The conformal boundary and the boundary of the convex core

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1 Introduction

In this note we investigate the relationship between the conformal boundary at infinity of a hyperbolic 3-manifold and the boundary of its convex core. In particular, we prove that the length of a curve in the conformal boundary gives an upper bound on the length of the corresponding curve in the boundary of the convex core.

A hyperbolic 3-manifold N is the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a group Γ of isometries. The "boundary at infinity" of $\mathbf{H}^3$ may be identified with the Riemann sphere $\widehat{\mathbf{C}}$ and Γ extends to act on $\widehat{\mathbf{C}}$ as a group of conformal automorphisms. The domain of discontinuity $\Omega(\Gamma)$ of Γ is the largest Γ-invariant open subset of $\widehat{\mathbf{C}}$ which Γ acts on properly discontinuously. If Γ is not abelian, then $\Omega(\Gamma)$ inherits a hyperbolic metric, called the Poincaré metric, which Γ acts on as a group of isometries. One may then consider $\partial_c N = \Omega(\Gamma)/\Gamma$ to be the "conformal boundary at infinity" of the hyperbolic 3-manifold N. The conformal boundary is the topological boundary of $\bar{N} = (\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$.

The first important result concerning the relationship between the geometry of the conformal boundary and the geometry of N is due to Bers [4] who proved that if each component of $\Omega(\Gamma)$ is simply connected and α is a closed curve in its conformal boundary $\partial_c N$, then the geodesic α^* in N in the homotopy class of α has length $l_N(\alpha^*)$ at most twice the length $l_{\partial_c N}(\alpha)$ of α in the conformal boundary. (If α is homotopic to arbitrarily short closed curves in N, then we say $l_N(\alpha^*) = 0$.) Canary [8] generalized this to the setting of arbitrary Kleinian groups, proving that given any $\epsilon > 0$ there exists $K > 0$ such that if $\Omega(\Gamma)$ has injectivity radius bounded below by ϵ (at each point), then if α is a closed curve in $\partial_c(N)$, then

$$l_N(\alpha^*) \leq K l_{\partial_c N}(\alpha).$$

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(If Γ is finitely generated, then there always exists some lower bound for the injectivity radius of $\Omega(\Gamma)$.) Sugawa [15] proved that, with no assumptions on $\Omega(\Gamma)$, if α is a closed curve in $\partial_c N$ of length L, then

$$l_N(\alpha^*) \leq 2Le^{\frac{L}{2}}.$$

The convex core $C(N)$ of a hyperbolic 3-manifold is the smallest convex submanifold such that the inclusion of $C(N)$ into N is a homotopy equivalence. The convex core is homeomorphic to $\bar{N}$ and $N - C(N)$ is homeomorphic to $\partial_c N \times (0, \infty)$. The intrinsic metric on the boundary $\partial C(N)$ of the convex core is hyperbolic and the nearest point retraction $r : \partial_c N \to \partial C(N)$ gives a proper homotopy equivalence.

Sullivan was the first to extensively study the relationship between the geometry of the conformal boundary and the geometry of the boundary of the convex core. He showed that there exists a constant $K > 0$ such that if each component of $\partial_c N$ is incompressible in $\bar{N}$ (equivalently, if each component of $\Omega(\Gamma)$ is simply connected), then the nearest point retraction is homotopic to a K-biLipschitz map (see Epstein-Marden [11] for details.) As a corollary, one obtains linear upper bounds on the length of a curve in the boundary of the convex core in terms of the length of the corresponding curve on the conformal boundary.

The main result of this note asserts that even when the conformal boundary is not incompressible, a bound on the length in the boundary of the conformal boundary implies a bound on the length of the curve in the boundary of the convex core. Our result may be thought of as a natural generalization or analogue of Sullivan's result in the spirit of the earlier work of Sugawa [15]. The main tool in the proofs of our results will be the characterization of the Poincaré metric due to Beardon and Pommerenke [3].

Main Theorem: *Suppose that N is a hyperbolic 3-manifold and $r : \partial_c N \to \partial C(N)$ is the nearest point retraction from its conformal boundary to the boundary of its convex core. If α is a closed curve in the conformal boundary of length L, then*

$$l_{\partial C(N)}(r(\alpha)^*) < \ 45L\ e^{\frac{L}{2}}$$

where $l_{\partial C(N)}(r(\alpha)^*)$ denotes the length of the closed geodesic in the intrinsic metric on $\partial C(N)$ in the homotopy class of $r(\alpha)$.

We will give a more precise statement in section 5. In particular, we will see that short compressible curves in the conformal boundary have "much shorter" representatives in the boundary of the convex core. In a final section we will exhibit a sequence of closed curves α_n in the conformal boundaries of hyperbolic 3-manifolds N_n such that $L_n = l_{\partial_c N}(\alpha_n)$ goes to infinity, but

$$l_{\partial C(N_n)}(r_n(\alpha_n)^*) \geq e^{\frac{L_n}{2}}$$

where $r_n : \partial_c N_n \to \partial C(N_n)$ is the nearest point retraction. These examples illustrate the necessity of the exponential term in our estimate.

Our main theorem was developed as a tool for use in our work [9], in collaboration with Culler, Hersonsky and Shalen, on approximation by maximal cusps in boundaries of quasiconformal deformation spaces of Kleinian groups. In that paper, we will be most interested in applying our main theorem in the case that γ is compressible. Recall that when γ is a compressible curve, then it has no geodesic representative but is homotopic to arbitrarily short curves, so one says that $l_N(\gamma^*) = 0$. In this case, the earlier results yield no information.

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2 The Key Lemma

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, which we regard as the group of orientation-preserving isometries of hyperbolic 3-space $\mathbf{H}^3$. We will assume throughout this paper that Γ is non-abelian and torsion-free, in which case $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold. The *limit set* L_Γ of Γ is the smallest, closed non-empty, Γ-invariant subset of $\widehat{\mathbf{C}}$, and its complement is the *domain of discontinuity* $\Omega(\Gamma)$. For simplicity we will always assume that $\infty \in L_\Gamma$. The domain of discontinuity inherits a unique Riemannian metric of constant curvature -1 which is conformal to the Euclidean metric, called the *Poincaré metric*, which we denote by $\rho(z)dz$.

The *convex core* $C(N)$ of N is the quotient of the convex hull $CH(L_\Gamma)$ of the limit set by Γ. One may define the nearest point retraction $\tilde{r} : \mathbf{H}^3 \to CH(L_\Gamma)$ to be the map which takes a point in $\mathbf{H}^3$ to the closest point in $CH(L_\Gamma)$. The nearest point retraction extends continuously to a map $\tilde{r} : \mathbf{H}^3 \cup \Omega(\Gamma) \to CH(L_\Gamma)$ which is Γ-equivariant. Thus $\tilde{r}$ descends to a map $r : \bar{N} \to C(N)$ which we also call the nearest point retraction. (See [10] and [11] for more details on the nearest point retracion and the convex core.)

The following elementary lemma encodes the key observation underlying all the other results. We recall that a closed curve γ which is homotopically non-trivial in $\partial_c N$ is said to be *compressible* if it is homotopically trivial in $\bar{N} = \partial_c N \cup N$. Otherwise, it is said to be *incompressible.* If γ is compressible, then it lifts to a closed curve $\tilde{\gamma}$ in $\Omega(\Gamma)$. If γ is incompressible, we think of it as the image of a map $f : S^1 \to \partial_c N$. If $p : \mathbf{R} \to S^1$ is the universal covering map, then $p \circ f$ lifts to a map $\tilde{f} : \mathbf{R} \to \Omega(\Gamma)$. In an abuse of notation, we will refer to the image of $\tilde{f}$ as a *lift* of γ. Any lift $\tilde{\gamma}$ of γ is an (open) arc with endpoints at the fixed points of an element of Γ which stabilizes $\tilde{\gamma}$.

Lemma 2.1 *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold. Let γ be a closed curve in its conformal boundary $\partial_c N$ and let $\tilde{\gamma}$ be a lift of γ in $\Omega(\Gamma)$. Suppose that $\tilde{\gamma}$ has an endpoint at ∞ if γ is incompressible. If $\rho(z) \geq \frac{1}{Kd(z,L_\Gamma)}$ for all $z \in \tilde{\gamma}$, then*

$$l_{\partial C(N)}(r(\gamma)^*) \leq \sqrt{2}K l_{\partial_c N}(\gamma)$$

where $r(\gamma)^*$ is the geodesic representative of $r(\gamma)$ in the intrinsic metric on $\partial C(N)$.

Proof of 2.1: Let $s : [0,a] \to \Omega(\Gamma)$ be a parameterization by Euclidean arc length of a fundamental domain for $\tilde{\gamma}$. Let $g \in \Gamma$ be the group element such that $g(s(0)) = s(a)$. (If γ is compressible, then g is the trivial element.) By our assumption, if g is non-trivial, g has a fixed point at ∞. Therefore, $g(z) = \lambda z + \mu$ for some $\lambda, \mu \in \mathbf{C}$. Since L_Γ is Γ-invariant, $\rho(g(z)) = \frac{\rho(z)}{|\lambda|}$ and $d(g(z), L_\Gamma) = |\lambda| d(z, L_\Gamma)$ for all $z \in \Omega(\Gamma)$.

Let $\widehat{s} : [0,a] \to \mathbf{H}^3$ be the curve given by $\widehat{s}(t) = (s(t), d(s(t), L_\Gamma))$. Since $CH(L_\Gamma)$ does not intersect the open ball of Euclidean radius $d(z, L_\Gamma)$ about z, $\widehat{s}([0,a]) \subset \overline{\mathbf{H}^3 - CH(L_\Gamma)}$. By our choice of normalization, $g(\widehat{s}(0)) = \widehat{s}(a)$ so that $\widehat{s}([0,a])$ projects to a closed curve $\widehat{\gamma}$ in N. Moreover, the vertical region between $s([0,a])$ and $\widehat{s}([0,a])$ projects to an annulus A in $\bar{N} - C(N)$ joining γ and $\widehat{\gamma}$. Therefore, $r(\gamma)$ and $r(\widehat{\gamma})$ are homotopic within $\partial C(N)$ (by the homotopy provided by $r(A)$.) Since r is distance non-increasing on N, $l_{\partial C(N)}(r(\gamma)^*) \leq l_N(\widehat{\gamma})$. It now suffices to simply estimate the length of $\widehat{\gamma}$.

Recall that

$$l(\gamma) = \int_0^a \rho(s(t))dt$$

and that

$$l(\widehat{\gamma}) = \int_0^a \frac{1}{d(s(t), L_\Gamma)} |\widehat{s}'(t)| dt$$

where $|\widehat{s}'(t)|$ is simply the norm of the (Euclidean) derivative of $\widehat{s}$. But since $d(z, L_\Gamma)$ is 1-Lipschitz and s is parameterized by Euclidean arc length, $|\widehat{s}'(t)| \leq \sqrt{2}$. Applying our assumption that $\rho(z) \geq \frac{1}{Kd(z,L_\Gamma)}$ for all $z \in \tilde{\gamma}$ we see that

$$l(\widehat{\gamma}) \leq \int_0^a \sqrt{2} K \rho(s(t))dt = \sqrt{2} K l(\gamma),$$

which completes the proof.

$\boxed{2.1}$

3 The Poincaré metric and conformal modulus

In order to verify that our assumption on the Poincaré metric is satisfied we make use of a characterization of the Poincaré metric which is due to Beardon and Pommerenke [3]. They introduce a quantity

$$\beta(z) = inf\left\{\left|log\frac{|z-a|}{|b-a|}\right| : a \in \partial\Omega, b \in \partial\Omega, |z-a| = d(z,\partial\Omega)\right\}.$$

In the proof of Corollary 3.3, we will observe that z lies on the core curve of an Euclidean annulus in $\Omega(\Gamma)$ of modulus $\frac{\beta(z)}{\pi}$. In fact, one could alternatively have defined $\beta(z)$ as π times the minimal modulus of an Euclidean annulus in $\Omega(\Gamma)$ which has z on its core curve, is centered at a point a such that $|z-a| = d(z, L_\Gamma)$ and whose closure intersects L_Γ.

We recall that an annulus A is *Euclidean* if it is bounded by two concentric circle. The *core curve* of A is the unique curve invariant by a conformal involution which interchanges the boundary components. The conformal *modulus* of a Euclidean annulus A bounded by circles of radius $r_2 > r_1$ is defined to be

$$\mathrm{mod}(A) = \frac{1}{2\pi}\log\left(\frac{r_2}{r_1}\right).$$

Any (open) topological annulus is conformally equivalent to a Euclidean annulus and we define its modulus to be equal to the modulus of the conformally equivalent Euclidean annulus.

Theorem 3.1 *Let Ω be a hyperbolic subdomain of $\mathbf{C}$ and $\rho(z)dz$ be its Poincaré metric. Then,*

$$\frac{1}{\sqrt{2}} \le \rho(z)d(z,\partial\Omega)[k+\beta(z)] \le 2k+\frac{\pi}{2}$$

where $k = 4 + log(3+2\sqrt{2}) \le 5.763$.

The core curve α of a Euclidean annulus A with large modulus is short. Moreover, if we let $r_A : A \to CH(\partial A)$ denote the nearest point retraction from A to the convex hull of its boundary, $r_A(\alpha)$ is much shorter. Theorem 2.16.1 in Epstein-Marden [11] makes these observations more precise.

Theorem 3.2 (Theorem 2.16.1 in [11]) *Let A be a Euclidean annulus of modulus* $\mathrm{mod}(A)$. *If α is the core curve of A and r_A is the nearest point retraction from A to the boundary of the convex core of ∂A, then*

$$l_A(\alpha) = \frac{\pi}{\mathrm{mod}(A)},$$

where l_A is measured in the Poincaré metric on A, and

$$l_{\partial CH(\partial A)}(r_A(\alpha)) = \frac{2\pi e^{\pi \mathrm{mod}(A)}}{e^{2\pi \mathrm{mod}(A)} - 1}.$$

As a corollary, we see that if $\beta(z)$ is large, then there is a short homotopically non-trivial curve in $\Omega(\Gamma)$ which passes through z. It is this corollary, in combination with the Margulis lemma, that we will use to bound $\beta(z)$ in our applications.

Corollary 3.3 *Let Ω be a hyperbolic subdomain of* $\mathbf{C}$. *If*

$$\beta(z) \geq M$$

then there is a homotopically non-trivial curve through z whose length in the Poincaré metric on Ω is at most $\frac{\pi^2}{M}$.

Proof of 3.3: By definition, there exists $a \in \partial\Omega$ such that $|z - a| = d(z, \partial\Omega)$ and the annulus

$$A = \{w|\ e^M|z - a| > |w - a| > e^{-M}|z - a|\}$$

is entirely contained within Ω. Let

$$\alpha = \{w|\ |w - a| = |z - a|\}$$

be the core curve of this annulus. The modulus mod(A) $= \frac{M}{\pi}$. Theorem 3.2 then implies that α has length $\frac{\pi^2}{M}$ in the Poincaré metric $\rho_A(z)dz$ on A. But $\rho_A(z) \geq \rho_\Omega(z)$ for all $z \in A$ where $\rho_\Omega(z)dz$ is the Poincaré metric on Ω. Therefore, α has length at most $\frac{\pi^2}{M}$ in Ω.

$\boxed{3.3}$

Remarks: (1) One may also use Theorem 3.2 to show that $r_\Omega(\alpha)^*$ has length at most $\frac{2\pi e^M}{e^{2M}-1}$ in $\partial CH(\partial\Omega)$.
(2) Corollary 3.3 is an improvement on Lemma 2.3 in [8].

4 The case where the geometry of $\Omega(\Gamma)$ is bounded

It is particularly simple to consider the case where there is a uniform lower bound on the injectivity radius in the domain of discontinuity. We recall that if Γ is finitely generated, then Ahlfors' finiteness theorem [1] implies that every homotopically non-trivial closed curve in $\Omega(\Gamma)$ is homotopic to a closed geodesic and that there exists some $\epsilon > 0$ such that no closed geodesic in $\Omega(\Gamma)$ has length less than ϵ.

Proposition 4.1 *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold. Suppose that $\Omega(\Gamma)$ contains no homotopically non-trival closed curves of length less than ϵ. If α is a closed curve in $\partial_c N$, then*

$$l_{\partial C(N)}(r(\alpha)^*) \le D l_{\partial_c N}(\alpha).$$

where

$$D = 2\left(k + \frac{\pi^2}{\epsilon}\right).$$

Proof of 4.1: We may assume that if α is incompressible, then one of its lifts $\tilde{\alpha}$ has an endpoint at ∞.

If $\Omega(\Gamma)$ contains no homotopically non-trivial closed curves of length less than ϵ, then Corollary 3.3 implies that $\beta(z) \le \frac{\pi^2}{\epsilon}$ for all $z \in \Omega(\Gamma)$. Therefore, $\rho(z) \ge \frac{1}{Kd(z,\partial\Omega)}$ where $K = \sqrt{2}\left(k + \frac{\pi^2}{\epsilon}\right)$. Lemma 2.1 then implies that

$$l_{\partial C(N)}(r(\alpha)^*) \le 2\left(k + \frac{\pi^2}{\epsilon}\right) l_{\partial_c N}(\alpha)$$

as claimed.

$\boxed{4.1}$

Thurston [16] has shown that if $f : X \to Y$ is a proper homotopy equivalence between two finite area hyperbolic surfaces and

$$\frac{l_Y(f(\gamma)^*)}{l_X(\gamma)} \le D$$

for all simple closed geodesics γ in X, then f is homotopic to a D-Lipschitz map. Recall that a hyperbolic 3-manifold is called *analytically finite* if its conformal boundary has finite area. Ahlfors' finiteness theorem [1] asserts that if a hyperbolic 3-manifold has finitely generated fundamental group, then it is analytically finite. Combining Thurston's result with Proposition 4.1 we obtain the following

Corollary 4.2 *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold. Suppose that $\Omega(\Gamma)$ contains no homotopically non-trival closed curves of length less than ϵ. Then $r : \partial_c N \to \partial C(N)$ is homotopic to a D-Lipschitz map, where*

$$D = 2\left(k + \frac{\pi^2}{\epsilon}\right).$$

As another corollary we obtain a new proof of Theorem 2.1 from [8] with a somewhat better estimate.

Corollary 4.3 *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold. Suppose that $\Omega(\Gamma)$ contains no homotopically non-trival closed curves of length less than ϵ. If γ is a closed curve in $\Omega(\Gamma)$, then*

$$l_N(\gamma^*) \le D l_{\partial_c N}(\gamma)$$

If every component of $\Omega(\Gamma)$ is simply connected, then $\rho(z) \ge \frac{1}{2d(z,L_\Gamma)}$ for all $z \in L_\Gamma$ (see [3] for example). In this case, we may argue just as in the proof of Proposition 4.1 to show that if α is a closed curve in the conformal boundary of N, then

$$l_{\partial C(N)}(r(\alpha)^*) \le 2\sqrt{2} l_{\partial_c N}(\alpha).$$

We may again apply Thurston's result to conclude:

Corollary 4.4 *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold. If each component of $\Omega(\Gamma)$ is simply connected, then $r : \partial_c N \to \partial C(N)$ is homotopic to a $2\sqrt{2}$-Lipschitz map.*

In section 2.3 of Epstein-Marden [11] they show that if each component of $\Omega(\Gamma)$ is simply connected, then r is 4-Lipschitz and that r is homotopic to a 66.3-biLipschitz map. Bridgeman [6] has shown that r has a homotopy inverse which is 6.8-Lipschitz. It is conjectured that r is homotopic to a 2-biLipschitz map.

One may combine the techniques of section 2.3 in Epstein-Marden [11] with the observations in this section to show directly that, with the same assumptions as in Corollary 4.2, the nearest point retraction is itself $\sqrt{2}D$-Lipschitz. We have chosen to keep our discussion more elementary, so we will not pursue this.

5 The general case

However, one often does not know, a priori, any uniform bounds on the geometry of $\Omega(\Gamma)$, so the results of the last section will not suffice in general. Our main result gives uniform bounds on the lengths of curves in the convex core boundary which are independent of the geometry of $\Omega(\Gamma)$. We will say that a closed curve is *primitive* if it is not homotopic to a non-trivial power of any other closed curve.

Theorem 5.1 *Let N be a hyperbolic 3-manifold and let γ be a (primitive) closed geodesic of length L in $\partial_c N$.*

1. *If $L \ge 1$, then*

$$l_{\partial C(N)}(r(\gamma)^*) \le C_1 L e^{\frac{L}{2}}$$

where $C_1 = 2\left(k + \pi^2\right) \le 31.265$.

2. *If γ is incompressible in $\bar{N}$ and $L \leq 1$, then*

$$l_{\partial C(N)}(r(\gamma)^*) \leq C_2 L$$

where $C_2 = 2(k + \sqrt{e}\pi^2) \leq 44.071$.

3. *If γ is compressible and $L \leq 1$, then*

$$l_{\partial C(N)}(r(\gamma)^*) \leq \frac{4\pi e^{(.502)\pi}}{e^{\frac{\pi^2}{\sqrt{e}L}}} \leq C_3 L$$

where $C_3 = \frac{4\pi e^{(.502)\pi}}{e^{\frac{\pi^2}{\sqrt{e}}}} \leq .153$.

Proof of 5.1: In order to be able to apply Lemma 2.1 we will always assume that if γ is incompressible, then one of its lifts $\tilde{\gamma}$ has an endpoint at ∞.

We will make repeated use of the following quantitative form of the Margulis lemma which appears as Theorem 8.3.1 in Beardon [2].

Theorem 5.2 *Suppose that α and γ are two (primitive) homotopically non-trivial closed curves passing through a point x on a complete hyperbolic surface S. If α and γ are not homotopic, then*

$$\sinh\left(\frac{l_S(\alpha)}{2}\right)\sinh\left(\frac{l_S(\gamma)}{2}\right) \geq 1.$$

Therefore, if α is any closed curve in $\partial_c N$ passing through a point $x \in \gamma$ which is not homotopic to a multiple of γ, then

$$\sinh\left(\frac{l_{\partial_c N}(\alpha)}{2}\right)\sinh\left(\frac{L}{2}\right) \geq 1.$$

In particular,

$$\mathrm{inj}_{\partial_c N}(x) \geq \min\left\{\frac{L}{2}, \sinh^{-1}\left(\frac{1}{\sinh(\frac{L}{2})}\right)\right\}$$

where $\mathrm{inj}_{\partial_c N}(x)$ denotes the injectivity radius of $\partial_c N$ at the point x.

It is easily checked that if $L \geq 1$, then

$$\sinh^{-1}\left(\frac{1}{\sinh(\frac{L}{2})}\right) \geq \frac{1}{2e^{\frac{L}{2}}}.$$

So, if $L \geq 1$ and $z \in \tilde{\gamma} \subset \Omega(\Gamma)$, then

$$\mathrm{inj}_{\Omega(\Gamma)}(z) \geq \frac{1}{2e^{\frac{L}{2}}}.$$

Applying Corollary 3.3, we see that this implies that

$$\beta(z) \leq \pi^2 e^{\frac{L}{2}}$$

for all $z \in \tilde{\gamma}$, and hence, by Theorem 3.1, that $\rho(z) \geq \frac{1}{Kd(z,L_\Gamma)}$ where

$$K = \sqrt{2}\left(k + \pi^2 e^{\frac{L}{2}}\right) \leq \sqrt{2}\left(k + \pi^2\right) e^{\frac{L}{2}}.$$

We then simply apply Lemma 2.1 to complete the proof of case 1.

In case (2), γ is incompressible and has length $L \leq 1$. Suppose that α is the shortest homotopically non-trivial curve through a point $z \in \tilde{\gamma}$. Let $p : \Omega(\Gamma) \to \partial_c N$ be the usual covering map. Since γ is incompressible, γ and $p(\alpha)$ are not homotopic. Theorem 5.2 implies that

$$\sinh\left(\frac{l_{\Omega(\Gamma)}(\alpha)}{2}\right)\sinh(.5) \geq 1.$$

Thus

$$l_{\Omega(\Gamma)}(\alpha) \geq 2\sinh^{-1}\left(\frac{1}{\sinh(.5)}\right) \geq \frac{1}{\sqrt{e}},$$

so $inj_{\Omega(\Gamma)}(z) \geq \frac{1}{2\sqrt{e}}$. Therefore, by Corollary 3.3 and Theorem 3.1, $\beta(z) \leq \pi^2\sqrt{e}$ and $\rho(z) \geq \frac{1}{Kd(z,L_\Gamma)}$, where $K = \sqrt{2}(k + \sqrt{e}\pi^2)$, for all $z \in \tilde{\gamma}$. Again, case 2 follows directly from Lemma 2.1.

We now suppose that γ is compressible and $L \leq 1$. Let $\tilde{\gamma}$ be a lift of γ. Theorem 5.2 of Sugawa [14] implies that $\tilde{\gamma}$ is homotopic to the core curve of a topological annulus $R \subset \Omega(\Gamma)$ with modulus

$$mod(R) \geq \frac{\pi}{Le^{\frac{L}{2}}} \geq \frac{\pi}{\sqrt{e}L}.$$

(See also Corollary 3 of Maskit [13] where it is shown that $\mathrm{mod}(R) \geq \frac{2}{Le^{\frac{L}{2}}}$.) We make use of a result of Herron, Liu and Minda [12], to guarantee that A contains an Euclidean annulus of modulus close to $\mathrm{mod}(R)$.

Theorem 5.3 (Corollary 3.5 of [12]) *Suppose that R is a topological annulus in $\widehat{\mathbb{C}}$ which separates 0 from ∞. If R has modulus $\mathrm{mod}(R) > .5$, then R contains a separating Euclidean annulus, centered at the origin with modulus*

$$\mathrm{mod}(A) \geq \mathrm{mod}(R) - \frac{1}{\pi}\log 2(1 + \sqrt{2}) \geq \mathrm{mod}(R) - .502.$$

Therefore, since $\mathrm{mod}(R) \geq \frac{\pi}{\sqrt{e}} > .5$, $\tilde{\gamma}$ is homotopic to the core curve γ' of a Euclidean annulus $A \subset \Omega(\Gamma)$ of modulus

$$\mathrm{mod(A)} \geq \frac{\pi}{\sqrt{e}L} - .502$$

Let $CH(\partial A)$ denote the convex core of ∂A and let $r_A : A \to \partial CH(\partial A)$ denote the nearest point retraction of A onto the boundary of the convex core of A. Theorem 3.2 implies that $\widehat{\gamma} = r_A(\gamma')$ has length

$$l_{\mathbf{H}^3}(\widehat{\gamma}) \leq \frac{2\pi e^{\pi \mathrm{mod}(A)}}{e^{2\pi \mathrm{mod}(A)} - 1} \leq \frac{4\pi}{e^{\pi \mathrm{mod}(A)}} \leq \frac{4\pi e^{(.502)\pi}}{e^{\frac{\pi^2}{\sqrt{e}L}}} \leq \frac{4\pi e^{(.502)\pi}}{e^{\frac{\pi^2}{\sqrt{e}}}}\ L.$$

But since the component of $\mathbf{H}^3 - CH(A)$ bounded by A is contained in $\mathbf{H}^3 - CH(L_\Gamma)$, $\widehat{\gamma}$ is homotopic to $\tilde{\gamma}$ within $(\mathbf{H}^3 \cup \Omega(\Gamma)) - CH(L_\Gamma)$. Therefore,

$$l_{\partial C(N)}(r(\gamma)^*) = l_{\partial CH(L_\Gamma)}(\tilde{r}(\tilde{\gamma})^*) \leq l_{\mathbf{H}^3}(\widehat{\gamma}) \leq \frac{4\pi e^{(.502)\pi}}{e^{\frac{\pi^2}{\sqrt{e}L}}}$$

which completes the proof of case (3).

$\boxed{5.1}$

In order to recover our main result from Theorem 5.1 it only remains to check that if γ is not homotopic to a closed geodesic, which implies that $l_{\partial_c N}(\gamma^*) = 0$, then $l_{\partial C(N)}(r(\gamma)^*) = 0$ as well.

Lemma 5.4 *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold. If γ is a homotopically non-trivial closed curve in $\partial_c N$ and $l_{\partial_c N}(\gamma^*) = 0$, then γ is incompressible and $l_{\partial C(N)}(r(\gamma)^*) = 0$.*

Proof of 5.4: Suppose that $l_{\partial_c N}(\gamma^*) = 0$. If γ is compressible, then $\tilde{\gamma}$ is a closed curve which is homotopic to arbitrarily short curves in $\Omega(\Gamma)$. This would imply that there is an isolated point of L_Γ, which does not occur for torsion-free nonabelian Kleinian groups. Thus, γ is incompressible.

Let $\{\gamma_n\}$ be a sequence of curves homotopic to γ such that $l_{\partial_c N}(\gamma_n) \leq \frac{1}{n}$ for all n. The arguments in case (2) of the proof of theorem 5.1, applied to γ_n, then imply that $l_{\partial C(N)}(r(\gamma)^*) \leq \frac{C_2}{n}$ for all n. Therefore, $l_{\partial C(N)}(r(\gamma)^*) = 0$.

$\boxed{5.4}$

Combining Theorem 5.1 and Lemma 5.4 we obtain the following version of our main result:

Corollary 5.5 *Let γ be a closed curve of length L in $\partial_c N$, then*

$$l_{\partial C(N)}(r(\gamma)^*) \leq C_2 L\, e^{\frac{L}{2}}.$$

where $C_2 = 2(k + \sqrt{e}\pi^2) \leq 44.071$. In particular, given $A > 0$ there exists $B > 0$, such that if γ has length less than A in $\partial_c N$, then $r(\gamma)^*$ has length less than B in $\partial C(N)$.

Since $l_N(\gamma^*) \leq l_{\partial C(N)}(r(\gamma)^*)$ we obtain a version of Sugawa's result from [15], although our constant is larger. In fact, Sugawa's result bounds the complex length of γ^*, so when L is small his result gives much more information.

Corollary 5.6 *Let γ be a closed geodesic of length L in $\partial_c N$, then*

$$l_N(\gamma^*) < 45L\, e^{\frac{L}{2}}.$$

6 Examples

In this section, we exhibit a sequence $\{\mu_n\}$ of curves in the conformal boundaries of hyperbolic manifolds N_n such that $l_{\partial_c N_n}(\mu_n)$ goes to infinity and

$$l_{\partial C(N_n)}(r_n(\mu_n)^*) \geq e^{\frac{l_{\partial_c N_n}(\mu_n)}{2}}$$

where $r_n : \partial_c N_n \to \partial C(N_n)$ is the nearest point retraction. These examples demonstrate that the exponential term is necessary in the statement of our main theorem.

Let S_n be a hyperbolic surface of genus 2 which is built from 2 pairs of pants P_1 and P_2, such that the boundary components of P_1 all have length 1 and the boundary components of P_2 have lengths 1, $\frac{1}{n}$ and $\frac{1}{n}$. We glue the two boundary components of P_2 which have length $\frac{1}{n}$ to each other in such a way that the endpoints of the unique common perpendicular joining the two boundary components are identified. Let α_n be the resulting curve of length $\frac{1}{n}$ and let μ_n be the closed geodesic obtained by identifying the endpoints of the common perpendicular. We glue the remaining boundary components together in such a way that a hyperbolic surface of genus 2 results.

Theorem 7.19.2 in Beardon [2] may be used to show that μ_n has length

$$L_n = l_{S_n}(\mu_n) = \cosh^{-1}\left(\frac{\cosh(.5) + \cosh^2(\frac{1}{2n})}{\sinh^2(\frac{1}{2n})}\right).$$

Since $\sinh(x) > x$ and $cosh^{-1}$ is increasing on $[0,\infty)$,

$$L_n \leq \cosh^{-1}\left(4n^2\left(\cosh(.5)+\cosh^2(\frac{1}{2n})\right)\right).$$

But

$$\cosh(.5)+\cosh^2(\frac{1}{2n}) < 3\ ,$$

for all n, so

$$L_n \leq \cosh^{-1}(12n^2).$$

Since $\cosh^{-1}(x) < \log(2x)$,

$$L_n \leq \log(24n^2) < 2\log(5n).$$

The quasiconformal deformation theory of Kleinian groups, see for example Bers [5], assures us that there exist Kleinian groups Γ_n such that if $N_n = \mathbf{H}^3/\Gamma_n$, then $\partial_c N_n$ is isometric to S_n, $\bar{N}_n = N_n \cup \partial_c N_n$ is homeomorphic to a handlebody of genus two and α_n is compressible in $\bar{N}_n$.

Let $\tilde{\alpha}_n$ denote the lift of α_n to $\Omega(\Gamma_n)$. Theorem 5.2 of Sugawa [14] implies that $\tilde{\alpha}_n$ is homotopic to the core curve of a separating topological annulus R_n in $\Omega(\Gamma_n)$ with modulus

$$mod(R_n) \geq \frac{\pi n}{e^{\frac{1}{2n}}} \geq 2n$$

if $n \geq 2$. Corollary 3.5 of [12] (stated above as Theorem 5.3), then guarantees that $\tilde{\alpha}_n$ is homotopic to the core curve $\tilde{\alpha}_n'$ of an Euclidean annulus $A_n \subset \Omega(\Gamma_n)$ of modulus $\mathrm{mod}(A_n) \geq 2n-1$.

Theorem 3.2 gives that $r_{A_n}(\tilde{\alpha}_n')$ has length

$$l_{\partial CH(\partial A_n)}(r_{A_n}(\tilde{\alpha}_n')) = \frac{2\pi e^{\pi(2n-1)}}{e^{2\pi(2n-1)}-1} \leq \frac{4\pi}{e^{\pi(2n-1)}}$$

if $n \geq 2$, where $r_{A_n} : A_n \to \partial CH(\partial A_n)$ is the nearest point retraction of A_n onto the convex hull of ∂A_n. Therefore, since $r_{A_n}(\tilde{\alpha}_n')$ is homotopic to α_n (in the closure of $\bar{N}_n - C(N_n)$),

$$l_{\partial C(N_n)}(r_n(\alpha_n)^*) \leq \frac{4\pi}{e^{\pi(2n-1)}}.$$

Since $r_n(\alpha_n)^*$ must intersect $r_n(\mu_n)^*$, Theorem 8.3.1 of Beardon (stated here as Theorem 5.2) gives that if

$$M_n = l_{\partial C(N_n)}(r_n(\mu_n)^*)\ ,$$

then

$$\sinh\left(\frac{M_n}{2}\right)\sinh\left(\frac{2\pi}{e^{\pi(2n-1)}}\right) \geq 1.$$

Therefore, since $\sinh(x) < 2x$ if $x \leq 2$ and $\sinh^{-1}$ is an increasing function,

$$M_n \geq 2\sinh^{-1}\left(\frac{1}{\sinh(\frac{2\pi}{e^{\pi(2n-1)}})}\right) \geq 2\sinh^{-1}\left(\frac{e^{\pi(2n-1)}}{4\pi}\right)$$

if $n \geq 2$. Since $\sinh^{-1}(x) > \log(2x)$, we see that

$$M_n \geq 2(2\pi n - \pi - \log(2\pi)) \geq 5n$$

if $n \geq 2$.

We then see that $M_n \geq e^{\frac{L_n}{2}}$ if $n \geq 2$ which says that

$$l_{\partial C(N_n)}(r_n(\mu_n)^*) \geq e^{\frac{l_{S_n}(\mu_n)}{2}}$$

as promised.

With a little more work one can also show that $l_{N_n}(\mu_n^*) > c_1 n$ for some constant c_1, which demonstrates the necessity of the exponential term in Sugawa's result from [15] as well. The basic idea is that the Margulis lemma provides an annular collar X_n on $\partial C(N_n)$ with $r(\alpha_n)^*$ as its core curve, such that its two boundary components, $\partial_1 X_n$ and $\partial_2 X_n$, have length at most 1 and $d_{X_n}(\partial_1 X_n, \partial_2 X_n) \geq b_1 n$ for some constant $b_1 > 0$ (see, for example, Theorem 4.1.1 in Buser [7].) The boundary components of X_n bound disks, D_n^1 and D_n^2 in $C(N_n)$ of diameter at most 1. Then $D_n^1 \cup D_n^2 \cup X_n$ bounds a ball B_n in $C(N_n)$ and one can show that $d_{B_n}(\partial_1 X_n, \partial_2 X_n) > c_1 n$ for some $c_1 > 0$. The geodesic μ_n^* must pass though B_n and intersect both D_n^1 and D_n^2, which guarantees that $l_{N_n}(\mu_n^*) > c_1 n$. Therefore,

$$l_{N_n}(\mu_n^*) \geq \frac{c_1}{5} e^{\frac{l_{S_n}(\mu_n)}{2}}.$$

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