THE CURIOUS MODULI SPACES OF UNMARKED KLEINIAN SURFACE GROUPS

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Abstract. Fixing a closed hyperbolic surface S, we define a moduli space $\mathrm{A}\mathcal{I}(S)$ of unmarked hyperbolic 3-manifolds homotopy equivalent to S. This 3-dimensional analogue of the moduli space $\mathcal{M}(S)$ of unmarked hyperbolic surfaces homeomorphic to S has bizarre local topology, possessing many points that are not closed. There is, however, a natural embedding $\iota : \mathcal{M}(S) \to \mathrm{A}\mathcal{I}(S)$ and compactification $\mathrm{A}\overline{\mathcal{I}}(S)$ such that ι extends to an embedding of the Deligne-Mumford compactification $\overline{\mathcal{M}}(S) \to \mathrm{A}\overline{\mathcal{I}}(S)$.

For a closed oriented hyperbolic surface S, the moduli space $\mathcal{M}(S)$ of hyperbolic (or Riemann) surfaces homeomorphic to S is a familiar and interesting object. Hyperbolic 3-manifolds homotopy equivalent to S have also been intensely studied since the foundational work of Ahlfors and Bers. An analogous moduli space of (unmarked) hyperbolic 3-manifolds homotopy equivalent to S is rarely considered. Here we define such a topological moduli space $\mathrm{A}\mathcal{I}(S)$, study a bit of its local topology, and define a natural compactification $\mathrm{A}\overline{\mathcal{I}}(S)$.

Perhaps a reason for its anonymity, the local topology of $\mathrm{A}\mathcal{I}(S)$ is bizarre. We can reinterpret interesting phenomena from the theory of Kleinian groups as statements about $\mathrm{A}\mathcal{I}(S)$. There exist singly-degenerate Kleinian groups whose ending lamination is fixed by a pseudo-Anosov homeomorphism. This beautiful fact implies $\mathrm{A}\mathcal{I}(S)$ is not well separated, possessing many points that are not closed. Compactness of Bers slices implies that even the geometrically finite points fail to form a Hausdorff space.

As penance for its local foibles, $\mathrm{A}\mathcal{I}(S)$ offers better global topological behavior. Recall the Deligne-Mumford compactification $\overline{\mathcal{M}}(S)$ is obtained by adding Riemann surfaces with nodes. The space $\mathrm{A}\mathcal{I}(S)$ has an analogous compactification $\mathrm{A}\overline{\mathcal{I}}(S)$ obtained by adding hyperbolic 3-manifolds N homotopy equivalent to S cut along simple closed curves, such that the corresponding "cut" homotopy classes in N are parabolic. We begin by constructing an augmented deformation space $\mathrm{A}\overline{\mathrm{H}}(S)$, which is the analogue of

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the augmented Teichmüller space, and then show that its quotient $\mathrm{A}\overline{\mathcal{I}}(S)$ is sequentially compact. Sequential compactness follows by combining theorems of Thurston [25, Thm.6.2] and Canary-Minsky-Taylor [10] to show that, up to re-marking, a sequence in $\mathrm{AH}(S)$ always converges on the complement of a multicurve in S.

There is an embedding $\mathcal{M}(S) \to \mathrm{A}\mathcal{I}(S)$ with image the closed set of Fuchsian manifolds. This embedding extends to an embedding $\overline{\mathcal{M}}(S) \to \mathrm{A}\overline{\mathcal{I}}(S)$ of the Deligne-Mumford compactification. Hopefully connoisseurs of the more familiar $\mathcal{M}(S)$ will find it interesting to see how the story changes abruptly in 3-dimensions, with local structure suffering while global compactness manages to survive.

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2. Preliminaries

Let S be a compact oriented hyperbolizable surface with (possibly empty) boundary. We do not assume S is connected. Denote the set of components of S by $c(S)$. For convenience, put a hyperbolic metric on S so that any boundary curves are very short horocycles. In this case S embeds isometrically into a complete finite area hyperbolic surface $\hat{S}$ homeomorphic to the interior of S. By choosing the boundary curves to be sufficiently short, we can assume the simple closed geodesics of $\hat{S}$ are contained in S. A multicurve b is a (necessarily finite) set of pairwise disjoint simple closed geodesics on S. (As geodesics, no component of b is homotopic into ∂S.) Let $\mathcal{N}b \subset S$ denote a small open collar neighborhood of b.

All hyperbolic 3-manifolds in this article are assumed to be oriented and complete. Unless otherwise stated, they have no boundary. However, our manifolds will not always be connected. Let $\mu_3 > 0$ be the Margulis constant for hyperbolic 3-manifolds. Recall that μ_3 is a number with the following property. If N is any hyperbolic 3-manifold then the subset of points in N with injectivity radius at most μ_3 is a disjoint union of two types of sets: an embedded solid torus neighborhood of a simple closed geodesic with length less than μ_3, or a properly embedded product region $T \times [0, \infty)$ where T is either a torus or a noncompact annulus without boundary [1]. Define the cusps of N to be the components of the form $T \times [0, \infty)$.

We now give two definitions of the set $\mathrm{H}(S, \partial S)$ of marked Kleinian surface groups associated to S; one algebraic and one geometric. These two perspectives are both useful. Many of the following definitions will have an algebraic and a geometric formulation. We begin with the algebraic definition. Let $P \subseteq \partial S$ be a subset of boundary components. (We are interested only in the two cases $P = \partial S$ and $P = \emptyset$.) When S is connected, the set $\mathrm{H}(S, P)$ of marked Kleinian surface groups associated to S is the set of conjugacy classes of discrete faithful representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$

such that if $\gamma \in \pi_1(S)$ can be represented by a curve freely homotopic into P, then $\rho(\gamma)$ is parabolic. If S is disconnected, $\mathrm{H}(S,P)$ is the Cartesian product of $\mathrm{H}(R, P\cap R)$ over the components R of S. The notation $\mathrm{H}(S)$ is shorthand for $\mathrm{H}(S,\emptyset)$.

Abusing notation slightly, we will often simply use ρ to denote an element of $\mathrm{H}(S,P)$. If necessary, we will distinguish the representations of the components of S using the notation $\{\rho_R\}_{R\in c(S)}$, where $c(S)$ is the set of components of S. An element ρ of $\mathrm{H}(S,P)$ determines a hyperbolic 3-manifold

$$N_\rho = \coprod_{R\in c(S)} \mathbf{H}^3/\rho_R(\pi_1(R)),$$

together with a homotopy equivalence $m_\rho : S \to N_\rho$, known as the marking of N_ρ, induced by the isomorphism $\pi_1(S) \to N_\rho$ (with the obvious modification for disconnected S). The marking sends P into the cusps of N_ρ. The phrase, "a hyperbolic manifold in $\mathrm{H}(S,\partial S)$," indicates such a pair (N_ρ, m_ρ).

With this pair of objects in mind, we may define $\mathrm{H}(S,P)$ more geometrically as follows. For an oriented hyperbolic 3-manifold N and a homotopy equivalence $m : S \to N$, $(N,m) \in \mathrm{H}(S,P)$ if m takes P into the cusps of N. Two pairs (N_1,m_1) and (N_2,m_2) are equal in $\mathrm{H}(S,P)$ if there exists an orientation preserving isometry $\iota : N_1 \to N_2$ such that $\iota\circ m_1$ is homotopic to m_2.

By Bonahon's tameness theorem, any hyperbolic manifold in $\mathrm{H}(S,\partial S)$ is homeomorphic to $S\times\mathbf{R}$ [4]. We could "stiffen" the above geometric definitions using homeomorphisms rather than homotopy equivalences. Nothing is gained from this, and we will keep the above more traditional definitions.

An element $\{\rho_R\}_{R\in c(S)}$ of $\mathrm{H}(S,\partial S)$ is Fuchsian if each ρ_R is conjugate to a representation with image in the group $\mathrm{PSL}_2(\mathbf{R})$ of isometries of the hyperbolic plane. Equivalently, a hyperbolic manifold in $\mathrm{H}(S,\partial S)$ is Fuchsian if it contains an embedded totally geodesic hyperbolic surface homotopy equivalent to N.

Define a topology on $\mathrm{H}(S,P)$ as follows. Assume first that S is connected. Consider the subset $\mathrm{Hom}_P(\pi_1(S),\mathrm{PSL}_2(\mathbf{C})) \subseteq \mathrm{Hom}(\pi_1(S),\mathrm{PSL}_2(\mathbf{C}))$ of homomorphisms taking conjugacy classes of P to parabolic elements of $\mathrm{PSL}_2(\mathbf{C})$. Put the compact-open topology on $\mathrm{Hom}_P(\pi_1(S),\mathrm{PSL}_2(\mathbf{C}))$. Let $\mathcal{D}$ denote the subset of faithful homomorphisms with discrete image. (The set $\mathcal{D}$ is closed [12, 14].) The isometry group $\mathrm{PSL}_2(\mathbf{C})$ acts by conjugation on $\mathrm{Hom}_P(\pi_1(S),\mathrm{PSL}_2(\mathbf{C}))$. There is a subset $\mathcal{O}$ of $\mathrm{Hom}_P(\pi_1(S),\mathrm{PSL}_2(\mathbf{C}))$ containing $\mathcal{D}$ such that the quotient of $\mathcal{O}$ by this action is a smooth complex manifold. (The set $\mathcal{O}$ is the set of nonradical representations. See [15, Sec. 4.3].) The subset $\mathcal{D}$ is preserved by the $\mathrm{PSL}_2(\mathbf{C})$-action, and it makes sense to define the topological quotient

$$\mathrm{AH}(S,P) := \frac{\mathcal{D}\cap \mathrm{Hom}_P(\pi_1(S),\mathrm{PSL}_2(\mathbf{C}))}{\mathrm{PSL}_2(\mathbf{C})}.$$

$\mathrm{AH}(S,P)$ is a subspace of the manifold $\mathcal{O}/\mathrm{PSL}_2(\mathbf{C})$, which in turn can be algebraically "completed" to the character variety $X(S,P)$, an algebraic variety which is intuitively the quotient of $\mathrm{Hom}_P(\pi_1(S),\mathrm{PSL}_2(\mathbf{C}))$ by the $\mathrm{PSL}_2(\mathbf{C})$-action. For more information on this construction see [13, 15]. If S is not connected then topologize $\mathrm{H}(S,P)$ as the topological product of the $\mathrm{H}(R,P\cap R)$ for $R\in c(S)$. This topology on $\mathrm{AH}(S,P)$ is called the algebraic topology.

More geometrically, we say that a sequence $\{(N_n,m_n)\}$ in $\mathrm{H}(S,\partial S)$ converges algebraically to (N,m) if there exist homotopy equivalences $h_n : N \to N_n$ such that m_n is homotopic to $h_n\circ m$ for all n and $\{h_n\}$ C^∞-converges to a local isometry on every compact subset of N. (The equivalence of these definitions is discussed in [20, Sec. 3.1].)

The interior $\mathrm{QF}(S)$ of $\mathrm{AH}(S,\partial S)$ (as a subset of $X(S,\partial S)$) consists of the quasifuchsian hyperbolic 3-manifolds [19, 23]. We recall that if $N = \mathbf{H}^3/\Gamma$ then its conformal boundary $\partial_c N$ is the quotient of the domain of discontinuity for the action of Γ on the Riemann sphere. A hyperbolic 3-manifold N in $\mathrm{AH}(S,\partial S)$ is quasifuchsian if the conformal bordification $N\cup \partial_c N$ of N is homeomorphic to $S\times[0,1]$. Bers [2] showed that a quasifuchsian hyperbolic 3-manifold is determined by the conformal structure on $S\times\{0,1\}$ and that any conformal structure arises. If $(X,Y)\in\mathcal{T}(S)\times\mathcal{T}(\overline{S})$, then we let $Q(X,Y)\in\mathrm{AH}(S,\partial S)$ be the quasifuchsian hyperbolic 3-manifold with conformal structure X on "top" and conformal structure Y on the "bottom." (Here $\overline{S}$ denotes S with the opposite orientation.)

3. The topology of the moduli space

We will assume throughout this section that S is not a thrice-punctured sphere, since in that case $\mathrm{H}(S,\partial S)$ is a point. The moduli set $\mathcal{I}(S,\partial S)$ of unmarked Kleinian surface groups is simply the quotient of $\mathrm{H}(S,\partial S)$ by the natural action of the mapping class group $\mathrm{Mod}(S)$ of isotopy classes of orientation-preserving homeomorphisms of S. We recall that if ϕ is a (representative of a) mapping class in $\mathrm{Mod}(S)$, then ϕ acts on $\mathrm{AH}(S,\partial S)$ by taking ρ to $\rho\circ\phi_*^{-1}$. An element of $\mathcal{I}(S,\partial S)$ is simply an oriented hyperbolic 3-manifold N which is homotopy equivalent to S (by a homotopy equivalence which takes ∂S into the cusps of N), where we do not keep track of the specific homotopy equivalence.

The moduli space $\mathcal{I}(S,\partial S)$ inherits an algebraic topology which we denote by $\mathrm{A}\mathcal{I}(S,\partial S)$. In the algebraic topology, this moduli space is rather badly behaved topologically. A first hint that this should be the case is the observation, first due to Thurston [25] (see also McMullen [20]), that there are points in $\mathrm{H}(S,\partial S)$ which are fixed by infinite order elements of $\mathrm{Mod}(S)$. These points arise naturally as the covers associated to the fibers of finite volume hyperbolic 3-manifolds which fiber over the circle. We recall the outline of Thurston's construction. If $\phi : S\to S$ is a pseudo-Anosov element of $\mathrm{Mod}(S)$ and $(X,Y)\in\mathcal{T}(S)\times\mathcal{T}(\overline{S})$, then Thurston considers the sequence of quasifuchsian Kleinian groups $\rho_n = Q(\phi^n(X), \phi^{-n}(Y))$ in $\mathrm{AH}(S, \partial S)$. He shows that $\{\rho_n\}$ converges to a Kleinian group ρ with empty domain of discontinuity which is a fixed point of the action of ϕ on $\mathrm{AH}(S, \partial S)$.

A minor variation on Thurston's construction allows us to construct points in $\mathrm{A}\mathcal{I}(S, \partial S)$ that are not closed.

Proposition 3.1. *There are points of* $\mathrm{A}\mathcal{I}(S, \partial S)$ *that are not closed.*

Proof. It suffices to consider the case when S is connected. Pick a pseudo-Anosov homeomorphism $\phi : S \to S$ and $(X, Y) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$. If we let $\rho_n = Q(X, \phi^{-n}(Y))$, then Bers [3, Thm.3] proved that $\{\rho_n\}$ has a convergent subsequence with limit $\rho \in \mathrm{H}(S, \partial S)$ which has non-empty domain of discontinuity. Consider the sequence $\rho \circ \phi^n_* \subset \mathrm{H}(S, \partial S)$. Using Thurston's Double Limit Theorem [25], one can prove that up to subsequence this converges to a manifold with an empty domain of discontinuity [20, 3.11]. Therefore the fiber in $\mathrm{AH}(S, \partial S)$ over the image of ρ in $\mathrm{A}\mathcal{I}(S, \partial S)$ is not closed, implying that the image of ρ is not a closed point. □

Remark: One may further show that if N is a degenerate hyperbolic 3-manifold in $\mathrm{A}\mathcal{I}(S, \partial S)$ with a lower bound on its injectivity radius, then N is not a closed point in $\mathrm{A}\mathcal{I}(S, \partial S)$. (More generally, one need only assume that there is a lower bound on the injectivity radius of N outside of cusps associated to ∂S.) We recall that N is degenerate if its domain of discontinuity has exactly one component and every cusp in N is associated to a component of ∂S. In this case, N has one geometrically infinite end and there exists a sequence $\{h_n : S \to N\}$ of pleated surfaces, each of which is a homotopy equivalence, exiting the geometrically infinite end of N. Since $h_n(S)$ has bounded geometry (away from the cusps), one may re-mark the maps h_n so that the associated sequence of representations $\rho_n = (h_n)_*$ in $AH(S, \partial S)$ has a convergent subsequence with limit ρ such that the domain of discontinuity of $\rho(\pi_1(S))$ is empty. Again, it follows that N is not a closed point of $\mathrm{A}\mathcal{I}(S, \partial S)$.

One can show that at least the geometrically finite points in $\mathrm{A}\mathcal{I}(S, \partial S)$ are closed.

Proposition 3.2. *If* $N \in \mathcal{I}(S, \partial S)$ *is geometrically finite then* N *is a closed point in* $\mathrm{A}\mathcal{I}(S, \partial S)$.

Proof. Let G be a graph in S which is a bouquet of circles associated to a minimal generating set of $\pi_1(S)$. Given an element (N, m) in the pre-image of N under the quotient map $\mathrm{H}(S, \partial S) \to \mathcal{I}(S, \partial S)$ one obtains a graph $m(G)$ in N, and the element (N, m) is determined by the homotopy class of this marked graph. If $\{(N, m_n)\}$ is an algebraically convergent sequence of elements of the fiber then one may assume that $m_n(G)$ has length less than K for some K (perhaps after homotoping the maps m_n). It follows that there exist constants $0 < \varepsilon_0 < \varepsilon_1$ so that if $x \in m_n(G)$ (for any n), then $\mathrm{inj}_N(x) \in [\varepsilon_0, \varepsilon_1]$. (The existence of ε_1 is obvious. If curves penetrate arbitrarily deeply into the thin part, but are not contained entirely within the thin part, then they must be growing arbitrarily long, hence the lower bound ε_0.) Let C be the set of points in N with injectivity radius in $[\varepsilon_0, \varepsilon_1]$. Since N is geometrically finite, C is compact. Therefore, there exists at most finitely many homotopy classes of immersions of G into C with total length at most K. It follows that there are at most finitely many distinct elements in the sequence $\{(N, m_n)\}$, so the sequence is eventually constant. It follows that the pre-image of N is closed, and hence that N is a closed point. $\square$

Surprisingly, the set of geometrically finite points in $\mathrm{A}\mathcal{I}(S, \partial S)$ is not Hausdorff.

Proposition 3.3. *The geometrically finite points of* $\mathrm{A}\mathcal{I}(S, \partial S)$ *do not form a Hausdorff space in the subspace topology.*

Proof. Let $D \in \mathrm{Mod}(S)$ be a Dehn twist about a non-peripheral simple closed curve on S. Pick $(X, Y) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ such that X and Y are not isometric. Let $\rho_n = Q(X, D^n(Y))$ for all n. By [3] and [16, Sec.3], $\{\rho_n\}$ converges algebraically to a manifold whose "top" conformal boundary component is isometric to X, and whose "bottom" conformal boundary has developed a rank one cusp. Similarly, $\rho_n \circ D_*^{-n} = Q(D^{-n}(X), Y)$ converges algebraically to a manifold whose "bottom" conformal boundary component is isometric to Y. In particular, the two limiting manifolds cannot be isometric, thus building a sequence in $\mathrm{A}\mathcal{I}(S, \partial S)$ with two distinct limits. $\square$

Historical Remarks. It follows from Bers' simultaneous uniformization [2] that the mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously, but not freely, on the interior $\mathrm{QF}(S)$ of $\mathrm{AH}(S, \partial S)$. If we parameterize $\mathrm{QF}(S)$ as $\mathcal{T}(S) \times \mathcal{T}(\overline{S})$, then the action is just the diagonal action where $\mathrm{Mod}(S)$ acts on each factor in the usual manner. Its quotient is thus naturally a bundle, in the orbifold sense, over the moduli space $\mathcal{M}(S)$ with generic fiber homeomorphic to $\mathcal{T}(S)$. It has recently been shown [7] that $\mathrm{AH}(S, \partial S)$ is the closure of $\mathrm{QF}(S)$ (see also [9, 6]). The examples above (based on the cited work of others) show that $\mathrm{Mod}(S)$ does not act properly discontinuously on $\mathrm{AH}(S, \partial S)$ and hence does not act properly discontinuously on $X(S, \partial S)$. Moreover, Souto and Storm [21] showed that $\mathrm{Mod}(S)$ acts topologically transitively on the closure of the set of points in the frontier of $\mathrm{QF}(S)$ whose conformal boundary does not contain a component homeomorphic to S. One can further show that any open, $\mathrm{Mod}(S)$-invariant open subset of $X(S, \partial S)$ on which $\mathrm{Mod}(S)$ acts properly discontinuously is disjoint from $\partial QF(S)$ (see Lee [17]). It is conjectured that every $\mathrm{Mod}(S)$-invariant open subset of $X(S, \partial S)$ on which $\mathrm{Mod}(S)$ acts properly discontinuously is a subset of $\mathrm{QF}(S)$.

In the case that S is a once-punctured torus, Bowditch [5] studied the subset Φ_Q of $X(S, \partial S)$ consisting of representations ρ such that if Ω is the set of simple closed curves on S, then $\rho(\gamma)$ is hyperbolic for all $\gamma \in \Omega$ and there are only finitely many elements of Ω such that $|\operatorname{tr}^2(\rho(\gamma))| \leq 4$. He conjectured that $\Phi_Q = \mathrm{QF}(S)$. Tan, Wong and Zhang [24] showed that Φ_Q is an open subset of $X(S, \partial S)$ on which $\mathrm{Mod}(S)$ acts properly discontinuously. Cantat [11] has used techniques from holomorphic dynamics to investigate the action of the mapping class group on the character variety associated to the once-punctured torus.

4. The augmented deformation set

The goal of this section is to enlarge $\mathrm{H}(S, \partial S)$ into an augmented deformation set $\overline{\mathrm{H}}(S, \partial S)$. The action of $\mathrm{Mod}(S)$ will extend to an action on $\overline{\mathrm{H}}(S, \partial S)$ and its quotient will give a compactification of $\mathrm{A}\mathcal{I}(S, \partial S)$. The augmentation described here is analogous to the augmentation of Teichmüller space by adding noded Riemann surfaces.

We again give two definitions of our augmented deformation set, with algebraic and geometric flavors. We begin with the algebraic definition. If a is a multicurve on S, let $c(a)$ denote the collection of components of $S - \mathcal{N}a$ where $\mathcal{N}a$ is an open collar neighborhood of a in S. An element $(\{\rho_R\}_{R \in c(a)}, a) \in \overline{\mathrm{H}}(S, \partial S)$ is a multicurve a together with an element $\rho_R \in \mathrm{H}(R, \partial R)$ for each $R \in c(A)$. An element of $\overline{\mathrm{H}}(S, \partial S)$ is geometrically finite (resp. Fuchsian) if each ρ_R is geometrically finite (resp. Fuchsian). Elements of $\overline{\mathrm{H}}(S, \partial S)$ using the multicurve a define the stratum corresponding to a.

The geometric definition is somewhat more difficult to formulate, but it gives more insight into the nature of elements of the set. Let $\mathcal{U}$ be the set of triples (N, a, m) where:

(1) N is a (possibly disconnected) oriented hyperbolic 3-manifold.
(2) $a \subset S$ is a multicurve with open collar neighborhood $\mathcal{N}a$.
(3) m is a homotopy equivalence
$$m : (S - \mathcal{N}a) \to N$$
taking $\partial(S - \mathcal{N}a)$ into the cusps of N.

A triple (N, a, m) is geometrically finite if each component of N is geometrically finite. Similarly, it is Fuchsian if every component of N is Fuchsian.

Of course this set $\mathcal{U}$ is too large. Form an equivalence relation by declaring two elements (L, a, ℓ) and (N, b, m) of $\mathcal{U}$ to be equivalent if $a = b$ and there exists an orientation preserving isometry $\iota : L \to N$ such that
$$m^{-1} \circ \iota \circ \ell : (S - \mathcal{N}a) \to (S - \mathcal{N}a)$$
is homotopic to the identity. (The map m^{-1} is a homotopy inverse of m.) The augmented deformation set $\overline{\mathrm{H}}(S, \partial S)$ is $\mathcal{U}$ modulo the above equivalence relation.

5. The algebraic topology on the augmented deformation set

We now define the algebraic topology for $\overline{\mathrm{H}}(S, \partial S)$, which extends the algebraic topology on $\mathrm{H}(S, \partial S)$. Its definition is motivated by Thurston's notion of a maximal subsurface of convergence.

Theorem 5.1. [25, Thm. 6.2] *Given any sequence $\{\rho_n\} \subset \mathrm{AH}(S, \partial S)$ there exists a subsequence $\{\rho_j\}$ and a (possibly empty, possibly disconnected) subsurface R of S with incompressible boundary such that:*

1. *For each component F of R the sequence of restrictions $\{\rho_j|_F\}$ converges in $\mathrm{AH}(F)$.*
2. *If Γ is a nontrivial subgroup of $\pi_1(S)$ such that the restrictions $\{\rho_j|_\Gamma\}$ converge (up to conjugacy) on a subsequence of $\{\rho_j\}$ then Γ is conjugate to a subgroup of $\pi_1(F)$ for some component F of R.*

We will be particularly interested in the special case of this theorem where the subsequence is the entire sequence and R is the complement of an open collar neighborhood of a multicurve.

Let $\mathcal{C}$ be the set of conjugacy classes of $\pi_1(S)$. For an element $(\{\rho_R\}_{R \in c(a)}, a) \in \overline{\mathrm{H}}(S, \partial S)$ and $\gamma \in \mathcal{C}$ we will define the square of the trace of an element γ in $(\{\rho_R\}_{R \in c(a)})$ as an element of the Riemann sphere $\mathbf{CP}^1$. If for some $R' \in c(a)$, γ can be realized by a closed curve in R', then define $\mathrm{tr}^2((\{\rho_R\}_{R \in c(a)}, a), \gamma)$ to be the square of the trace of $\rho_{R'}(\gamma)$. (Note that the trace of an element of $\mathrm{PSL}_2(\mathbf{C})$ is not well-defined but the square of its trace is well-defined.) Otherwise, define $\mathrm{tr}^2((\{\rho_R\}_{R \in c(a)}, a), \gamma)$ to be ∞. This case occurs if and only if γ essentially intersects the multicurve a. Using this extended length function we define the set map

$$\begin{aligned} t_* : \overline{\mathrm{H}}(S, \partial S) &\to (\mathbf{CP}^1)^{\mathcal{C}} \\ (\{\rho_R\}_{R \in c(a)}, a) &\mapsto \left\{ \gamma \mapsto \mathrm{tr}^2((\{\rho_R\}_{R \in c(a)}, a), \gamma) \right\}, \end{aligned}$$

where $(\mathbf{CP}^1)^{\mathcal{C}}$ is the space of functions from $\mathcal{C}$ to $\mathbf{CP}^1$ in the product topology (which is equivalent to pointwise convergence).

Lemma 5.2. *The set map t_* is injective.*

Proof. If $(\{\rho_R\}_{R \in c(a)}, a) \in \overline{\mathrm{H}}(S, \partial S)$, then a is the unique multicurve such that $t_*((\{\rho_R\}_{R \in c(a)}, a))(\gamma) = \infty$ if and only if γ intersects a essentially (for all $\gamma \in \mathcal{C}$). Therefore, the image of t_* determines the multicurve. Theorem 1.3 in [13] implies that for each component $R \in c(a)$, $\rho_R \in H(R, \partial R)$ is determined by the restriction of $t_*((\{\rho_R\}_{R \in c(a)}, a))$ to the set of conjugacy classes of elements of $\pi_1(R)$. Therefore, $(\{\rho_R\}_{R \in c(a)}, a)$ is entirely determined by $t_*((\{\rho_R\}_{R \in c(a)}, a))$. $\square$

Topologize $\overline{\mathrm{H}}(S, \partial S)$ using t_* as a subspace of $(\mathbf{CP}^1)^{\mathcal{C}}$. Let us temporarily call the resulting topological space $t_*(\overline{\mathrm{H}}(S, \partial S))$. As a subspace of a metric space, $t_*(\overline{\mathrm{H}}(S, \partial S))$ is Hausdorff and second countable. To understand this topology better, we now give a more intrinsic formulation of its convergence.

Definition 5.3. *A sequence* $\{\rho_n\} \subset \mathrm{AH}(S, \partial S)$ *is a shattering sequence with shattering multicurve* a *if:*

(1) *For each component* F *of* $S - \mathcal{N}a$ *the restrictions* $\{\rho_n|_F\}$ *converge in* $\mathrm{AH}(F)$.
(2) *If* Γ *is a nontrivial subgroup of* $\pi_1(S)$ *such that the restrictions* $\{\rho_n|_\Gamma\}$ *converge (up to conjugacy) on a subsequence of* $\{\rho_n\}$ *then* Γ *is conjugate to a subgroup of* $\pi_1(F)$ *for some component* F *of* $S - \mathcal{N}a$.

In other words, in Theorem 5.1 there is no need to pass to a subsequence, and R *is the complement of a multicurve.*

A sequence $(\{\rho_R^n\}_{R\in c(a_n)}, a_n)$ in $\overline{\mathrm{H}}(S, \partial S)$ is a *stable sequence* if the multicurve a_n is constant, i.e. there exists a multicurve a_{stable}, called the *stable multicurve*, such that $a_n = a_{\mathrm{stable}}$ for all n.

Definition 5.4. *A stable sequence* $\{(\{\rho_R^n\}_{R\in c(a_{stable})}, a_{stable})\}$ *in* $\overline{H}(S, \partial S)$ *converges algebraically to* $(\{\rho_F\}_{F\in c(a)}, a)$ *if*

(1) $a_{stable} \subseteq a$.
(2) *If* $F \in c(a)$*, then* $\{\rho_R^n|_{\pi_1(F)}\}$ *converges to* ρ_F *in* $\mathrm{AH}(F)$ *(where* R *is the element of* $c(a_{stable})$ *containing* F*).*
(3) *For all* $R \in c(a_{stable})$*,* $\{\rho_R^n\}$ *is a shattering sequence in* $\mathrm{AH}(R, \partial R)$ *with shattering multicurve* $a \cap R$.

In particular, a sequence $\{(\rho_S^n, \emptyset)\}$ in $\overline{\mathrm{H}}(S, \partial S)$ converges algebraically to $(\{\rho_F\}_{F\in c(a)}, a)$ if and only if it is a shattering sequence with shattering multicurve a and $\{\rho_S^n|_{\pi_1(F)}\}$ converges algebraically to ρ_F for all $F \in c(a)$.

A (not necessarily stable) sequence in $\overline{\mathrm{H}}(S, \partial S)$ converges algebraically to $(\{\rho_F\}_{F\in c(a)}, a) \in \overline{\mathrm{H}}(S, \partial S)$ if, after possibly discarding finitely many elements, it can be partitioned into stable subsequences, with distinct stable multicurves, all converging algebraically to $(\{\rho_F\}_{F\in c(a)}, a)$. (Since the stable multicurve for any stable subsequence lies in a, this partition must be finite.)

Proposition 5.5. *A sequence in* $\overline{\mathrm{H}}(S, \partial S)$ *converges in* $t_*(\overline{\mathrm{H}}(S, \partial S))$ *if and only if it converges algebraically.*

Proof. Suppose a sequence in $\overline{\mathrm{H}}(S, \partial S)$ converges algebraically to $(\{\rho_R\}_{R\in c(a)}, a)$. Without loss of generality, we can assume the sequence is stable. Then condition (2) of Definition 5.3 guarantees that any curve essentially intersecting a has trace going to ∞. Condition (2) of Definition 5.4 guarantees the other traces converge, establishing convergence in $t_*(\overline{\mathrm{H}}(S, \partial S))$.

Next suppose a sequence $\left(\{\rho_R^n\}_{R\in c(a_n)}, a_n\right)$ converges to $(\{\rho_R\}_{R\in c(a)}, a)$ in $t_*(\overline{\mathrm{H}}(S, \partial S))$. We must first check that, after possibly discarding finitely many terms, the sequence can be partitioned into stable sequences. To begin, suppose there is a subsequence (denoted without subscripts) such that a_n always essentially intersects a. Then, up to subsequence, $i(a_n, a^0) > 0$ for some component a^0 of a, so $t_*(\{\rho_R\}_{R\in c(a)}, a)(a^0) = \infty$, since $t_*(\{\rho_R^n\}_{R\in c(a_n)}, a_n)(a^0) = \infty$ for all n, which is a contradiction. Therefore $i(a_n, a) = 0$ for all $n \gg 0$.

We next claim that $a_n \subset a$ for all $n \gg 0$. If not, we may pass to a subsequence and choose a component a_n^0 of $a_n - a$ such that $\{a_n^0\}$, viewed as a sequence of projective measured laminations, converges to a (projective class of a) measured lamination λ on S. See [4, Sec. 4] for a discussion of measured laminations and intersection number. Since each a_n^0 is a simple closed curve in $S - a$, λ is disjoint from a. Pick an essential simple closed curve b in $S - \mathcal{N}a$ such that $i(b, \lambda) > 0$. Then by continuity of intersection number, for all $n \gg 0$ we have $i(b, a_n) > 0$, implying that $t_*(\{\rho_R\}_{R \in c(a)}, a)(b) = \infty$. This is a contradiction, proving that $a_n \subseteq a$ for $n \gg 0$.

After discarding finitely many terms the sequence can therefore be partitioned into a finite set of stable sequences. Suppose that $F \in c(a)$. On any stable subsequence, with associated multicurve a_{stable}, there exists $R \in c(a_{\text{stable}})$ such that $F \subset R$. Since $\operatorname{tr}^2(\rho_R^n(c))$ converges to $\operatorname{tr}^2(\rho_F(c))$ for any conjugacy class c of an element of $\pi_1(F)$, it follows that $\{\rho_R^n|_{\pi_1(F)}\}$ converges to ρ_F in $AH(F)$ (see Corollary 2.3 in [13]). From here the definition of our extended trace function implies condition (2) of Definition 5.3 for each stable subsequence. $\square$

With their equivalence established, we refer to the topology on $t_*(\overline{\mathrm{H}}(S, \partial S))$ as the algebraic topology and denote it by $\mathrm{A}\overline{\mathrm{H}}(S, \partial S)$. This topology is closely related to the notion of algebraic convergence on subsurfaces which played a role in the proof of the Ending Lamination Conjecture [7, Sec, 6]. One expects the augmented space to be at least as topologically complicated as $\mathrm{AH}(S, \partial S)$, which is known, for example, not to be locally connected [8, 18].

6. The augmented moduli space

The goal of this section is to define the quotient augmented moduli space and establish that it is sequentially compact.

If $\phi \in \mathrm{Mod}(S)$ and $(\{\rho_F\}_{F \in c(a)}, a) \in \mathrm{A}\overline{\mathrm{H}}(S, \partial S)$, then we can choose a representative, also called ϕ, so that $\phi(a)$ is a (geodesic) multicurve. The mapping class ϕ takes $(\{\rho_F\}_{F \in c(a)}, a)$ to $(\{\rho_F \circ \phi_*^{-1}\}_{\phi(F) \in c(\phi(a))}, \phi(a)\})$. Stated geometrically, ϕ takes (N, a, m) to $(N, \phi(a), m \circ \phi^{-1}|_{S - \mathcal{N}\phi(a)})$. It is easy to check that the each mapping class induces a homeomorphism of $\mathrm{A}\overline{\mathrm{H}}(S, \partial S)$ and that we obtain a continuous extension of the action of $\mathrm{Mod}(S)$ on $\mathrm{AH}(S, \partial S)$.

We then define the natural quotient space, with its induced quotient algebraic topology by

$$\mathrm{A}\overline{\mathcal{I}}(S, \partial S) := \mathrm{A}\overline{\mathrm{H}}(S, \partial S) / \mathrm{Mod}(S).$$

The key feature of our augmented moduli space is that it is sequentially compact.

Theorem 6.1. *$\mathrm{A}\overline{\mathcal{I}}(S, \partial S)$ is a sequentially compact topological space.*

This compactness result is essentially a corollary of a result of Canary-Minsky-Taylor, together with Theorem 5.1. We restate the result of Canary-Minsky-Taylor in the setting of $\mathrm{AH}(S, \partial S)$.

Theorem 6.2. [10, Thm.5.5] *Let $\{\rho_n\}$ be a sequence in $\mathrm{AH}(S, \partial S)$. Then there exists a subsequence $\{\rho_j\}$, a sequence $\{\phi_j\}$ in $\mathrm{Mod}(S)$, and a multicurve a in S, such that if R is a component of $S - \mathcal{N}a$ then $\{\rho_j \circ (\phi_j)_*^{-1}|_{\pi_1(R)}\}$ converges in $\mathrm{AH}(R)$ to an element of $\mathrm{AH}(R, \partial R)$.*

If we combine the above result with Theorem 5.1, we see that we may always re-mark a subsequence to find a shattering subsequence. The proof consists simply of applying Theorem 5.1 to the sequence produced by Theorem 6.2. The multicurve b in the statement of Corollary 6.3 is always contained in the multicurve associated to the subsequence produced by Theorem 6.2.

Corollary 6.3. *Let $\{\rho_n\}$ be a sequence in $\mathrm{AH}(S, \partial S)$. Then there exists a subsequence $\{\rho_j\}$, a sequence $\{\phi_j\}$ in $\mathrm{Mod}(S)$, and a multicurve b in S, such that $\{\rho_j \circ (\phi_j)_*^{-1}\}$ is a shattering sequence with shattering multicurve b.*

Proof of Theorem 6.1. Let $\{(\{\rho^n_{R_n}\}_{R_n \in c(a_n)}, a_n)\}$ be a sequence in $\mathrm{A}\overline{\mathrm{H}}(S, \partial S)$. Since there exists only a finite number of multicurves, up to homeomorphism, on a compact surface, we may pass to a subsequence $(\{\rho^j_{R_j}\}_{R_j \in c(a_j)}, a_j)$ and find a sequence $\{\phi_j\}$ in $\mathrm{Mod}(S)$ so that $\phi_j(a_j)$ is the same multicurve, say a, for all j. Then each $\phi_j(\{\rho^j_{R_j}\}_{R_j \in c(a_j)}, a_j)$ can be rewritten as $(\{\sigma^j_F\}_{F \in c(a)}, a)$. We then apply Corollary 6.3 to the sequence $\{\sigma^j_F\}$ successively for each component F of $c(a)$, possibly further remarking the surface F in the process, to find an algebraically convergent subsequence of $\{(\{\sigma^j_F\}_{F \in c(a)}, a)\}$. Since every sequence in $\mathrm{A}\overline{\mathrm{H}}(S, \partial S)$ has a subsequence which can be re-marked by elements of $\mathrm{Mod}(S)$ so that it converges in $\mathrm{A}\overline{\mathrm{H}}(S, \partial S)$, we conclude that $\mathrm{A}\overline{\mathcal{I}}(S, \partial S)$ is sequentially compact. $\square$

Remark. Geometrically finite points in $\mathrm{A}\overline{\mathcal{I}}(S, \partial S)$ are not necessarily closed. (Recall Proposition 3.2 proved that geometrically finite points are closed in $\mathrm{A}\mathcal{I}(S, \partial S)$.) To see this, consider the following example. Let $b \subset S$ be a simple closed separating geodesic. Let $\rho \in \mathrm{H}(S, \partial S)$ be geometrically finite with exactly one rank one parabolic corresponding to the curve $b \subset S$. Let $D : S \to S$ indicate a Dehn twist along b. Consider the two component element $(\{\rho|_R\}_{R \in c(b)}, b) \in \overline{\mathrm{H}}(S, \partial S)$. Clearly ρ and $(\{\rho|_R\}_{R \in c(b)}, b)$ project to distinct points in $\mathrm{A}\overline{\mathcal{I}}(S, \partial S)$. Nonetheless, the sequence $\{\rho \circ D^n\}$ converges algebraically to $(\{\rho|_R\}_{R \in c(b)}, b)$. This shows the projection of ρ to $\mathrm{A}\overline{\mathcal{I}}(S, \partial S)$ is not a closed point.

7. The Fuchsian locus and the Deligne-Mumford compactification

One forms the augmented Teichmüller space $\overline{\mathcal{T}}(S)$ by appending the Teichmüller space $\mathcal{T}(S-\mathcal{N}a)$ associated to the complement of every multicurve a on S. Notice that one may associate to any point in $\mathcal{T}(S-\mathcal{N}a)$ a unique Fuchsian element $(\{\rho_R\}_{R\in c(a)}, a)$ of $\overline{\mathrm{H}}(S,\partial S)$. Therefore one may identify $\overline{\mathcal{T}}(S)$ with the set $\overline{\mathcal{F}}(S,\partial S) \subset \overline{\mathrm{H}}(S,\partial S)$ of Fuchsian elements of $\overline{\mathrm{H}}(S,\partial S)$. We call $\overline{\mathcal{F}}(S,\partial S)$ the Fuchsian locus. One may check that $\overline{\mathcal{F}}(S,\partial S)$ (with the algebraic topology) is actually homeomorphic to $\overline{\mathcal{T}}(S)$. Since the Deligne-Mumford compactification $\overline{\mathcal{M}}(S)$ of $\mathcal{M}(S)$ arises as the quotient of the augmented Teichmüller space $\overline{\mathcal{T}}(S)$ under the action of $\mathrm{Mod}(S)$, we may identify $\overline{\mathcal{M}}(S)$ with the quotient of the Fuchsian locus $\overline{\mathcal{F}}(S,\partial S)$ in $\overline{\mathcal{I}}(S,\partial S)$. We refer the reader to Wolpert's survey article [27] for a discussion of the basic properties of augmented Teichmüller space and its relationship to the Deligne-Mumford compactification.

It is easy to check that the Fuchsian locus is closed in $\mathrm{A}\overline{\mathrm{H}}(S,\partial S)$ and invariant under the action of $\mathrm{Mod}(S)$, implying that $\overline{\mathcal{M}}(S)$ is identified with a closed subset of $\mathrm{A}\overline{\mathcal{I}}(S,\partial S)$.

Proposition 7.1. *The natural embedding $\iota : \mathcal{M}(S) \to \mathrm{A}\mathcal{I}(S,\partial S)$, sending a hyperbolic surface to its corresponding Fuchsian 3-manifold, extends to an embedding of the Deligne-Mumford compactification $\overline{\mathcal{M}}(S) \to \mathrm{A}\overline{\mathcal{I}}(S,\partial S)$ with image the set of Fuchsian 3-manifolds.*

Remark: We recall that if the augmented Teichmüller space is not a point, then it fails to be locally compact. Since $\overline{\mathcal{T}}(S)$ is homeomorphic to a closed subset of $\mathrm{A}\overline{\mathrm{H}}(S,\partial S)$, it follows that $\mathrm{A}\overline{\mathrm{H}}(S,\partial S)$ also fails to be locally compact.

8. Other topologies

In [26], Thurston discusses two other topologies on $\mathrm{H}(S,\partial S)$, the strong topology and the quasi-isometric topology. Both extend naturally to topologies on $\overline{\mathrm{H}}(S,\partial S)$.

A sequence $\{(N_n, m_n)\}$ in $\mathrm{H}(S,\partial S)$ converges strongly to $(N,m) \in \mathrm{H}(S,\partial S)$ if there exists a sequence $\{h_n : N \to N_n\}$ of homotopy equivalences which C^∞-converge to an isometry on every compact subset of N such that $h_n \circ m$ is homotopic to m_n for all n. The key difference with the definition of algebraic convergence is that $\{h_n\}$ converges to an isometry, rather than just a local isometry. The deformation space $\mathrm{H}(S,\partial S)$ equipped with the strong topology is denoted $\mathrm{GH}(S,\partial S)$. The "G" is due to Thurston, who called this the geometric topology.

We may readily generalize this to the setting of the augmented deformation space $\overline{\mathrm{H}}(S,\partial S)$. We say that a sequence $\{(N_n, a_n, m_n)\}$ in $\overline{\mathrm{H}}(S,\partial S)$ converges strongly to (N,a,m) if $\{(N_n, a_n, m_n)\}$ converges algebraically to

(N, a, m) and there exists a sequence of continuous maps $\{h_n : N \to N_n\}$ such that $h_n \circ m$ is homotopic to $m_n|_F$, for all n, on every component F of $S - \mathcal{N}a$, and $\{h_n\}$ C^∞-converges to an isometry on every compact subset of N. The deformation space $\overline{\mathrm{H}}(S, \partial S)$ equipped with the strong topology is denoted $\mathrm{G}\overline{\mathrm{H}}(S, \partial S)$.

A sequence $\{(N_n, m_n)\}$ in $\mathrm{H}(S, \partial S)$ converges in the quasi-isometric topology to $(N, m) \in \mathrm{H}(S, \partial S)$ if, for all large enough n, there exists a K_n-bilipschitz diffeomorphism $h_n : N \to N_n$ such that $h_n \circ m$ is homotopic to m_n with $\lim K_n = 1$. The deformation space $\mathrm{H}(S, \partial S)$ equipped with the quasi-isometric topology is denoted $\mathrm{QH}(S, \partial S)$. More generally, a sequence $\{(N_n, a_n, m_n)\}$ in $\overline{\mathrm{H}}(S, \partial S)$ converges in the quasi-isometric topology to (N, a, m) if, for all large enough n, $a_n = a$ and there exists a K_n-bilipschitz diffeomorphism $h_n : N \to N_n$ such that $h_n \circ m$ is homotopic to m_n and $\lim K_n = 1$. The deformation space $\overline{\mathrm{H}}(S, \partial S)$ equipped with the quasi-isometric topology is denoted $\mathrm{Q}\overline{\mathrm{H}}(S, \partial S)$.

One may readily check that $\mathrm{Mod}(S)$ acts on both $\mathrm{G}\overline{\mathrm{H}}(S, \partial S)$ and $\mathrm{Q}\overline{\mathrm{H}}(S, \partial S)$ as a group of homeomorphisms. So one obtains strong and quasi-isometric topologies on the quotient space $\overline{\mathcal{I}}(S, \partial S)$.

$$\mathrm{G}\overline{\mathcal{I}}(S, \partial S) := \mathrm{G}\overline{\mathrm{H}}(S, \partial S)/\operatorname{Mod}(S),$$
$$\mathrm{Q}\overline{\mathcal{I}}(S, \partial S) := \mathrm{Q}\overline{\mathrm{H}}(S, \partial S)/\operatorname{Mod}(S).$$

It follows from the analogous fact for $\mathrm{H}(S, \partial S)$ that the identity maps

$$\mathrm{Q}\overline{\mathcal{I}}(S, \partial S) \to \mathrm{G}\overline{\mathcal{I}}(S, \partial S) \to \mathrm{A}\overline{\mathcal{I}}(S, \partial S)$$

are continuous, but the inverse maps are not (see [26]).

The space $\mathrm{Q}\overline{\mathcal{I}}(S, \partial S)$ is locally nice and globally terrible. If S is not a thrice-punctured sphere, it is a disjoint union of an uncountable collection of noncompact orbifolds of various dimensions and an uncountable number of isolated points. (This follows from Sullivan's extension of the Quasiconformal Parametrization Theorem, also known as Sullivan rigidity [22].) In particular, $\mathrm{Q}\overline{\mathcal{I}}(S, \partial S)$ is Hausdorff and noncompact.

Proposition 8.1. *The space* $\mathrm{G}\overline{\mathcal{I}}(S, \partial S)$ *is not sequentially compact.*

Proof. Consider the sequence $\{\rho_n\}$ from the proof of Proposition 3.3. It determines a sequence of hyperbolic manifolds N_n whose geometric limit X is homeomorphic to $S \times (0, 1)$ minus $b \times \{1/2\}$, where b is a simple closed curve of S [16]. The manifold X is not homotopy equivalent to S. No matter how the sequence N_n is marked, this geometric limit will not change. This implies that no subsequence converges in $\mathrm{G}\overline{\mathcal{I}}(S, \partial S)$. $\square$

Finally, the examples in Proposition 3.1 also converge strongly, see Theorem 3.12 in [20], so we see that there are points in $\mathrm{G}\overline{\mathcal{I}}(S, \partial S)$ that are not closed.

One may define a refinement of the algebraic topology on $\overline{\mathrm{H}}(S, \partial S)$, which is still coarser than the strong topology, so that geometrically finite points are closed in the resulting quotient topology on $\overline{\mathcal{I}}(S, \partial S)$, yet the resulting quotient topology on the augmented moduli space is still sequentially compact. We say that a stable sequence $\{(\{\rho_R^n\}_{R\in c(a_{\text{stable}})}, a_{\text{stable}})\}$ converges maximally algebraically to $(\{\rho_F\}_{F\in c(a)}, a)$ if it converges algebraically and there does not exist a subsequence $\{(\{\rho_R^j\}_{R\in c(a_{\text{stable}})}, a_{\text{stable}})\}$ and a sequence $\{\phi_j\}$ in $\mathrm{Mod}(S)$, each of which is a product of Dehn twists about elements of $a - a_{\text{stable}}$, such that $\{\phi_j(\{\rho_R^j\}_{R\in c(a_{\text{stable}})}, a_{\text{stable}})\}$ converges algebraically to $(\{\rho_B\}_{F\in c(b)}, b)$ where b is a proper subset of a. We denote $\overline{\mathrm{H}}(S, \partial S)$ with the topology of maximally algebraic convergence by $\mathrm{B}\overline{\mathrm{H}}(S, \partial S)$. Its quotient by the action of $\mathrm{Mod}(S)$ is denoted by $\mathrm{B}\overline{\mathcal{I}}(S, \partial S)$. (Note that this topology is designed specifically to disallow examples like those described in the remark terminating Section 6.) The proof of Theorem 6.1 can be easily modified to verify the sequential compactness of $\mathrm{B}\overline{\mathcal{I}}(S, \partial S)$.

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