, \mathbb{P}(\mathcal{E})$ be as above.

(a) if rank $\mathcal{E} \geq 2$ $\exists$ a canonical Born:

$$\mathrm{Sym}^\bullet \mathcal{E} \cong \bigoplus_{\ell \in \mathbb{Z}} \pi_* \left(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)\right).$$

$$\left( \text{so } \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell) = \begin{cases} 0 & \ell < 0 \\ \mathcal{O}_X & \ell = 0. \\ \mathrm{Sym}^\ell \mathcal{E} & \ell > 0. \end{cases} \right)$$

(b) $\exists$ a natural surjective morphism

$$\pi^*(\mathcal{E}) \twoheadrightarrow \mathcal{O}(1).$$

(c) Let $Y \xrightarrow{\varphi} X$ be a morphism. Then there is a natural bijection:

$$\left\{ \begin{matrix} \text{maps} \\ \psi: Y \to \mathbb{P}(\mathcal{E}) \\ \text{over } X \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{line bundles } \mathcal{L} \\ \text{on } Y + \text{surjections} \\ f: \varphi^* \mathcal{E} \twoheadrightarrow \mathcal{L} \end{matrix} \right\}.$$

**Sketch of Proof:**

Let $\mathcal{U} \subseteq X$ be an affine open set w/ $\mathcal{U} = \mathrm{Spec}\, A$ s.t. $\mathcal{E}(\mathcal{U})$ is free of rank $r$ on $\mathcal{U}$.

Know: $\mathbb{P}(\mathcal{E})_{\mathcal{U}} := \mathrm{Proj}_A (A \oplus \mathcal{E}|_{\mathcal{U}} \oplus \mathrm{Sym}^2 \mathcal{E}|_{\mathcal{U}} \oplus \cdots) = \mathbb{P}^n_A.$

**Hart. II.5.13** $\Rightarrow$ $(\pi_{\mathcal{U}})_* (\mathcal{O}(\ell)) \underset{\text{nat.}}{\cong} \mathrm{Sym}^\ell \mathcal{E}(\mathcal{U}).$

$$\underset{\text{adjunction}}{\Longrightarrow} \quad \mathrm{Hom}_{\mathcal{U}}(\mathrm{Sym}^{\ell}\mathcal{E}(\mathcal{U}), (\pi_{\mathcal{U}})_{*}(\mathcal{O}(\ell)))$$
$$\|\!\wr$$
$$\mathrm{Hom}_{\mathbb{P}(\mathcal{E})_{\mathcal{U}}}(\pi^{*}\mathrm{Sym}^{\ell}(\mathcal{E}(\mathcal{U})), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)).$$

$\Rightarrow \exists$ a map $\pi^{*}\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Locally: over affine opens $\mathcal{U} \subset X$
we have $\pi^{*}\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is surjective.

Lastly:

For any map: $Y \xrightarrow{\varphi} \mathbb{P}(\mathcal{E})$ consider

$$\varphi^{*}\mathcal{E} \cong \varphi^{*}(\pi^{*}\mathcal{E}) \twoheadrightarrow \mathcal{O}(1).$$

Likewise: Given a surjection:

$$f : \varphi^{*}\mathcal{E} \twoheadrightarrow \mathcal{L}.$$

take affine opens $\mathcal{U} \subset X$ that trivialize $\mathcal{E}$.

There is a unique map:

$$\varphi^{-1}(\mathcal{U}) \xrightarrow[\varphi_{\mathcal{U}}]{} \mathbb{P}^{n}_{\mathcal{U}} =: \mathbb{P}(\mathcal{E})_{\mathcal{U}}.$$

These maps glue. $\blacksquare$

# Examples of Projective Bundles

Let $\mathbb{P}^1 = \mathbb{P}^1_k$.

Let $\mathcal{E}_{a,b} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ for $a, b \in \mathbb{Z}$.

Q. When is $\mathbb{P}(\mathcal{E}_{a,b}) \cong \mathbb{P}(\mathcal{E}_{c,d})$?

**Easier question:**

when is $\mathbb{P}(\mathcal{E}_{a,b}) \cong \mathbb{P}(\mathcal{E}_{c,d})$ as schemes$/\mathbb{P}^1$?

(HW: Hartshorne II.7.9).

$\Rightarrow \mathbb{P}(\mathcal{E}_{a,b}) \cong \mathbb{P}(\mathcal{E}_{c,d}) \iff \exists\, \mathcal{L} \in \mathrm{Pic}(\mathbb{P}^1)$ s.t. $\mathcal{E}_{a,b} \otimes \mathcal{L}$

$$\left( \mathrm{Pic}\,\mathbb{P}^1 = \underset{(l \in \mathbb{Z})}{\{\mathcal{O}(l)\}}. \right)$$

$$\iff |a-b| = |c-d|.$$

(ASIDE: If $k = \mathbb{C}$, the complex manifolds $\mathbb{P}(\mathcal{E}_{a,b})$, $\mathbb{P}(\mathcal{E}_{c,d})$ are diffeomorphic $\iff a+b \equiv c+d \pmod 2$).

**Example** $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$.

$$\begin{array}{c} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)). \\ \big\downarrow \pi \\ \mathbb{P}^1 \end{array}$$

What are the global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$?

Have: $\mathcal{E} \cong \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

$$\Rightarrow \Gamma(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong \Gamma(X, \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$$

$$\cong \Gamma(X, \mathcal{E})$$

Know: $\mathcal{E}$ globally generated.

(if $\mathbb{P}^1$ has coord. $x, y$:

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & y \end{bmatrix}} \mathcal{O} \oplus \mathcal{O}(1)$$

is surjective).

$$\Rightarrow \underset{\substack{\| \\ \mathcal{O}^{\oplus 3}_{\mathbb{P}(\mathcal{E})}}}{\pi^*(\mathcal{O}^{\oplus 3})} \to \mathcal{E} \quad \text{surj.}$$

$$\Rightarrow \begin{tikzcd} \pi^* \mathcal{O}^{\oplus 3} \arrow[r] \arrow[rr, bend left, "\text{surjective.}"] & \mathcal{E} \arrow[r] & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \end{tikzcd}$$

Determines a map: $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^2_k$. $\blacksquare$

# Blowing Up.

$Z \subset X.$ $\quad$ $\tilde{X} = \mathrm{Proj}(\mathcal{O} \oplus I_Z \oplus I_Z^2 \oplus \cdots)\ldots$

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"] \\ X \end{tikzcd}$$

<u>Defn</u>: Let $f : Y \to X$ be a map and let $I \subset \mathcal{O}_X$ be an ideal sheaf. The inverse image ideal $f^{-1}I \subset \mathcal{O}_Y$ is the image of

$$f^*I \to f^*\mathcal{O}_X \cong \mathcal{O}_Y.$$

**Theorem** $X$ a scheme,
$I$ a coherent ideal sheaf.
$\pi : \tilde{X} \to X$ the blow-up.

(a) $f^{-1}(I)$ is invertible on $\tilde{X}$: ($f^{-1}(I)$ defines the <u>exceptional</u> divisor).

(b) If $Z \subset X$ corresponds to $I$ & $U = X \setminus Z$ then
$$\pi|_U : \pi^{-1}(U) \xrightarrow{\cong} U.$$

(c) If $f : Y \to X$ is a map of Noetherian schemes and $f^{-1} I = I_Y$.
Let $\tilde{Y}$ be the blow-up of $Y$ at $I_Y$.
$\exists!$ morphism:

$$\begin{tikzcd} \tilde{Y} \arrow[r, dashed, "\tilde{f}"] \arrow[d, "\pi_Y"'] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If $f$ is a closed immersion, then so is $\tilde{f}$. (In this case, we call $\tilde{Y}$ the <u>strict transform</u> of $Y$.)

If $X$ is a variety/$k$ & $I \neq 0$, then:

(d) $\tilde{X}$ is also a variety.

(e) $\pi$ is birational/proper, surjective.

**Proof.**

(a) Want to check $f^{-1}I$ is invertible.

Locally have

$$\mathcal{U} = \operatorname{Spec} A \subset X.$$

$$I(U) = (f_1, \dots, f_\ell) \subset A.$$

$$\tilde{X}_{\mathcal{U}} = \operatorname{Proj}(\underbrace{A \oplus I \oplus I^2 \oplus \cdots}_{S.})$$

$$\cup$$

$$\operatorname{Spec}(S_{f_1})_0$$

$$\begin{array}{ccc} A & \longrightarrow & (S_{f_1})_0 \\ \cup & & \\ I & \longrightarrow & (S_{f_1})_0 . \\ f_1 & \longrightarrow & f_1 \\ f_2 & \longrightarrow & f_2 = f_1 \cdot \frac{f_2}{f_1} \\ \vdots & & \\ f_\ell & \longrightarrow & f_1 \cdot \frac{f_\ell}{f_1} . \end{array}$$

Observe (as we said)

$$f^{-1}I = \mathcal{O}_{\tilde{X}}(1).$$

(c) We'll check this locally for closed immersions:

$$\begin{tikzcd} \operatorname{Spec} B = Y \arrow[r, hook] & X = \operatorname{Spec} A \\ \operatorname{Proj}(\underbrace{B \oplus f^{-1}I \oplus \cdots}_{S_\cdot'}) \arrow[u] & \operatorname{Proj}(\underbrace{A \oplus I \oplus \cdots}_{S_\cdot}) = \tilde{X} \arrow[u] \end{tikzcd}$$

For $f \in I$ have:

$$S_\cdot = A \oplus I \oplus \cdots \twoheadrightarrow B \oplus f^{-1}I \oplus \cdots =: S_\cdot'$$

Given: $(S_\cdot)_f \twoheadrightarrow (S_\cdot')_f$

AND: $(S_\cdot)_{f\,0} \twoheadrightarrow (S_\cdot')_{f\,0}$. ✓

(d) $X$ a variety $\Longrightarrow$ $\tilde{X}$ a variety.

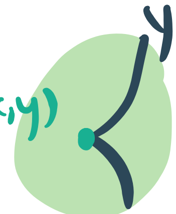
Check locally. $\mathcal{U} = \operatorname{Spec} A \subset X$.

1. $S_\cdot = A \oplus I \oplus I^2 \oplus \cdots$ is an integral domain
2. $\Rightarrow (S_\cdot)_f$ is integral.
3. $\Rightarrow (S_\cdot)_{f\,0}$ is integral $\square$

## EXAMPLE.

$Y = (y^2 - x^3 = 0)$.

$I = (x,y)$



$X = \mathbb{A}^2_k$

$$\tilde{X} = \mathrm{Proj}(k[x,y] \oplus I \oplus I^2 \oplus \cdots).$$

<u>NOTE</u>: There is a quotient map:

$$\underbrace{k[x,y][u,v]}_{\text{degree}=1} \longrightarrow k[x,y] \oplus I \oplus I^2 \oplus \cdots$$

$\left( u \to \underset{\overset{\wedge}{I}}{x} \quad v \to y \in_I \right)$.

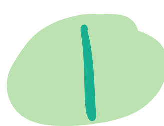
$$\Rightarrow \tilde{X} \subseteq \mathrm{Proj}(k[x,y][u,v]) = \mathbb{A}^2 \times \mathbb{P}^1.$$

Ideal is $xv = yu$.

$\mathbb{P}^1$

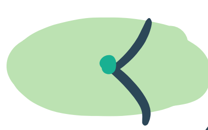
2 charts:

$$k[x,y][v/u] \to k[x,y,w=v/u] \big/ (xw = y) \cong k[x,w].$$



$(x,w)$ $\mathbb{A}^2$

$(x,xw)$



$\mathbb{A}^2$

$$k[x,y][u/v] \longrightarrow k[x,y,z=\tfrac{u}{v}] \big/ (x = yz) \cong k[y,z].$$

$(yz,y)$

$(z,y)$

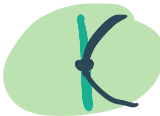
$\mathbb{A}^2$

$$K \subset k[x,y][u,v] \twoheadrightarrow k[x,y]/_{y^2-x^3} \oplus f^{-1}I \oplus (f^{-1}I)^2 \oplus \cdots$$

$$K = (y^2-x^3,\ xv-yu,\ yv-x^2u,\ v^2-xu^2).$$

1st chart

$$k[x,w]/_{w^2-x}$$



2nd chart

$$k[z,y]/_{1-xz^2 = 1-yz^3}$$

$yz$.