MODULI SPACES OF HYPERBOLIC 3-MANIFOLDS AND DYNAMICS ON CHARACTER VARIETIES

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ABSTRACT. The space $AH(M)$ of marked hyperbolic 3-manifold homotopy equivalent to a compact 3-manifold with boundary M sits inside the $\mathrm{PSL}_2(\mathbf{C})$-character variety $X(M)$ of $\pi_1(M)$. We study the dynamics of the action of $\mathrm{Out}(\pi_1(M))$ on both $AH(M)$ and $X(M)$. The nature of the dynamics reflects the topology of M.

The quotient $AI(M) = AH(M)/\mathrm{Out}(\pi_1(M))$ may naturally be thought of as the moduli space of unmarked hyperbolic 3-manifolds homotopy equivalent to M and its topology reflects the dynamics of the action.

1. INTRODUCTION

For a compact, orientable, hyperbolizable 3-manifold M with boundary, the deformation space $AH(M)$ of marked hyperbolic 3-manifolds homotopy equivalent to M is a familiar object of study. This deformation space sits naturally inside the $\mathrm{PSL}_2(\mathbf{C})$-character variety $X(M)$ and the outer automorphism group $\mathrm{Out}(\pi_1(M))$ acts by homeomorphisms on both $AH(M)$ and $X(M)$. The action of $\mathrm{Out}(\pi_1(M))$ on $AH(M)$ and $X(M)$ has largely been studied in the case when M is an interval bundle over a closed surface (see, for example, [8, 22, 49, 18]) or a handlebody (see, for example, [43, 54]). In this paper, we initiate a study of this action for general hyperbolizable 3-manifolds.

We also study the topological quotient

$$AI(M) = AH(M)/\mathrm{Out}(\pi_1(M))$$

which we may think of as the moduli space of unmarked hyperbolic 3-manifolds homotopy equivalent to M. The space $AH(M)$ is a rather pathological topological object itself, often failing to even be locally connected (see Bromberg [12] and Magid [35]). However, since $AH(M)$ is a closed subset of an open submanifold of the character variety, it does retain many nice topological properties. We will see that the topology of $AI(M)$ can be significantly more pathological.

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The first hint that the dynamics of $\mathrm{Out}(\pi_1(M))$ on $AH(M)$ are complicated, was Thurston's [51] proof that if M is homeomorphic to $S \times I$, then there are infinite order elements of $\mathrm{Out}(\pi_1(M))$ which have fixed points in $AH(M)$. (These elements are pseudo-Anosov mapping classes.) One may further show that $AI(S \times I)$ is not even T_1, see [18] for a closely related result. Recall that a topological space is T_1 if all its points are closed. On the other hand, we show that in all other cases $AI(M)$ is T_1.

Theorem 1.1. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. Then the moduli space $AI(M)$ is T_1 if and only if M is not an untwisted interval bundle.*

We next show that $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$ if M contains a primitive essential annulus. A properly embedded annulus in M is a primitive essential annulus if it cannot be properly isotoped into the boundary of M and its core curve generates a maximal abelian subgroup of $\pi_1(M)$. In particular, if M has compressible boundary and no toroidal boundary components, then M contains a primitive essential annulus (see Corollary 7.5).

Theorem 1.2. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. If M contains a primitive essential annulus then $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$. Moreover, if M contains a primitive essential annulus, then $AI(M)$ is not Hausdorff.*

On the other hand, if M is acylindrical, i.e. has incompressible boundary and contains no essential annuli, then $\mathrm{Out}(\pi_1(M))$ is finite (see Johannson [29, Proposition 27.1]), so $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $AH(M)$ and $X(M)$. It is easy to see that $\mathrm{Out}(\pi_1(M))$ fails to act properly discontinuously on $X(M)$ if M is not acylindrical, since it will contain infinite order elements with fixed points in $X(M)$.

If M is a compact hyperbolizable 3-manifold which is not acylindrical, but does not contain any primitive essential annuli, then $\mathrm{Out}(\pi_1(M))$ is infinite. However, if, in addition, M has no toroidal boundary components, we show that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH(M)$ in $X(M)$. In particular, we see that $AI(M)$ is Hausdorff in this case.

Theorem 1.3. *If M is a compact hyperbolizable 3-manifold with no primitive essential annuli whose boundary has no toroidal boundary components, then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant neighborhood $W(M)$ of $AH(M)$ in $X(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$. In particular, $AI(M)$ is Hausdorff.*

If M is a compact hyperbolizable 3-manifold with no primitive essential annuli whose boundary has no toroidal boundary components, then $\mathrm{Out}(\pi_1(M))$ is virtually abelian (see the discussion in sections 5 and 9). However, we note that the conclusion of Theorem 1.3 relies crucially on the

topology of M, not just the group theory of $\mathrm{Out}(\pi_1(M))$. In particular, if M is a compact hyperbolizable 3-manifold M with incompressible boundary, such that every component of its characteristic submanifold is a solid torus, then $\mathrm{Out}(\pi_1(M))$ is always virtually abelian, but M may contain primitive essential annuli, in which case $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$.

One may combine Theorems 1.2 and 1.3 to completely characterize when $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $AH(M)$ in the case that M has no toroidal boundary components.

Corollary 1.4. *Let M be a compact hyperbolizable 3-manifold with no toroidal boundary components and non-abelian fundamental group. The group $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $AH(M)$ if and only if M contains no primitive essential annuli. Moreover, $AI(M)$ is Hausdorff if and only if M contains no primitive essential annuli.*

It is a consequence of the classical deformation theory of Kleinian groups (see Bers [5] or Canary-McCullough [17, Chapter 7] for a survey of this theory) that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on the interior $\mathrm{int}(AH(M))$ of $AH(M)$. If H_n is the handlebody of genus $n \geq 2$, Minsky [43] exhibited an explicit $\mathrm{Out}(\pi_1(H_n))$-invariant open subset $PS(H_n)$ of $X(H_n)$ such that $\mathrm{int}(AH(H_n))$ is a proper subset of $PS(H_n)$ and $\mathrm{Out}(\pi_1(H_n))$ acts properly discontinuously on $AH(H_n)$.

If M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components, which is not an interval bundle, then we find an open set $W(M)$ strictly bigger than $\mathrm{int}(AH(M))$ which $\mathrm{Out}(\pi_1(M))$ acts properly discontinuosly on. See Theorem 9.1 and its proof for a more precise description of $W(M)$. We further observe, see Lemma 8.1, that $W(M) \cap \partial AH(M)$ is a dense open subset of $\partial AH(M)$ in this setting.

Theorem 1.5. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components, which is not an interval bundle. Then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant subset $W(M)$ of $X(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$ and $\mathrm{int}(AH(M))$ is a proper subset of $W(M)$.*

It is conjectured that if M is an untwisted interval bundle over a closed surface S, then $\mathrm{int}(AH(M))$ is the maximal open $\mathrm{Out}(\pi_1(M))$-invariant subset of $X(M)$ on which $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously. One may show that no open domain of discontinuity can intersect $\partial AH(S \times I)$ (see [34]). Further evidence for this conjecture is provided by results of Bowditch [8], Goldman [21], Souto-Storm [49], Tan-Wong-Zhang [54] and Cantat [19].

Michelle Lee [34] has recently shown that if M is an twisted interval bundle over a closed surface, then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant subset W of $X(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on

W and $\mathrm{int}(AH(M))$ is a proper subset of W. Moreover, W contains points in $\partial AH(M)$. As a corollary, she proves that if M has incompressible boundary and no toroidal boundary components, then there is open $\mathrm{Out}(\pi_1(M))$-invariant subset W of $X(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on W, $\mathrm{int}(AH(M))$ is a proper subset of W, and $W \cap \partial AH(M) \neq \emptyset$ if and only if M is not an untwisted interval bundle.

Outline of paper: In section 2, we recall background material from topology and hyperbolic geometry which will be used in the paper.

In section 3, we prove Theorem 1.1. The proof that $AI(S \times I)$ is not T_1 follows the arguments in [18, Proposition 3.1] closely. We now sketch the proof that $AI(M)$ is T_1 otherwise. In this case, let $N \in AI(M)$ and let R be a compact core for N. We show that N is a closed point, by showing that any convergent sequence $\{\rho_n\}$ in the pre-image of N is eventually constant. For all n, there exists a homotopy equivalence $h_n : M \to N$ such that $(h_n)_* = \rho_n$. If G is a graph in M carrying $\pi_1(M)$, then, since $\{\rho_n\}$ is convergent, we can assume that the length of $h_n(G)$ is at most K, for all n and some K. But, we observe that $h_n(G)$ cannot lie entirely in the complement of R, if R is not a compression body. In this case, each $h_n(G)$ lies in the compact neighborhood of radius K of R, so there are only finitely many possible homotopy classes of maps of G. Thus, there are only finitely many possibilities for ρ_n, so $\{\rho_n\}$ is eventually constant. The proof in the case that R is a compression body is somewhat more complicated and uses the Covering Theorem.

In section 4, we prove Theorem 1.2. Let A be a primitive essential annulus in M. If α is a core curve of A, then the complement $\hat{M}$ of a regular neighborhood of α in M is hyperbolizable. We consider a geometrically finite hyperbolic manifold $\hat{N}$ homeomorphic to the interior of $\hat{M}$ and use the Hyperbolic Dehn Filling Theorem to produce a convergent sequence $\{\rho_n\}$ in $AH(M)$ and a sequence $\{\varphi_n\}$ of distinct elements of $\mathrm{Out}(\pi_1(M))$ such that $\{\rho_n \circ \varphi_n\}$ also converges. Therefore, $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$. Moreover, we show that $\{\rho_n\}$ projects to a sequence in $AI(M)$ with two distinct limits, so $AI(M)$ is not Hausdorff.

In section 5 we recall basic facts about the characteristic submanifold and the mapping class group of compact hyperbolizable 3-manifolds with incompressible boundary and no toroidal boundary components. We identify a finite index subgroup $J(M)$ of $\mathrm{Out}(\pi_1(M))$ and a projection of $J(M)$ onto the direct product of mapping class groups of the base surfaces whose kernel $K(M)$ is the free abelian subgroup generated by Dehn twists in frontier annuli of the characteristic submanifold.

In section 6, we organize the frontier annuli of the characteristic submanifold into characteristic collections of annuli and describe free subgroups of $\pi_1(M)$ which register the action of the subgroup of $\mathrm{Out}(\pi_1(M))$ generated by Dehn twists in the annuli in such a collection.

In section 7, we show that compact hyperbolizable 3-manifolds with compressible boundary and no toroidal boundary components contain primitive essential annuli.

In section 8, we introduce a subset $AH_n(M)$ of $AH(M)$ which contains all purely hyperbolic representations. We see that $\text{int}(AH(M))$ is a proper subset of $AH_n(M)$ and that $AH_n(M) = AH(M)$ if M does not contain any primitive essential annuli.

In section 9, we prove that if M has incompressible boundary and no toroidal boundary components, but is not an interval bundle, there is an open neighborhood $W(M)$ of $AH_n(M)$ in $X(M)$ such that $\text{Out}(\pi_1(M))$ preserves and acts properly discontinuously on $W(M)$. Theorems 1.3 and 1.5 are immediate corollaries. We finish the outline by sketching the proof in a special case.

Let X be an acylindrical, compact hyperbolizable 3-manifold and let A be an incompressible annulus in its boundary. Let V be a solid torus and let $\{B_1, \dots, B_n\}$ be a collection of disjoint parallel annuli in ∂V whose core curves are homotopic to the n^{th} power of the core curve of V where $|n| \geq 2$. Let $\{M_1, \dots, M_n\}$ be copies of X and let $\{A_1, \dots, A_n\}$ be copies of A in M_i. We form M by attaching each M_i to V by identifying A_i and B_i. Then M contains no primitive essential annuli, is hyperbolizable, and $\text{Out}(\pi_1(M))$ has a finite index subgroup $J(M)$ generated by Dehn twists about $\{A_1, \dots, A_n\}$. In particular, $J(M) \cong \mathbf{Z}^{n-1}$.

In this case, $\{A_1, \dots, A_n\}$ is the only characteristic collection of annuli. We say that a group H registers $J(M)$ if it is freely generated by the core curve of V and, for each i, a curve contained in $V \cup M_i$ which is not homotopic into V. So $H \cong F_{n+1}$. There is a natural map $r_H : X(M) \to X(H)$ where $X(H)$ is the $\text{PSL}_2(\mathbf{C})$-character variety of the group H. Notice that $J(M)$ preserves H and injects into $\text{Out}(H)$. Let

$$\mathcal{S}_{n+1} = \text{int}(AH(H)) \subset X(H)$$

denote the space of Schottky representations (i.e. representations which are purely hyperbolic and geometrically finite.) Since $\text{Out}(H)$ acts properly discontinuously on $\mathcal{S}_{n+1}$, we see that $J(M)$ acts properly discontinuously on

$$W_H = r_H^{-1}(\mathcal{S}_{n+1})$$

Let $W(M) = \bigcup W_H$ where the union is taken over all subgroups which register $J(M)$. Notice that $W(M)$ is open and $J(M)$ acts properly discontinuously on $W(M)$. One may use a ping pong argument to show that $AH(M) \subset W(M)$, see Lemma 8.3. Johannson's Classification Theorem is used to show that $W(M)$ is invariant under $\text{Out}(\pi_1(M))$, see Lemma 9.3. (Actually, we define a somewhat larger set, in general, by using the space of primitive-stable representations in place of Schottky space.)

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2. Preliminaries

As a convention, the letter M will denote a compact connected oriented hyperbolizable 3-manifold with boundary. We recall that M is said to be hyperbolizable if the interior of M admits a complete hyperbolic metric. We will use N to denote a hyperbolic 3-manifold. All hyperbolic 3-manifolds are assumed to be oriented, complete, and connected.

2.1. **The deformation spaces.** Recall that $\mathrm{PSL}_2(\mathbf{C})$ is the group of orientation-preserving isometries of $\mathbf{H}^3$. Given a 3-manifold M, a discrete, faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ determines a hyperbolic 3-manifold $N_\rho = \mathbf{H}^3/\rho(\pi_1(M))$ and a homotopy equivalence $m_\rho : M \to N_\rho$, called the marking of N_ρ.

We let $D(M)$ denote the set of discrete, faithful representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbf{C})$. The group $\mathrm{PSL}_2(\mathbf{C})$ acts by conjugation on $D(M)$ and we let

$$AH(M) = D(M)/\mathrm{PSL}_2(\mathbf{C}).$$

Elements of $AH(M)$ are hyperbolic 3-manifolds homotopy equivalent to M equipped with (homotopy classes of) markings.

The space $AH(M)$ is a closed subset of the character variety

$$X(M) = \mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))//\mathrm{PSL}_2(\mathbf{C}),$$

which is the Mumford quotient of the space $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ of representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(g)$ is parabolic if $g \neq id$ lies in a rank two free abelian subgroup of $\pi_1(M)$. If M has no toroidal boundary components, then $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ is simply $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$. Moreover, each point in $AH(M)$ is a smooth point of $X(M)$ (see Kapovich [30, Sections 4.3 and 8.8] and Heusener-Porti [24] for more details on this construction).

The group $\mathrm{Aut}(\pi_1(M))$ acts naturally on $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ via

$$(\varphi \cdot \rho)(\gamma) := \rho(\varphi^{-1}(\gamma)).$$

This descends to an action of $\mathrm{Out}(\pi_1(M))$ on $AH(M)$ and $X(M)$. This action is not free, and it often has complex dynamics. Nonetheless, we can define the topological quotient space

$$AI(M) = AH(M)/\mathrm{Out}(\pi_1(M)).$$

Elements of $AI(M)$ are naturally oriented hyperbolic 3-manifolds homotopy equivalent to M without a specified marking.

2.2. **Topological background.** A compact 3-manifold M is said to have *incompressible boundary* if whenever S is a component of ∂M, the inclusion map induces an injection of $\pi_1(S)$ into $\pi_1(M)$. In our setting, this is equivalent to $\pi_1(M)$ being freely indecomposable. A properly embedded annulus A in M is said to be *essential* if the inclusion map induces an injection of $\pi_1(A)$ into $\pi_1(M)$ and A cannot be properly homotoped into ∂M (i.e. there does not exist a homotopy of pairs of the inclusion $(A, \partial A) \to (M, \partial M)$ to a map with image in ∂M). An essential annulus A is said to be *primitive* if the image of $\pi_1(A)$ in $\pi_1(M)$ is a maximal abelian subgroup.

If M does not have incompressible boundary, it is said to have *compressible boundary.* The fundamental examples of 3-manifolds with compressible boundary are compression bodies. A *compression body* is either a handlebody or is formed by attaching 1-handles to disjoint disks on the boundary surface $R \times \{1\}$ of a 3-manifold $R \times [0, 1]$ where R is a closed, but not necessarily connected, surface (see, for example, Bonahon [6]). The resulting 3-manifold C (assumed to be connected) will have a single boundary component $\partial_+ C$ intersecting $R \times \{1\}$, called the positive (or external) boundary of C. If C is not an untwisted interval bundle over a closed surface, then $\partial_+ C$ is the unique compressible boundary component of C. Notice that the induced homomorphism $\pi_1(\partial_+ C) \to \pi_1(C)$ is surjective. In fact, a compact irreducible 3-manifold M is a compression body if and only if there exists a component S of ∂M such that $\pi_1(S) \to \pi_1(M)$ is surjective.

Every compact hyperbolizable 3-manifold can be constructed from compression bodies and manifolds with incompressible boundary. Bonahon [6] and McCullough-Miller [40] showed that there exists a neighborhood C_M of ∂M, called the *characteristic compression body*, such that each component of C_M is a compression body and each component of $\partial C_M - \partial M$ is incompressible in M.

Dehn filling will play a key role in the proof of Theorem 1.2. Let F be a toroidal boundary component of compact 3-manifold M and let (m, l) be a choice of meridian and longitude for F. Given a pair (p, q) of relatively prime integers, we may form a new manifold $M(p, q)$ by attaching a solid torus V to M by an orientation-reversing homeomorphism $g\colon \partial V \to F$ so that, if c is the meridian of V, then $g(c)$ is a (p, q) curve on F with respect to the chosen meridian-longitude system. We say that $M(p, q)$ is obtained from M by (p, q)-*Dehn filling along* F.

2.3. **Hyperbolic background.** If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, then $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ acts on $\widehat{\mathbb{C}}$ as a group of conformal automorphisms. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open Γ-invariant subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously. Note that $\Omega(\Gamma)$ may be empty. Its complement $\Lambda(\Gamma) = \widehat{\mathbb{C}} - \Omega(\Gamma)$ is called the *limit set.* The quotient $\partial_c N = \Omega(\Gamma)/\Gamma$ is naturally a Riemann surface called the *conformal boundary.*

Thurston's Hyperbolization theorem, see Morgan [44, Theorem B'], guarantees that if M is compact and hyperbolizable, then there exists a hyperbolic 3-manifold N and a homeomorphism

$$\psi : M - \partial_T M \to N \cup \partial_c N$$

where $\partial_T M$ denotes the collection of toroidal boundary components of M.

The convex core $C(N)$ of N is the smallest convex submanifold whose inclusion into N is a homotopy equivalence. More concretely, it is obtained as the quotient, by Γ, of the convex hull, in $\mathbb{H}^3$, of the limit set $\Lambda(\Gamma)$. There is a well-defined retraction $r : N \to C(N)$ obtained by taking x to the (unique) point in $C(N)$ closest to x. The nearest point retraction r is a homotopy equivalence and is $\frac{1}{\cosh s}$-Lipschitz on the complement of the neighborhood of radius s of $C(N)$.

There exists a universal constant μ, called the Margulis constant, such that if $\epsilon < \mu$, then each component of the ϵ-thin part

$$N_{thin(\epsilon)} = \{x \in N \mid \mathrm{inj}_N(x) < \epsilon\}$$

(where $\mathrm{inj}_N(x)$ denotes the injectivity radius of N at x) is either a metric regular neighborhood of a geodesic or is homeomorphic to $T \times (0, \infty)$ where T is either a torus or an open annulus (see Benedetti-Petronio [4] for example). The ϵ-thick part of N is defined simply to be the complement of the ϵ-thin part

$$N_{thick(\epsilon)} = N - N_{thin(\epsilon)}.$$

It is also useful to consider the manifold N^0_ϵ obtained from N by removing the non-compact components of $N_{thin(\epsilon)}$.

If N is a hyperbolic 3-manifold with finitely generated fundamental group, then it admits a compact core, i.e. a compact submanifold whose inclusion into M is a homotopy equivalence (see Scott [48]). More generally, if $\epsilon < \mu$, then there exists a *relative compact core* R for N^0_ϵ, i.e. a compact core which intersects each component of ∂N^0_ϵ in a compact core for that component (see Kulkarni-Shalen [33] or McCullough [38]). Let $P = \partial R - \partial N^0_\epsilon$ and let P^0 denote the interior of P. The Tameness Theorem of Agol [1] and Calegari-Gabai [14] assures us that we may choose R so that $N^0_\epsilon - R$ is homeomorphic to $(\partial R - P^0) \times (0, \infty)$. In particular, the ends of N^0_ϵ are in one-to-one correspondence with the components of $\partial R - P^0$. (We will blur this distinction and simply regard an end as a component of $N^0_\epsilon - R$ once we have chosen ϵ and a relative compact core R for N^0_ϵ.) We say that an end U of N^0_ϵ is *geometrically finite* if the intersection of $C(N)$ with U is bounded (i.e. admits a compact closure). N is said to be geometrically finite if all the ends of N^0_ϵ are geometrically finite.

Thurston [53] showed that if M is a compact hyperbolizable 3-manifold whose boundary is a torus F, then all but finitely many Dehn fillings of M are hyperbolizable. Moreover, as the Dehn surgery coefficients approach ∞, the resulting hyperbolic manifolds "converge" to the hyperbolic 3-manifold homeomorphic to $\mathrm{int}(M)$. If M has other boundary components, then there

is a version of this theorem where one begins with a geometrically finite hyperbolic 3-manifold homeomorphic to $\mathrm{int}(M)$ and one is allowed to perform the Dehn filling while fixing the conformal structure on the non-toroidal boundary components of M. The proof uses the cone-manifold deformation theory developed by Hodgson-Kerckhoff [25] in the finite volume case and extended to the infinite volume case by Bromberg [11] and Brock-Bromberg [9]. (The first statement of a Hyperbolic Dehn Filling Theorem in the infinite volume setting was given by Bonahon-Otal [7], see also Comar [20].) For a general statement of the Filling Theorem, and a discussion of its derivation from the previously mentioned work, see Bromberg [12] or Magid [35].

Hyperbolic Dehn Filling Theorem: *Let M be a compact, hyperbolizable 3-manifold and let F be a toroidal boundary component of M. Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold admitting an orientation-preserving homeomorphism $\psi : M - \partial_T M \to N \cup \partial_c N$. Let $\{(p_n, q_n)\}$ be an infinite sequence of distinct pairs of relatively prime integers.*

Then, for all sufficiently large n, there exists a (non-faithful) representation $\beta_n : \Gamma \to \mathrm{PSL}_2(\mathbf{C})$ with discrete image such that

1. *$\{\beta_n\}$ converges to the identity representation of Γ, and*
2. *if $i_n : M \to M(p_n, q_n)$ denotes the inclusion map, then for each n, there exists an orientation-preserving homeomorphism*

$$\psi_n : M(p_n, q_n) - \partial_T M(p_n, q_n) \to N_{\beta_n} \cup \partial_c N_{\beta_n}$$

such that $\beta_n \circ \psi_*$ is conjugate to $(\psi_n)_* \circ (i_n)_*$, and the restriction of $\psi_n \circ i_n \circ \psi^{-1}$ to $\partial_c N$ is conformal.

3. Points are usually closed

If S is a closed orientable surface, we showed in [18] that

$$\mathcal{AI}(S) = AH(S \times I)/\mathrm{Mod}_+(S)$$

is not T_1 where $\mathrm{Mod}_+(S)$ is the group of (isotopy classes of) orientation-preserving homeomorphisms of S. We recall that a topological space is T_1 if all points are closed sets. Since $\mathrm{Mod}_+(S)$ is identified with an index two subgroup of $\mathrm{Out}(\pi_1(S))$, one also expects that

$$AI(S \times I) = AH(S \times I)/\mathrm{Out}(\pi_1(S))$$

is not T_1.

In this section, we show that if M is an untwisted interval bundle, which also includes the case that M is a handlebody, then $AI(M)$ is not T_1, but that $AI(M)$ is T_1 for all other compact, hyperbolizable 3-manifolds.

Theorem 1.1. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. Then the moduli space $AI(M)$ is T_1 if and only if M is not an untwisted interval bundle.*

Proof. We first show that $AI(M)$ is T_1 if M is not an untwisted interval bundle. Let $p : AH(M) \to AI(M)$ be the quotient map and let N be a hyperbolic manifold in $AI(M)$. We must show that $p^{-1}(N)$ is a closed subset of $AH(M)$. Since $AH(M)$ is Hausdorff and second countable, it suffices to show that if $\{\rho_n\}$ is a convergent sequence in $p^{-1}(N)$, then $\lim \rho_n \in p^{-1}(N)$.

An element $\rho \in p^{-1}(N)$ is a representation such that N_ρ is isometric to N. Let $\{\rho_n\}$ be a convergent sequence of representations in $p^{-1}(N)$. Let $G \subset M$ be a finite graph such that the inclusion map induces a surjection of $\pi_1(G)$ onto $\pi_1(M)$. Each ρ_n gives rise to a homotopy equivalence $h_n : M \to N$, and hence to a map $j_n = h_n|_G : G \to N$, both of which are only well-defined up to homotopy. Since $\{\rho_n\}$ is convergent, there exists K such that $j_n(G)$ has length at most K for all n, after possibly altering h_n by a homotopy.

Let R be a compact core for N. Assume first that R is not a compression body. In this case, if S is any component of ∂R, then the inclusion map does not induce a surjection of $\pi_1(S)$ to $\pi_1(R)$ (see the discussion in section 2). Since $j_n(G)$ carries the fundamental group it cannot lie entirely outside of R. It follows that $j_n(G)$ lies in the closed neighborhood $\mathcal{N}_K(R)$ of radius K about R. By compactness, there are only finitely many homotopy classes of maps of G into $\mathcal{N}_K(R)$ with total length at most K. Hence, there are only finitely many different representations among the ρ_n, up to conjugacy. The deformation space $AH(M)$ is Hausdorff, and the sequence $\{\rho_n\}$ converges, implying that $\{\rho_n\}$ is eventually constant. Therefore $\lim \rho_n$ lies in the preimage of N, implying that the fiber $p^{-1}(N)$ is closed and that N is a closed point of $AI(M)$.

Next we assume that R is a compression body. If R were an untwisted interval bundle, then M would also have to be a untwisted interval bundle (by Theorems 5.2 and 10.6 in Hempel [23]) which we have disallowed. So R must have at least one incompressible boundary component and only one compressible boundary component $\partial_+ R$. We are free to assume that M is homeomorphic to R, since the definition of $AI(M)$ depends only on the homotopy type of M. Let D denote the union of R and the component of $N - R$ bounded by $\partial_+ R$. Since the fundamental group of a component of $N - D$ never surjects onto $\pi_1(N)$, with respect to the map induced by inclusion, we see as above that each $j_n(G)$ must intersect D, so is contained in the neighborhood of radius K of D.

Recall that there exists $\epsilon_K > 0$ so that the distance from the ϵ_K-thin part of N to the μ-thick part of N is greater than K (where μ is the Margulis constant). It follows that $j_n(G)$ must be contained in the ϵ_K-thick part of N.

Let F be an incompressible boundary component of M. Then $h_n(F)$ is homotopic to an incompressible boundary component of R (see, for example, the proof of Proposition 9.2.1 in [17]). As there are finitely many possibilities, we may pass to a subsequence so that $h_n(F)$ is homotopic to a fixed boundary component F'. We may choose G so that there is a proper

subgraph $G_F \subset G$ such that the image of $\pi_1(G_F)$ in $\pi_1(M)$ (under the inclusion map) is conjugate to $\pi_1(F)$. Let $p_F : N_F \to N$ be the covering map associated to $\pi_1(F') \subset \pi_1(N)$. Then $j_n|_{G_F}$ lifts to a map k_n of G_F into N_F.

Assume first that F is a torus. Then $k_n(G_F)$ must lie in the portion X of N_F with injectivity radius between ϵ_K and $K/2$, which is compact. It follows that $j_n(G)$ must lie in the closed neighborhood of radius K of $p_F(X)$. Since $p_F(X)$ is compact, we may conclude, as in the general case, that $\{\rho_n\}$ is eventually constant and hence that $p^{-1}(N)$ is closed.

We now suppose that F has genus at least 2. We first establish that there exists L such that $k_n(G_F)$ must be contained in a neighborhood of radius L of the convex core $C(N_F)$. It is a consequence of the thick-thin decomposition, that if G' is a graph in N_F which carries the fundamental group then G' must have length at least μ. We also recall that the nearest point retraction $r_F : N_F \to C(N_F)$ is a homotopy equivalence which is $\frac{1}{\cosh s}$-Lipschitz on the complement of the neighborhood of radius s of $C(N)$. Therefore, if $k_n(G_F)$ lies outside of $\mathcal{N}_s(C(N_F))$, then $r_F(k_n(G_F))$ has length at most $\frac{K}{\cosh s}$. It follows that $k_n(G_F)$ must intersect the neighborhood of radius $\cosh^{-1}(\frac{K}{\mu})$ of $C(N_F)$, so we may choose $L = K + \cosh^{-1}(\frac{K}{\mu})$.

If N_F is geometrically finite, then $X = C(N_F) \cap N_{thick(\epsilon_K)}$ is compact and $j_n(G)$ must be contained in the neighborhood of radius $L+K$ of $p_F(X)$ which allows us to complete the proof as before.

If N_F is not geometrically finite, we will need to invoke the Covering Theorem to complete the proof. Let $\tilde{F}$ denote the lift of F' to N_F. Then $\tilde{F}$ divides N_F into two components, one of which, say A_-, is mapped homeomorphically to the component of $N - R$ bounded by F'. Let $A_+ = N_F - A_-$. We may choose a a relative compact core R_F for $(N_F)^0_\epsilon$ (for some $\epsilon < \epsilon_K$) so that $\tilde{F}$ is contained in the interior of R_F. Since p_F is infinite-to-one on each end of $(N_F)^0_\epsilon$ which is contained in A_+, the Covering Theorem (see [15] or [53]) implies that all such ends are geometrically finite. Therefore,

$$Y = A_+ \cap C(N_F) \cap (N_F)_{thick(\epsilon_K)}$$

is compact. If we let $Z = A_- \cup Y$, then we see that $k_n(G_F)$ is contained in the closed neighborhood of radius L about Z (since $C(N_F) \cap N_{thick(\epsilon_K)} \subset Z$). Therefore, $j_n(G)$ is contained in the closed $(L+K)$-neighborhood of

$$D \cap p_F(Z) = D \cap p_F(Y).$$

Since $D \cap p_F(Y)$ is compact, we conclude, exactly as in the previous cases, that $p^{-1}(N)$ is closed. This case completes the proof that $AI(M)$ is T_1 if M is not an untwisted interval bundle.

We now deal with the case where $M = S \times I$ is an untwisted interval bundle over a compact surface S. (In the special case that M is a handlebody of genus 2, we choose S to be the punctured torus.) In our previous paper [18], we consider the space $AH(S)$ of (conjugacy classes of) discrete

faithful representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ such that if $g \in \pi_1(S)$ is peripheral, then $\rho(g)$ is parabolic. In Proposition 3.1, we use work of Thurston [51] and McMullen [41] to exhibit a sequence $\{\rho_n\}$ in $AH(S)$ which converges to $\rho \in AH(S)$ such that $\Lambda(\rho) = \widehat{\mathbb{C}}$, $\Lambda(\rho_1) \neq \widehat{\mathbb{C}}$ and for all n there exists $\varphi_n \in \mathrm{Mod}_+(S)$ such that $\rho_n = \rho_1 \circ \varphi_n$. Since $AH(S) \subset AH(S \times I)$ and $\mathrm{Mod}_+(S)$ is identified with a subgroup of $\mathrm{Out}(\pi_1(S))$, we see that $\{\rho_n\}$ is a sequence in $p^{-1}(N_{\rho_1})$ which converges to a point outside of $p^{-1}(N_{\rho_1})$. Therefore, N_{ρ_1} is a point in $AI(S \times I)$ which is not closed. □

Remark: One may further show, as in the remark after Proposition 3.1 in [18], that if $N \in AI(S \times I)$ is a degenerate hyperbolic 3-manifold with a lower bound on its injectivity radius, then N is not a closed point in $AI(S \times I)$. We recall that $N = \mathbb{H}^3/\Gamma$ is degenerate if $\Omega(\Gamma)$ is connected and simply connected and Γ is finitely generated.

4. Primitive essential annuli and the failure of proper discontinuity

In this section, we show that if M contains a primitive essential annulus, then $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$. We do so by using the Hyperbolic Dehn Filling Theorem to produce a convergent sequence $\{\rho_n\}$ in $AH(M)$ and a sequence $\{\varphi_n\}$ of distinct element of $\mathrm{Out}(\pi_1(M))$ such that $\{\rho_n \circ \varphi_n\}$ is also convergent. The construction is a generalization of a construction of Kerckhoff-Thurston [31]. One may also think of the argument as a simple version of the "wrapping" construction (see Anderson-Canary [2]) which was also used to show that components of $\mathrm{int}(AH(M))$ self-bump whenever M contains a primitive essential annulus (see McMullen [42] and Bromberg-Holt [13]).

Theorem 1.2. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. If M contains a primitive essential annulus then $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$. Moreover, if M contains a primitive essential annulus, then $AI(M)$ is not Hausdorff.*

Proof. Let A be a primitive essential annulus in M with core curve α. Let $\hat{M} = M - \mathcal{N}(\alpha)$ where $\mathcal{N}(\alpha)$ is an open regular neighborhood of α. Lemma 10.2 in [3] observes that $\hat{M}$ is hyperbolizable. Since $\hat{M}$ is hyperbolizable, Thurston's Hyperbolization Theorem implies that there exists a hyperbolic manifold $\hat{N}$ and a homeomorphism $\psi : \hat{M} - \partial_T \hat{M} \to \hat{N} \cup \partial_c \hat{N}$. The classical deformation theory of Kleinian groups (see Bers [5] or [17]) implies that we may choose any conformal structure on $\partial_c \hat{N}$.

Let A_0 and A_1 denote the components of $A \cap \hat{M}$. Let M_i be the complement in $\hat{M}$ of a regular neighborhood of A_i. Let $h_i : M \to \hat{M}$ be an embedding with image M_i which agrees with the identity map off of a (somewhat larger) regular neighborhood of A.

Let F be the toroidal boundary component of $\hat{M}$ which is the boundary of $\mathcal{N}(\alpha)$ in M. Choose a meridian-longitude system for F so that the meridian for F bounds a disk in M and the longitude is isotopic to $A_1 \cap F$. Lemma 10.3 in [3] implies that if $i_n : \hat{M} \to \hat{M}(1,n)$ is the inclusion map, then $i_n \circ h_i : M \to \hat{M}(1,n)$ is homotopic to a homeomorphism for each $i = 0, 1$ and all $n \in \mathbb{Z}$. Moreover, we may similarly check that $i_n \circ h_1$ is homotopic to $i_n \circ h_0 \circ D_A^n$ for all n, where D_A denotes a Dehn twist along A. Notice first that $j_n = D_{A_0}^n$ takes a $(1,0)$-curve on F to a $(1,n)$-curve on F, so extends to a homeomorphism $j_n : M = \hat{M}(1,0) \to \hat{M}(1,n)$. Therefore, since $i_0 \circ h_0$ and $i_0 \circ h_1$ are homotopic, so are $j_n \circ i_0 \circ h_0$ and $j_n \circ i_0 \circ h_1$. But, $j_n \circ i_0 \circ h_0$ is homotopic to $i_n \circ h_0 \circ D_A^n$ and $j_n \circ i_0 \circ h_1 = i_n \circ h_1$, which completes the proof that $i_n \circ h_1$ is homotopic to $i_n \circ h_0 \circ D_A^n$ for all n.

Let $\rho_0 = (\psi \circ h_0)_*$ and $\rho_1 = (\psi \circ h_1)_*$. Since $(h_i)_*$ induces an injection of $\pi_1(M)$ into $\pi_1(\hat{M})$, $\rho_i \in AH(M)$. We next observe that one can choose $\hat{N}$ so that N_{ρ_0} and N_{ρ_1} are not isometric. Let $a_i = A_i \cap (\partial M - \partial_T \hat{M})$ and let a_i^* denote the geodesic representative of $\psi(a_i)$ in $\partial_c \hat{N}$. Notice that for each $i = 0, 1$ there is a conformal embedding of $\partial_c \hat{N} - a_i^*$ into $\partial_c N_{\rho_i}$ such that each component of the complement of the image of $\partial_c \hat{N} - a_i^*$ is a neighborhood of a cusp. One may therefore choose the conformal structure on $\partial_c \hat{N}$ so that there is not a conformal homeomorphism from $\partial_c N_{\rho_0}$ to $\partial_c N_{\rho_1}$. Therefore, N_{ρ_0} and N_{ρ_1} are not isometric.

Let $\{N_n = N_{\beta_n}\}$ be the sequence of hyperbolic 3-manifolds provided by the Hyperbolic Dehn Filling Theorem applied to the sequence $\{(1,n)\}_{n \in \mathbb{Z}_+}$ and let $\{\psi_n : \hat{M}(1,n) - \partial_T \hat{M}(1,n) \to N_n \cup \partial_c N_n\}$ be the homeomorphisms such that $\psi_n \circ i_n \circ \psi^{-1}$ is conformal on $\partial_c N$. Let

$$\rho_{n,i} = \beta_n \circ \rho_i$$

for all n large enough that N_n and ψ_n exist. Since $\beta_n \circ \psi_*$ is conjugate to $(\psi_n \circ i_n)_*$ (by applying part (2) of the Hyperbolic Dehn Filling Theorem) and $i_n \circ h_i$ is homotopic to a homeomorphism, we see that $\rho_{n,i} = (\psi_n \circ i_n \circ h_i)_*$ lies in $AH(M)$ for all n and each i. It follows from part (1) of the Hyperbolic Dehn Filling Theorem that $\{\rho_{n,i}\}$ converges to ρ_i for each i. Moreover, $\rho_{n,1} = \rho_{n,0} \circ (D_A)_*^n$ for all n, since $i_n \circ h_1$ is homotopic to $i_n \circ h_0 \circ D_A^n$ for all n. Therefore, $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$.

Moreover, $\{\rho_{n,0}\}$ and $\{\rho_{n,1}\}$ project to the same sequence in $AI(M)$ and both N_{ρ_0} and N_{ρ_1} are limits of this sequence. Since N_{ρ_0} and N_{ρ_1} are distinct manifolds in $AI(M)$, it follows that $AI(M)$ is not Hausdorff. □

Remark: One can also establish Theorem 1.2 using deformation theory of Kleinian groups and convergence results of Thurston [52]. This version of the argument follows the same outline as the proof of Proposition 3.3 in [18].

We provide a brief sketch of this argument. The classical deformation theory of Kleinian groups (in combination with Thurston's Hyperbolization Theorem) guarantees that there exists a component B of $\mathrm{int}(AH(M))$ such

that if $\rho \in B$, then there exists a homeomorphism $\bar{h}_\rho : M - \partial_T M \to N_\rho \cup \partial_c N_\rho$ and the point ρ is determined by the induced conformal structure on $\partial M - \partial_T M$. Moreover, every possible conformal structure on $\partial M - \partial_T M$ arises in this manner.

Let a_0 and a_1 denote the components of ∂A and let t_{a_0} and t_{a_1} denote Dehn twists about a_0 and a_1 respectively. We choose orientations so that D_A induces $t_{a_0} \circ t_{a_1}$ on ∂M. We then let $\rho_{n,0} \in B$ have associated conformal structure $t^n_{a_1}(X)$ and let $\rho_{n,1}$ have associated conformal structure $t^{-n}_{a_0}(X)$ for some conformal structure X on ∂M. Thurston's convergence results [51, 52] can be used to show that there exists a subsequence $\{n_j\}$ of $\mathbb{Z}$ such that $\{\rho_{n_j,0}\}$ and $\{\rho_{n_j,1}\}$ both converge. One can guarantee, roughly as above, that the limiting hyperbolic manifolds are not isometric. Moreover, $\rho_{n,1} = \rho_{n,0} \circ (D_A)^n_*$ for all n, so we are the same situation as in the proof above.

5. The characteristic submanifold and mapping class groups

In order to further analyze the case where M has incompressible boundary we will make use of the characteristic submanifold (developed by Jaco-Shalen [27] and Johannson [29]) and the theory of mapping class groups of 3-manifolds developed by Johannson [29] and extended by McCullough and his co-authors [39, 26, 17].

We begin by recalling the definition of the characteristic submanifold, specialized to the hyperbolic setting. In the general setting, the components of the characteristic submanifold are interval bundles and Seifert fibred spaces. In the hyperbolic setting, the only Seifert fibred spaces which occur are the solid torus and the thickened torus (see Morgan [44, Sec. 11] or Canary-McCullough [17, Chap. 5]).

Theorem 5.1. *Let M be a compact oriented hyperbolizable 3-manifold with incompressible boundary. There exists a codimension zero submanifold $\Sigma(M) \subseteq M$ with frontier $Fr(\Sigma(M)) = \overline{\partial\Sigma(M) - \partial M}$ satisfying the following properties:*

1. *Each component Σ_i of $\Sigma(M)$ is either*
   (i) an interval bundle over a compact surface with negative Euler characteristic which intersects ∂M in its associated ∂I-bundle,
   (ii) a thickened torus such that $\partial M \cap \Sigma_i$ contains a torus, or
   (iii) a solid torus.
2. *The frontier $Fr(\Sigma(M))$ is a collection of essential annuli.*
3. *Any essential annulus or incompressible torus in M is properly isotopic into $\Sigma(M)$.*
4. *If X is a component of $M - \Sigma(M)$, then either $\pi_1(X)$ is non-abelian or $(\overline{X}, Fr(X)) \cong (S^1 \times [0,1] \times [0,1], S^1 \times [0,1] \times \{0,1\})$ and X lies between an interval bundle component of $\Sigma(M)$ and a thickened or solid torus component of $\Sigma(M)$.*

Moreover, such a $\Sigma(M)$ is unique up to isotopy, and is called the characteristic submanifold *of M.*

The existence and the uniqueness of the characteristic submanifold in general follows from The Characteristic Pair Theorem in [27] or Proposition 9.4 and Corollary 10.9 in [29]. Theorem 5.1(1) follows from [17, Theorem 5.3.4], part (2) follows from (1) and the definition of the characteristic submanifold, part (3) follows from [29, Theorem 12.5], and part (4) follows from [17, Theorem 2.9.3].

Johannson's Classification Theorem [29] asserts that every homotopy equivalence between compact, irreducible 3-manifolds with incompressible boundary may be homotoped so that it preserves the characteristic submanifold and is a homeomorphism on its complement. Therefore, the study of $\mathrm{Out}(\pi_1(M))$ often reduces to the study of mapping class groups of interval bundles and Seifert-fibered spaces.

Johannson's Classification Theorem [29, Theorem 24.2]**.** *Let M and Q be irreducible 3-manifolds with incompressible boundary and let $h : M \to Q$ be a homotopy equivalence. Then h is homotopic to a map $g : M \to Q$ such that*

(1) $g^{-1}(\Sigma(Q)) = \Sigma(M)$,
(2) $g|_{\Sigma(M)} : \Sigma(M) \to \Sigma(Q)$ *is a homotopy equivalence, and*
(3) $g|_{\overline{M-\Sigma(M)}} : \overline{M - \Sigma(M)} \to \overline{Q - \Sigma(Q)}$ *is a homeomorphism.*

Moreover, if h is a homeomorphism, then g is a homeomorphim.

We let the mapping class group $\mathrm{Mod}(M)$ denote the group of isotopy classes of self-homeomorphisms of M. Since M is irreducible and has (non-empty) incompressible boundary, any two homotopic homeomorphisms are isotopic (see Waldhausen [55, Theorem 7.1]), so $\mathrm{Mod}(M)$ is naturally a subgroup of $\mathrm{Out}(\pi_1(M))$. For simplicity, we will assume that M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. Notice that this implies that $\Sigma(M)$ contains no thickened torus components. Let Σ be the characteristic submanifold of M and denote its components by $\{\Sigma_1, \ldots, \Sigma_k\}$.

Following McCullough [39], we let $\mathrm{Mod}(\Sigma_i, Fr(\Sigma_i))$ denote the group of homotopy classes of homeomorphisms $h : \Sigma_i \to \Sigma_i$ such that $h(F) = F$ for each component F of $Fr(\Sigma_i)$. We let $G(\Sigma_i, Fr(\Sigma_i))$ denote the subgroup consisting of (homotopy classes of) homeomorphisms which have representatives which are the identity on $Fr(\Sigma_i)$. Define

$$G(\Sigma, Fr(\Sigma)) = \oplus_{i=1}^{k} G(\Sigma_i, Fr(\Sigma_i)).$$

Notice that using these definitions, the restriction of a Dehn twist along a component of $Fr(\Sigma)$ is trivial in $G(\Sigma, Fr(\Sigma))$.

In our case, each Σ_i is either an interval bundle over a compact surface F_i with negative Euler characteristic or a solid torus. If Σ_i is a solid torus, then $G(\Sigma_i, Fr(\Sigma_i))$ is finite (see Lemma 10.3.2 in [17]). If Σ_i is an interval bundle over a compact surface F_i, then $G(\Sigma_i, Fr(\Sigma_i))$ is isomorphic to the group $G(F_i, \partial F_i)$ of proper isotopy classes of self-homeomorphisms of F which are the identity on ∂F (see Proposition 3.2.1 in [39] and Lemma 6.1 in

[26]). Moreover, $G(\Sigma_i, Fr(\Sigma_i))$ injects into $\mathrm{Out}(\pi_1(\Sigma_i))$ (see Proposition 5.2.3 in [17] for example). We say that Σ_i is *tiny* if its base surface F_i is either a thrice-punctured sphere or a twice-punctured projective plane. If Σ_i is not tiny, then F_i contains a 2-sided, non-peripheral homotopically non-trivial simple closed curve, so $G(\Sigma_i, Fr(\Sigma_i))$ is infinite. If Σ_i is tiny, then $G(\Sigma_i, Fr(\Sigma_i))$ is finite (see Korkmaz [32] for the case when F_i is a twice-punctured projective plane).

Let $J(M)$ be the subgroup of $\mathrm{Mod}(M)$ consisting of classes represented by homeomorphisms fixing $M - \Sigma$ pointwise. Lemma 4.2.1 of McCullough [39] implies that $J(M)$ has finite index in $\mathrm{Mod}(M)$. (Instead of $J(M)$, McCullough writes $\mathcal{K}(M, \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$.) Lemma 4.2.2 of McCullough [39] implies that the kernel $K(M)$ of the natural surjective homomorphism

$$p_\Sigma : J(M) \to G(\Sigma, Fr(\Sigma))$$

is abelian and is generated by Dehn twists about the annuli in $Fr(\Sigma)$.

We summarize the discussion above in the following statement.

Theorem 5.2. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. Then there is a finite index subgroup $J(M)$ of $\mathrm{Mod}(M)$ and an exact sequence*

$$1 \longrightarrow K(M) \longrightarrow J(M) \xrightarrow{p_\Sigma} G(\Sigma, Fr(\Sigma)) \longrightarrow 1$$

such that $K(M)$ is an abelian group generated by Dehn twists about essential annuli in $Fr(\Sigma)$.

Suppose that Σ_i is a component of $\Sigma(M)$. If Σ_i is a solid torus or a tiny interval bundle, then $G(\Sigma_i, Fr(\Sigma_i))$ is finite. Otherwise, $G(\Sigma_i, Fr(\Sigma_i))$ is infinite and injects into $\mathrm{Out}(\pi_1(\Sigma_i))$.

6. Characteristic collections of annuli

We continue to assume that M has incompressible boundary and no toroidal boundary components and that $\Sigma(M)$ is its characteristic submanifold. In this section, we organize $K(M)$ into subgroups generated by collections of annuli with homotopic core curves, called characteristic collection of annuli, and define a class of free subgroups of $\pi_1(M)$ which "register" these subgroups of $K(M)$.

A *characteristic collection of annuli* for M is either a) the collection of all frontier annuli in a solid torus component of $\Sigma(M)$, or b) an annulus in the frontier of an interval bundle component of $\Sigma(M)$ which is not properly isotopic to a frontier annulus of a solid torus component of $\Sigma(M)$.

If C_j is a characteristic collection of annuli for M, let K_j be the subgroup of $K(M)$ generated by Dehn twists about the annuli in C_j. Notice that $K_i \cap K_j = \{id\}$ for $i \neq j$, since each element of K_j fixes any curve disjoint from C_j. Then $K(M) = \oplus_{j=1}^m K_j$, since every frontier annulus of $\Sigma(M)$ is properly isotopic to a component of some characteristic collection of annuli. Let $q_j : K(M) \to K_j$ be the projection map.

We next introduce free subgroups of $\pi_1(M)$, called C_j-registering subgroups, which are preserved by K_j and such that K_j acts effectively on the subgroup.

We first suppose that $C_j = Fr(T_j)$ where T_j is a solid torus component of $\Sigma(M)$. Let $\{A_1, \ldots, A_l\}$ denote the components of $Fr(T_j)$. For each $i = 1, \ldots, l$, let X_i be the component of $M - (T_j \cup C_1 \cup C_2 \cup \ldots \cup C_m)$ abutting A_i. (Notice that each X_i is either a component of $M - \Sigma(M)$ or properly isotopic to the interior of an interval bundle component of $\Sigma(M)$.) Let a be a core curve for T_j and let x_0 be a point on a. We say that a subgroup H of $\pi_1(M, x_0)$ is *C_j-registering* if it is freely (and minimally) generated by a and, for each $i = 1, \ldots, l$, a loop g_i in $T_j \cup X_i$ based at x_0 intersecting A_i exactly twice. In particular, every C_j-registering subgroup of $\pi_1(M, x_0)$ is isomorphic to F_{l+1}.

Notice that a Dehn twist D_{A_i} along any A_i preserves H in $\pi_1(M, x_0)$. It acts on H by the map t_i which fixes a and g_m for $m \neq i$, and conjugates g_i by a^n (where the core curve of A_i is homotopic to a^n). Let $s_H : K_j \to \mathrm{Out}(H)$ be the homomorphism which takes each D_{A_i} to t_i. Simultaneously twisting along all l annuli induces conjugation by a^n, which is an inner automorphism of H. Moreover, it is easily checked that $s_H(K_j)$ is isomorphic to $\mathbf{Z}^{l-1}$ and is generated by $\{t_1, \ldots, t_{l-1}\}$. The set $\{a, g_1, \ldots, g_l\}$ may be extended to a generating set for $\pi_1(M, x_0)$ by appending curves which intersect $Fr(T_j)$ exactly twice, so $D_{A_1} \circ \cdots D_{A_l}$ acts as conjugation by a^n on all of $\pi_1(M, x_0)$. Therefore, K_j itself is isomorphic to $\mathbf{Z}^{l-1}$ and s_H is injective. (In particular, if C_j is a single annulus in the boundary of a solid torus component of $\Sigma(M)$, then K_j is trivial and we could have omitted C_j.)

Now suppose that $C_j = \{A\}$ is a frontier annulus of an interval bundle component Σ_i of Σ which is not properly isotopic into a solid torus component of Σ. Let a be a core curve for A and let x_0 be a point on a. We say that a subgroup H of $\pi_1(M, x_0)$ is *C_j-registering* if it is freely (and minimally) generated by a and two loops g_1 and g_2 based at x_0 each of whose interiors misses A, and which lie in the two distinct components of $M - (C_1 \cup C_2 \cup \ldots \cup C_m)$ abutting A. In this case, H is isomorphic to F_3. Arguing as above, it follows that K_j is an infinite cyclic subgroup of $\mathrm{Out}(\pi_1(M))$ and there is an injection $s_H : K_j \to \mathrm{Out}(H)$.

In either situation, if H is a C_j-registering group for a characteristic collection of annuli C_j, then we may consider the map

$$r_H : X(M) \to X(H)$$

simply obtained by taking ρ to $\rho|_H$. (Here, $X(H)$ is the $\mathrm{PSL}_2(\mathbf{C})$-character variety of the abstract group H.) One easily checks from the description above that if $\alpha \in K_j$, then $r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(\alpha)$ for all $\rho \in X(M)$. Notice that if $\varphi \in K_l$ and $j \neq l$, then K_l acts trivially on H, since each generating curve of H is disjoint from C_l. Therefore,

$$r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(q_j(\alpha))$$

for all $\rho \in X(M)$ and $\alpha \in K(M)$.

We summarize the key points of this discussion for use later:

Lemma 6.1. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. If C_j is a characteristic collection of annuli for M and H is a C_j-registering subgroup of $\pi_1(M)$, then H is preserved by each element of K_j and there is a natural injective homomorphism $s_H : K_j \to \mathrm{Out}(H)$. Moreover, if $\alpha \in K(M)$, then $r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(q_j(\alpha))$ for all $\rho \in X(M)$.*

7. Primitive essential annuli and manifolds with compressible boundary

In this section we use a result of Johannson [29] to show that every compact hyperbolizable 3-manifolds with compressible boundary and no toroidal boundary components contains a primitive essential annulus. It then follows from Theorem 1.2 that if M has compressible boundary and no toroidal boundary components, then $\mathrm{Out}(\pi_1(M))$ fails to act properly discontinuously on $AH(M)$ and $AI(M)$ is not Hausdorff.

We first find indivisible curves in the boundary of compact hyperbolizable 3-manifolds with incompressible boundary and no toroidal boundary components. We call a curve a in M *indivisible* if it generates a maximal cyclic subgroup of $\pi_1(M)$.

Lemma 7.1. *Let M be a compact hyperbolizable 3-manifold with (non-empty) incompressible boundary. Then, if F is a component of ∂M, there exists an indivisible simple closed curve in F.*

Proof. We use a special case of a result of Johannson [29] (see also Jaco-Shalen [28]) which characterizes divisible simple closed curves in ∂M.

Lemma 7.2. ([29, Lemma 32.1]) *Let M be a compact hyperbolizable 3-manifold with incompressible boundary. An essential simple closed curve α in ∂M which is not indivisible is either isotopic into a solid torus component of $\Sigma(M)$ or is isotopic to a boundary component of an essential Möbius band in an interval bundle component of $\Sigma(M)$.*

Therefore, if $\Sigma(M)$ is not all of M, then any simple closed curve in F which cannot be isotoped into a solid torus or interval bundle component of $\Sigma(M)$ is indivisible.

If $\Sigma(M) = M$, then M is an interval bundle over a closed surface with negative Euler characteristic and the proof is completed by the following lemma, whose full statement will be used later in the paper.

Lemma 7.3. *Let M be a compact hyperbolizable 3-manifold with no toroidal boundary components. Let Σ_i be an interval bundle component of $\Sigma(M)$ which is not tiny, then there is a primitive essential annulus (for M) contained in Σ_i.*

Proof. Let F_i be the base surface of Σ_i. Since Σ_i is not tiny, F_i contains a non-peripheral simple closed curve a which is two-sided and homotopically non-trivial. Then a is an indivisible curve in F_i and hence in M. The sub-interval bundle A over a is thus a primitive essential annulus. □

□

We are now prepared to prove the main result of the section.

Proposition 7.4. *If M is a compact hyperbolizable 3-manifold with compressible boundary and no toroidal boundary components, then M contains a primitive essential annulus.*

Proof. We first observe that under our assumptions every maximal abelian subgroup of $\pi_1(M)$ is cyclic (since every non-cyclic abelian subgroup of the fundamental group of a compact hyperbolizable 3-manifold is conjugate into the fundamental group of a toroidal component of ∂M, see [44, Corollary 6.10]). Therefore, in our case an essential annulus is primitive if and only if its core curve is indivisible.

We first suppose that M is a compression body. If M is a handlebody, then it is an interval bundle, so contains a primitive essential annulus by Lemma 7.3. Otherwise, M is formed from $R \times I$ by appending 1-handles to $R \times \{1\}$, where R is a closed, but not necessarily connected, orientable surface. Let α be an essential simple closed curve in $R \times \{1\}$ which lies in ∂M. Let D be a disk in $R \times \{1\} - \partial M$. We may assume that α intersects ∂D in exactly one point. Let $\beta \subset (\partial M \cap R \times \{1\})$ be a simple closed curve homotopic to $\alpha * \partial D$ (in ∂M) and disjoint from α. Then α and β bound an embedded annulus in $R \times \{1\}$, which may be homotoped to a primitive essential annulus in M (by pushing the interior of the annulus into the interior of $R \times I$).

If M is not a compression body, let C_M be a characteristic compression body neighborhood of ∂M (as discussed in Section 2). Let C be a component of C_M which has a compressible boundary component $\partial_+ C$ and an incompressible boundary component F. Let X be the component of $\overline{M - C_M}$ which contains F in its boundary and let α be an essential simple closed curve in F which is indivisible in X (which exists by Lemma 7.1). Let α' be a curve in $\partial_+ C \subset \partial M$ which is homotopic to α. One may then construct as above a primitive essential annulus A in C with α' as one boundary component. It is clear that A remains essential in M. Since $\pi_1(M) = \pi_1(X) * H$ for some group H, the core curve of A, which is homotopic to α, is indivisible in $\pi_1(M)$. Therefore, A is our desired primitive essential annulus in M. □

Remark: The above argument is easily extended to the case where M is allowed to have toroidal boundary components (but is still hyperbolizable), unless M is a compression body all of whose boundary components are tori. In fact, the only counterexamples in this situation occur when M is obtained from one or two untwisted interval bundles over tori by attaching exactly one 1-handle.

We have thus already established Corollary 1.4 in the case that M has compressible boundary.

Corollary 7.5. *If M is a compact hyperbolizable 3-manifold with compressible boundary, no toroidal boundary components, and non-abelian fundamental group, then $\mathrm{Out}(\pi_1(M))$ does not act properly discontinuously on $AH(M)$. Moreover, the moduli space $AI(M)$ is not Hausdorff.*

8. The space $AH_n(M)$

In this section, we assume that M has incompressible boundary and no toroidal boundary components. We identify a subset $AH_n(M)$ of $AH(M)$ which contains all purely hyperbolic representations in $AH(M)$. We will see later that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH_n(M)$ in $X(M)$ if M is not an interval bundle.

We define $AH_n(M)$ to be the set of (conjugacy classes of) representations $\rho \in AH(M)$ such that

(1) If Σ_i is a component of the characteristic submanifold which is not a tiny interval bundle, then $\rho(\pi_1(\Sigma_i))$ is purely hyperbolic (i.e. if g is a non-trivial element of $\pi_1(M)$ which is conjugate into $\pi_1(\Sigma_i)$, then $\rho(g)$ is hyperbolic), and
(2) if Σ_i is a tiny interval bundle, then $\rho(\pi_1(Fr(\Sigma_i)))$ is purely hyperbolic.

We observe that $\mathrm{int}(AH(M))$ is a proper subset of $AH_n(M)$ and that $AH(M) = AH_n(M)$ if and only if M contains no primitive essential annuli.

Lemma 8.1. *Let M be a compact hyperbolizable 3-manifold with non-empty incompressible boundary and no toroidal boundary components. Then*

(1) *the interior of $AH(M)$ is a proper subset of $AH_n(M)$,*
(2) *$AH_n(M)$ contains a dense subset of $\partial AH(M)$, and*
(3) *$AH_n(M) = AH(M)$ if and only if M contains no primitive essential annuli.*

Proof. Sullivan [50] proved that all representations in $\mathrm{int}(AH(M))$ are purely hyperbolic (if M has no toroidal boundary components), so clearly $\mathrm{int}(AH(M))$ is contained in $AH_n(M)$. On the other hand, $\partial AH(M)$ is non-empty (see Lemma 4.1 in Canary-Hersonsky [16]) and purely hyperbolic representations are dense in $\partial AH(M)$ (which follows from Lemma 4.2 in [16] and the Density Theorem [9, 10, 45, 47]). This establishes claims (1) and (2).

If M contains a primitive essential annulus A, then there exist $\rho \in AH(M)$ such that $\rho(\alpha)$ is parabolic (where α is the core curve of A), so $AH_n(M)$ is not all of $AH(M)$ in this case (see Ohshika [46]).

Now suppose that M contains no primitive essential annuli. We first note that every component of $\Sigma(M)$ is a solid torus or tiny interval bundle (by Lemma 7.3). Moreover, if Σ_i is a tiny interval bundle component of $\Sigma(M)$, then any component A of its frontier must be isotopic to a component of the frontier of a solid torus component of $\Sigma(M)$. Otherwise, A would be a

primitive essential annulus (by Lemma 7.2). Therefore, it suffices to prove that $\rho(\Sigma_i)$ is purely hyperbolic whenever Σ_i is a solid torus component of $\Sigma(M)$.

Let T be a solid torus component of $\Sigma(M)$. A frontier annulus A of T is an essential annulus in M, so it must not be primitive. It follows that the core curve a of T is not peripheral in M (see [29, Theorem 32.1]).

Let $\rho \in AH(M)$ and let R be a relative compact core for $(N_\rho)^0_\epsilon$ (for some $\epsilon < \mu$). Let $h : M \to R$ be a homotopy equivalence in the homotopy class determined by ρ. By Johannson's Classification Theorem [29, Thm.24.2], h may be homotoped so that $h(T)$ is a component T' of $\Sigma(R)$, $h|_{Fr(T)}$ is an embedding with image $Fr(T')$ and $h|_T : (T, Fr(T)) \to (T', Fr(T'))$ is a homotopy equivalence of pairs. It follows that $h(a)$ is homotopic to the core curve of T' which is not peripheral in R.

If $\rho(a)$ were parabolic, then $h(a)$ would be homotopic into a non-compact component of $(N_\rho)_{thin(\epsilon)}$ and hence into $P = R \cap \partial(N_\rho)^0_\epsilon \subset \partial R$, so $h(a)$ would be peripheral in R. It follows that $\rho(a)$ is hyperbolic. Since a generates $\pi_1(T)$, we see that $\rho(\pi_1(T))$ is purely hyperbolic. Since T is an arbitrary solid torus component of $\Sigma(M)$, we see that $\rho \in AH_n(M)$. □

We next check that the restriction of $\rho \in AH_n(M)$ to the fundamental group of an interval bundle component of $\Sigma(M)$ (which is not tiny) is Schottky. By definition, a Schottky group is a free, geometrically finite, purely hyperbolic subgroup of $\mathrm{PSL}_2(\mathbf{C})$ (see Maskit [36] for a discussion of the equivalence of this definition with more classical definitions).

Lemma 8.2. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary with no toroidal boundary components which is not an interval bundle. If Σ_i is an interval bundle component of $\Sigma(M)$ which is not tiny and $\rho \in AH_n(M)$, then $\rho(\pi_1(\Sigma_i))$ is a Schottky group.*

Proof. By definition $\rho(\pi_1(\Sigma_i))$ is purely hyperbolic, so it suffices to prove it is free and geometrically finite. Since Σ_i is an interval bundle whose base surface F_i has non-empty boundary, $\pi_1(\Sigma_i) \cong \pi_1(F_i)$ is free. Let $\pi_i : N_i \to N_\rho$ be the cover of N_ρ associated to $\rho(\pi_1(\Sigma_i))$. Since $\pi_1(\Sigma_i)$ has infinite index in $\pi_1(M)$, $\pi_i : N_i \to N$ is a covering with infinite degree. Let R_i be a compact core for N_i. Since $\pi_1(R_i)$ is free and R_i is irreducible, R_i is a handlebody ([23, Theorem 5.2]). Therefore, $N_i = (N_i)^0_\epsilon$ has one end and π_i is infinite-to-one on this end, so the Covering Theorem (see [15]) implies that this end is geometrically finite, and hence that N_i is geometrically finite. Therefore, $\rho(\pi_1(\Sigma_i))$ is geometrically finite, completing the proof that it is a Schottky group. □

Finally, we check that if $\rho \in AH_n(M)$ and C_j is a characteristic collection of annuli, then there exists a C_j-registering subgroup whose image under ρ is Schottky.

Lemma 8.3. *Suppose that M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components and C_j is a*

characteristic collection of frontier annuli for M. If $\rho \in AH_n(M)$, then there exists a C_j-registering subgroup H of $\pi_1(M)$ such that $\rho(H)$ is a Schottky group.

Proof. We first suppose that $C_j = \{A\}$ is a frontier annulus of an interval bundle component of $\Sigma(M)$ (and that A is not properly isotopic to a frontier annulus of a solid torus component of $\Sigma(M)$) and let $x_0 \in A$. We identify $\pi_1(M)$ with $\pi_1(M, x_0)$. Let X_1 and X_2 be the (distinct) components of $M - Fr(\Sigma)$ abutting A. Notice that each X_i must have non-abelian fundamental group, since it either contains (the interior of) an interval bundle component of $\Sigma(M)$ or (the interior of) a component of $M - \Sigma(M)$ which is not a solid torus lying between an interval bundle component of $\Sigma(M)$ and a solid torus component of $\Sigma(M)$.

Let a be the core curve of A (based at x_0). By assumption, $\rho(a)$ is a hyperbolic element. Let F be a fundamental domain for the action of $< \rho(a) >$ on $\Omega(< \rho(a) >)$ which is an annulus in $\widehat{\mathbb{C}}$. Since each $\rho(\pi_1(\overline{X_i}, x_0))$ is discrete, torsion-free and non-abelian, hence non-elementary, we may choose hyperbolic elements $\gamma_i \in \rho(\pi_1(\overline{X_i}, x_0))$ whose fixed points lie in the interior of F. There exists $s > 0$ such that one may choose (round) disks $D_i^{\pm} \subset \mathrm{int}(F)$ about the fixed points of γ_i, such that $\gamma_i^s(\mathrm{int}(D_i^-)) = \widehat{\mathbb{C}} - D_i^+$, and D_1^+, D_1^-, D_2^+ and D_2^- are disjoint. Then, the Klein Combination Theorem (commonly referred to as the ping pong lemma), guarantees that $\rho(a)$, γ_1^s and γ_2^s freely generate a Schottky group, see, for example, Theorem C.2 in Maskit [37]. Then each $\rho^{-1}(\gamma_i^s)$ is represented by a curve g_i in $\overline{X_i}$ based at x_0 and a, g_1 and g_2 generate a C_j-registering subgroup H such that $\rho(H)$ is Schottky.

Now suppose that $C_j = \{A_1, \ldots, A_l\}$ is the collection of frontier annuli of a solid torus component T_j of $\Sigma(M)$. Let X_i be the component of $M - (T_j \cup C_1 \cup \ldots \cup C_m)$ abutting A_i. Pick x_0 in T_j and let a be a core curve of T_j passing through x_0. Again each X_i must have non-abelian fundamental group.

Let F be an annular fundamental domain for the action of $< \rho(a) >$ on the complement in $\widehat{\mathbb{C}}$ of the fixed points of $\rho(a)$. For each i, let $Y_i = X_i \cup A_i \cup \mathrm{int}(T_j')$ and pick a hyperbolic element γ_i in $\rho(\pi_1(Y_i, x_0))$ both of whose fixed points lie in the interior of F. (Notice that even though it could be the case that $X_i = X_k$ for $i \neq k$, we still have that $\pi_1(Y_i, x_0)$ intersects $\pi_1(Y_k, x_0)$ only in the subgroup generated by a, so these hyperbolic elements are all distinct.) Then, just as in the previous case, there exists $s > 0$ such that the elements $\{\rho(a), \gamma_1^s, \ldots, \gamma_l^s\}$ freely generate a Schottky group. Each $\rho^{-1}(\gamma_i^s)$ can be represented by a loop g_i based at x_0 which lies in Y_i and intersects A_i exactly twice. Therefore, the group H generated by $\{a, g_1, \ldots, g_l\}$ is C_j-registering and $\rho(H)$ is Schottky. □

9. Proper discontinuity on $AH_n(M)$

We are finally prepared to prove that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH_n(M)$ if M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components which is not an interval bundle.

Theorem 9.1. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components which is not an interval bundle. Then there exists an open $\mathrm{Out}(\pi_1(M))$-invariant neighborhood $W(M)$ of $AH_n(M)$ in $X(M)$ such that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$.*

Notice that Theorem 1.3 is an immediate consequence of Proposition 7.4, Lemma 8.1 and Theorem 9.1. Moreover, Theorem 1.5 is an immediate corollary of Lemma 8.1 and Theorem 9.1.

We now provide a brief outline of the section. In section 9.1 we recall Minsky's work which shows that $\mathrm{Out}(\pi_1(H_n))$ acts properly discontinuously on the open set $PS(H_n)$ of primitive-stable representations in $X(H_n)$ where H_n is the handlebody of genus g. In section 9.2, we consider the set $Z(M) \subset X(M)$ such that if $\rho \in Z(M)$ and C_j is a characteristic collection of annuli, then there exists a C_j-registering subgroup H of $\pi_1(M)$ such that $\rho|_H$ is primitive stable. We use Minsky's work to show that $K(M)$ acts properly discontinuously on $Z(M)$. In section 9.3, we consider the set $V(M)$ of all representation such that $\rho|_{\pi_1(\Sigma_i)}$ is primitive-stable whenever Σ_i is an interval bundle component of $\Sigma(M)$ which is not tiny. We show that if $\{\alpha_n\}$ is a sequence in $J(M)$ such that $\{\rho_\Sigma(\alpha_n)\}$ is a sequence of distinct elements and K is compact subset of $V(M)$, then $\{\alpha_n(K)\}$ leaves every compact set. In section 9.4, we let $W(M) = Z(M) \cap V(M)$ and combine the work in the previous sections to show that $J(M)$ acts properly discontinuously on $W(M)$. Since $J(M)$ has finite index in $\mathrm{Out}(\pi_1(M))$ (see [17]), this immediately implies Theorem 9.1. Johannson's Classification Theorem is used to show that $J(M)$ is invariant under $\mathrm{Out}(\pi_1(M))$.

9.1. **Schottky groups and primitive-stable groups.** In this section, we recall Minsky's work [43] on primitive-stable representations of the free group F_n, where $n \geq 2$. An element of F_n is called *primitive* if it is an element of a minimal free generating set for F_n. Let X be a bouquet of n circles with base point b and fix a specific identification of $\pi_1(X, b)$ with F_n. To a conjugacy class $[w]$ in F_n one can associated an infinite geodesic in X which is obtained by concatenating infinitely many copies of a cyclically reduced representative of w (here the cyclic reduction is in the generating set associated to the natural generators of $\pi_1(X, b)$). Let $\mathcal{P}$ denote the set of infinite geodesics in the universal cover $\tilde{X}$ of X which project to geodesics associated to primitive words of F_n.

Given a representation $\rho : F_n \to \mathrm{PSL}_2(\mathbf{C})$, $x \in \mathbb{H}^3$ and a lift $\tilde{b}$ of b, one obtains a unique ρ-equivariant map $\tau_{\rho,x} : \tilde{X} \to \mathbb{H}^3$ which takes $\tilde{b}$ to x and maps each edge of $\tilde{X}$ to a geodesic. A representation $\rho : F_n \to \mathrm{PSL}_2(\mathbf{C})$ is *primitive-stable* if there are constants $K, \delta > 0$ such that $\tau_{\rho,x}$ takes all the geodesics in $\mathcal{P}$ to (K, δ)-quasi-geodesics in $\mathbb{H}^3$. We let $PS(H_n)$ denote the set of (conjugacy classes) of primitive-stable representations in $X(H_n)$ where H_n is the handlebody of genus n.

We summarize the key points of Minsky's work which we use in the remainder of the section. We recall that Schottky space $\mathcal{S}_n \subset X(H_n)$ is the space of discrete faithful representations whose image is a Schottky group and that $\mathcal{S}_n$ is the interior of $AH(H_n)$.

Theorem 9.2. (Minsky [43]) *If $n \geq 2$, then*

(1) $\mathrm{Out}(F_n)$ *acts properly discontinuously on $PS(H_n)$,*
(2) *$PS(H_n)$ is an open subset of $X(H_n)$, and*
(3) *Schottky space $\mathcal{S}_n$ is a proper subset of $PS(H_n)$.*

Moreover, if K is any compact subset of $PS(H_n)$, and $\{\alpha_n\}$ is a sequence of distinct elements of $\mathrm{Out}(F_n)$*, then $\{\alpha_n(K)\}$ exits every compact subset of $X(H_n)$ (i.e. for any compact subset C of $X(H_n)$ there exists N such that if $n \geq N$, then $\alpha_n(K) \cap C = \emptyset$).*

Remark: In order to prove our main theorem it would suffice to use Schottky space $\mathcal{S}_n$ in place of $PS(H_n)$. However, the subset $W(M)$ we obtain using $PS(H_n)$ is larger than we would obtain using simply $\mathcal{S}_n$.

9.2. **Characteristic collection of annuli.** We will assume for the remainder of the section that M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components which is not an interval bundle. Main Topological Theorem 2 in Canary and McCullough [17] (which is itself an exercise in applying Johannson's theory) implies that that if M has incompressible boundary and no toroidal boundary components, then $\mathrm{Mod}(M)$ has finite index in $\mathrm{Out}(\pi_1(M))$. Therefore, applying Theorem 5.2, we see that $J(M)$ has finite index in $\mathrm{Out}(\pi_1(M))$. In particular, if M is acylindrical, then $J(M)$ is trivial and $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $X(M)$.

Let C_j be a characteristic collection of annuli in M. If H is a C_j-registering subgroup of $\pi_1(M)$, then the inclusion of H in $\pi_1(M)$ induces a natural injection $s_H : K_j \to \mathrm{Out}(H)$ such that if $\alpha \in K(M)$, then

$$r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(q_j(\alpha))$$

where $r_H(\rho) = \rho|_H$ (see Lemma 6.1). Let

$$Z_H = r_H^{-1}(PS(H))$$

where $PS(H) \subset X(H)$ is the set of (conjugacy classes of) primitive-stable representations of H. Let

$$Z(C_j) = \bigcup Z_H$$

where the union is taken over all C_j-registering subgroups H of $\pi_1(M)$.

If $\{C_1, \ldots, C_m\}$ is the set of all characteristic collections of annuli for M, then we define

$$Z(M) = \bigcap_{i=1}^{m} Z(C_j).$$

If there are no characteristic collection of annuli, then M is acylindrical and we set $Z(M) = X(M)$.

We use Lemma 8.3, Theorem 9.2, and Johannson's Classification Theorem to prove:

Lemma 9.3. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components. Then*

1. *$Z(M)$ is an $\mathrm{Out}(\pi_1(M))$-invariant open neighborhood of $AH_n(M)$ in $X(M)$, and*
2. *if $K \subset Z(M)$ is compact and $\{\alpha_n\}$ is a sequence of distinct elements of $K(M)$, then $\{\alpha_n(K)\}$ exits every compact set of $X(M)$.*

Proof. Lemma 8.3 implies that $AH_n(M) \subset Z(C_j)$ for each j, so $AH_n(M) \subset Z(H)$. Moreover, since r_H is continuous for all H, each $Z(C_j)$ is open, and hence $Z(M)$ is open.

Johannson's Classification Theorem implies that if C_j is a characteristic collection of annuli for M and $\varphi \in \mathrm{Out}(\pi_1(M))$, then there exists a homotopy equivalence $h : M \to M$ such that $h_* = \varphi$ and $h(C_j)$ is also a characteristic collection of annuli for M. Moreover, if H is a C_j-registering subgroup of $\pi_1(M)$, then $\varphi(H)$ is a $h(C_j)$-registering subgroup of $\pi_1(M)$. Therefore, $Z(M)$ is $\mathrm{Out}(\pi_1(M))$-invariant, completing the proof of claim (1).

If (2) fails to hold, then there is a compact subset K of $Z(M)$, a compact subset C of $X(M)$ and a sequence $\{\alpha_n\}$ of distinct elements of $K(M)$ such that $\alpha_n(K) \cap C$ is non-empty for all n. We may pass to a subsequence, still called $\{\alpha_n\}$, so that there exists j such that $\{q_j(\alpha_n)\}$ is a sequence of distinct elements. Since $X(M)$ is locally compact, for each $x \in K$, there exists an open neighborhood U_x of x and a C_j-registering subgroup H_x such that the closure $\bar{U}_x$ is a compact subset of Z_{H_x}. Since K is compact, there exists a finite collection of points $\{x_1, \ldots, x_r\}$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_r}$. Therefore, again passing to subsequence if necessary, there must exists x_i such that $\alpha_n(U_{x_i}) \cap C$ is non-empty for all n. Let $U' = U_{x_i}$ and $H' = H_{x_i}$. Lemma 6.1 implies that $\{s_{H'}(q_j(\alpha_n))\}$ is a sequence of distinct elements of $\mathrm{Out}(H')$ and that $s_{H'}(q_j(\alpha_n))(r_{H'}(\bar{U}')) = r_{H'}(\alpha_n(\bar{U}'))$. Theorem 9.2 then implies that $\{s_{H'}(q_j(\alpha_n))(r_{H'}(\bar{U}'))\} = \{r_{H'}(\alpha_n(\bar{U}'))\}$ exits every compact subset of $X(H')$. Therefore, $\{\alpha_n(U')\}$ exits every compact subset of $X(M)$ which is a contradiction. We have thus established (2). □

9.3. **Interval bundle components of $\Sigma(M)$.** Let Σ_i be an interval bundle component of $\Sigma(M)$ with base surface F_i and let $X(\Sigma_i)$ be its associated character variety. There exists a natural restriction map $r_i : X(M) \to X(\Sigma_i)$ taking ρ to $\rho|_{\pi_1(\Sigma_i)}$. Recall that $G(\Sigma_i, Fr(\Sigma_i))$ injects into $\mathrm{Out}(\pi_1(\Sigma_i))$ (by

Lemma 5.2), so acts effectively on $X(\Sigma_i)$. Moreover, if $\alpha \in J(M)$, then $r_i(\rho \circ \alpha) = r_i(\rho) \circ p_i(\alpha)$ where p_i is the projection of $J(M)$ onto $G(\Sigma_i, Fr(\Sigma_i))$. If Σ_i is not tiny, we define

$$V(\Sigma_i) = r_i^{-1}(PS(\Sigma_i)).$$

If $\{\Sigma_1, \ldots, \Sigma_n\}$ denotes the collection of all interval bundle components of $\Sigma(M)$ which are not tiny, then we let

$$V(M) = \bigcap_{i=1}^{n} V(\Sigma_i).$$

If every interval bundle component of $\Sigma(M)$ is tiny, then we let $V(M) = X(M)$.

We use Lemma 8.2, Theorem 9.2, and Johannson's Classification Theorem to prove:

Lemma 9.4. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components which is not an interval bundle. Then*

1. *$V(M)$ is an $\mathrm{Out}(\pi_1(M))$-invariant open neighborhood of $AH_n(M)$ in $X(M)$, and*
2. *if K is a compact subset of $V(M)$ and $\{\alpha_n\}$ is a sequence in $J(M)$ such that $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma, Fr(\Sigma))$, then $\{\alpha_n(K)\}$ exits every compact subset of $X(M)$.*

Proof. Lemma 8.2 implies that $AH_n(M) \subset V(\Sigma_i)$, for each i, and each $V(\Sigma_i)$ is open since r_i is continuous. Therefore, $V(M)$ is an open neighborhood of $AH_n(M)$.

Johannson's Classification Theorem implies that if $\varphi \in \mathrm{Out}(\pi_1(M))$, then there exists a homotopy equivalence $h : M \to M$ such that $h(\Sigma(M)) \subset \Sigma(M)$, $h|_{Fr(\Sigma)}$ is a self-homeomorphism of $Fr(\Sigma)$ and h induces φ. Therefore, if Σ_i is an interval bundle component of $\Sigma(M)$, then $\varphi(\pi_1(\Sigma_i))$ is conjugate to $\pi_1(\Sigma_j)$ where Σ_j is also an interval bundle component of $\Sigma(M)$. Moreover, if Σ_i is not tiny, then $\pi_1(\Sigma_j)$ is also not tiny (since $h|_{\Sigma_i} : \Sigma_i \to \Sigma_j$ is a homotopy equivalence which is a homeomorphism on the frontier). It follows that $V(M)$ is invariant under $\mathrm{Out}(\pi_1(M))$, completing the proof of claim (1).

If (2) fails to hold, then there is a compact subset K of $Z(M)$, a compact subset C of $X(M)$ and a sequence $\{\alpha_n\}$ of elements of $J(M)$ such that $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma, Fr(\Sigma))$ and $\alpha_n(K) \cap C$ is non-empty for all n. If a component Σ_i of $\Sigma(M)$ is a tiny interval bundle or a solid torus, then $G(\Sigma_i, Fr(\Sigma_i))$ is finite, by Lemma 5.2. So, we may pass to a subsequence, so that there exists an interval bundle Σ_i which is not tiny such that $\{p_i(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma_i, Fr(\Sigma_i))$. Theorem 9.2 then implies that $\{p_i(\alpha_n)(r_i(K))\}$ leaves every compact subset of $X(\Sigma_i)$. Therefore, since $r_i(\alpha_n(K)) = p_i(\alpha_n)(r_i(K))$ for all n, $\{\alpha_n(K)\}$ leaves every compact subset of $X(M)$. This contradiction establishes claim (2). □

9.4. **Assembly.** Let $W(M) = V(M) \cap Z(M)$. Since $V(M)$ and $Z(M)$ are open $\mathrm{Out}(\pi_1(M))$-invariant neighborhoods of $AH_n(M)$, so is $W(M)$. It remains to prove that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on $W(M)$. Since $J(M)$ is a finite index subgroup of $\mathrm{Out}(\pi_1(M))$, it suffices to prove that $J(M)$ acts properly discontinuously on $W(M)$. We will actually establish the following stronger fact, which will complete the proof of Theorem 9.1.

Lemma 9.5. *If K is a compact subset of $W(M)$ and $\{\alpha_n\}$ is a sequence of distinct elements of $J(M)$, then $\{\alpha_n(K)\}$ leaves every compact subset of $X(M)$.*

Proof. If our claim fails, then there exists a compact subset K of $W(M)$, a compact subset C of $X(M)$ and a sequence $\{\alpha_n\}$ of distinct elements of $J(M)$ such that $\alpha_n(K) \cap C$ is non-empty. We may pass to an infinite subsequence, still called $\{\alpha_n\}$, such that either $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements or $\{\rho_\Sigma(\alpha_n)\}$ is constant.

If $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements, Lemma 9.4 immediately implies that $\{\alpha_n(K)\}$ leaves every compact subset of $X(M)$ and we obtain the desired contradiction.

If $\{\rho_\Sigma(\alpha_n)\}$ is constant, then, by Theorem 5.2, there exists a sequence $\{\beta_n\}$ of distinct elements of $K(M)$ such that $\alpha_n = \alpha_1 \circ \beta_n$ for all n. Lemma 9.3 implies that $\{\beta_n(K)\}$ exits every compact subset of $X(M)$. Since α_1 induces a homeomorphism of $X(M)$, it follows that $\{\alpha_n(K) = \alpha_1(\beta_n(K))\}$ also leaves every compact subset of $X(M)$. This contradiction completes the proof. □

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