A new foreword for Notes on Notes of Thurston

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The paper "Notes on Notes of Thurston" was intended as an exposition of some portions of Thurston's lecture notes *The Geometry and Topology of Three-Manifolds*. The work described in Thurston's lecture notes revolutionized the study of Kleinian groups and hyperbolic manifolds, and formed the foundation for parts of Thurston's proof of his Geometrization theorem. At the time, much of the material in these notes was unavailable in a published form. In this foreword, we hope to point the reader to some more recently published places where detailed explanations of the material in Thurston's original lecture notes are available. We will place a special emphasis on the material in Thurston's chapters 8 and 9. This material was the basis for much of our original article and it still represents the material which has been least well-digested by the mathematical community. This is also the material which has been closest to the author's subsequent interests, so the selection will, by necessity, reflect some of his personal biases.

We hope this foreword will be useful to students and working mathematicians who are attempting to come to grips with the very beautiful, but also sparingly described, mathematics in these notes. No attempt has been made to make this foreword self-contained. It is simply a rough-and-ready guide to some of the relevant literature. In particular, we will not have space to define all the mathematical terms used, but we hope the reader will make use of the many references to sort these out. In particular, we will assume that the reader has a copy of Thurston's notes on hand. We would also like to suggest that it would be valuable for a publisher to make available Thurston's lecture notes, in their original form. The author would like to apologize, in advance, to the mathematicians whose relevant articles have been omitted due to the author's ignorance.

In a final section, we describe some recent progress on the issues dealt with in sections 8 and 9 of Thurston's notes.

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1 General References

Before focusing in a more detailed manner on the material in chapters 8 and 9, we will discuss some of the more general references which have appeared since the publication of our paper. In order to conserve space, we will be especially telegraphic in this section.

Thurston [153] recently published volume I of a new version of his lecture notes under the title *Three-dimensional Geometry and Topology*. This new volume contains much of the material in chapter 1, 2, and 3 of the original book, as well as material which comes from sections 5.3 and 5.10. However, the most exciting and novel portions of his original notes have been left for future volumes. A number of other books on Kleinian groups and hyperbolic manifolds have been published in the last 15 years, including books by Apanasov [14], Benedetti and Petronio [23], Buser [50], Kapovich [83], Katok [84], Maskit [104], Matsuzaki and Taniguchi [105], Ohshika [128] and Ratcliffe [137].

There are now several complete published proofs of Thurston's Geometrization Theorem for Haken 3-manifolds available. McMullen [107] outlined a proof of of the Geometrization theorem for Haken 3-manifolds which do not fiber over the circle which used his proof of Kra's Theta conjecture. A more complete version of this approach is given by Otal [132], who also incorporates work of Barrett and Diller [15]. Kapovich [83] has recently published a book on the proof of the Geometrization theorem. His approach to the main portion of the proof is based on work of Rips (see [21]) on the actions of groups on **R**-trees. (Morgan and Shalen [120, 121, 122] first used the theory of **R**-trees to prove key portions of the Geometrization theorem. See Bestvina [20] or Paulin [133] for a more geometric viewpoint on how actions of groups on **R**-trees arise as limits of divergent sequences of discrete faithful representations.) An outline of Thurston's original proof of the main portion of the Geometrization theorem was given by Morgan in [119]. Portions of this proof are available in Thurston's article [152] and preprint [155]. Thurston's original proof develops much more structural theory of Kleinian groups than the later proofs.

Otal [131] also published a proof of the Geometrization theorem in the case where the 3-manifold fibers over the circle. Otal's proof makes use of the theory of **R**-trees, and in particular uses a deep theorem of Skora [144] which characterizes certain types of actions of surface groups on **R**-trees. (Kleineidam and Souto [90] used some of Otal's techniques to prove a spectacular generalization of Thurston's Double Limit Theorem to the setting of hyperbolic structures on compression bodies.) Thurston's original proof of the Geometrization theorem for 3-manifolds which fibre over the circle is available at [154]. (A survey of this proof is given by Sullivan [146], see also McMullen [108].)

We will now briefly indicate where one might look for details on some of the material in Thurston's notes which is not in chapters 8 and 9. The material in sections 4.1–4.7 of Thurston's notes is discussed in chapter E of Benedetti-Petronio [23]. The material in sections 4.8 and 4.9 was further developed by Epstein in [65]. The results in sections 4.10 and 4.11 were generalized in Floyd-Hatcher [73] and Hatcher-Thurston [76].

The material in section 5.1 is the subject matter of sections 1.5–1.7 of [57]. In sections 5.2, 5.5 and 5.6, Thurston develops a useful estimate for the dimension of the representation

variety, which was proven carefully by Culler and Shalen in section 3 of [62]. Thurston's Hyperbolic Dehn Surgery theorem is established in section 5.8, using the dimension count established in the previous sections and the theory developed in section 5.1. This version of the proof is discussed in Hodgson-Kerckhoff [77] and, in more detail, in Bromberg [46]. Bromberg also develops generalizations of Thurston's Hyperbolic Dehn Surgery theorem to the infinite volume setting, see also Bonahon-Otal [31] and Comar [61]. A complete proof of the Hyperbolic Dehn Surgery theorem using ideal triangulations is given by Petronio and Porti [135]. The proof of the Mostow-Prasad rigidity theorem given in section 5.9 follows the same outline as Mostow's original proof [123], see also Marden [98], Mostow [124] and Prasad [136]. In sections 5.11 and 5.12, Thurston proves Jørgensen's theorem that given a bound C, there exists a finite collection of manifolds, such that every hyperbolic 3-manifold of volume at most C is obtained from one of the manifolds in the collection by Dehn Filling, see also Chapter E in [23].

In sections 6.1–6.5, Thurston gives Gromov's proof of the Mostow-Prasad rigidity theorem and develops Gromov's theory of simplicial volume, see Gromov [74] and Chapter C of Benedetti and Petronio [23]. In section 6.6, Thurston proves that the set of volumes of hyperbolic 3-manifolds is well-ordered, again see Chapter E of Benedetti and Petronio [23]. Dunbar and Meyerhoff [64] generalized Thurston's arguments to show that the set of volumes of hyperbolic 3-orbifolds is well-ordered.

Chapter 7 of the original notes, concerning volumes of hyperbolic manifolds, was written by John Milnor and much of the work in this chapter appears in appendices to [110] and [111]. Portions of the material in the incomplete chapter 11 appear in Appendix B of McMullen [108]. Chapter 13 begins with the theory of orbifolds, see for example Scott [142] and Kapovich [83]. Scott [142] also discusses the orbifold viewpoint on Seifert fibered spaces and the geometrization of Seifert fibered spaces. The remainder of Chapter 13 concerns Andreev's theorem and its generalizations. Andreev's original work appeared in [11] and [12]. Andreev's theorem has been generalized by Rivin-Hodgson [138] and Rivin [139].

2 Chapter 8 of Thurston's notes

Sections 8.1 and 8.2 largely deal with basic properties of the domain of discontinuity and the limit set of a Kleinian group. Variations on this material can be found in any text on Kleinian groups, for example [104] or [105].

2.1 Geometrically finite hyperbolic 3-manifolds

In section 8.3, Thurston offers a new viewpoint on two of the main results in Marden's seminal paper "The geometry of finitely generated Kleinian groups." Marden's Stability theorem (Proposition 9.1 in [98]) asserts that any small deformation of a convex cocompact Kleinian group is itself convex cocompact and is quasiconformally conjugate to the original group. Thurston's version of this theorem (Proposition 8.3.3 in his notes) appears as Proposition 2.5.1 in our article [57]. Marden's Stability theorem also includes a relative version of this

result, which asserts that any small deformation of a geometrically finite Kleinian groups, which preserves parabolicity, is itself geometrically finite and is quasiconformally conjugate to the original manifold.

Marden's Isomorphism Theorem (Theorem 8.1 in [98]) asserts that any homotopy equivalence between two geometrically finite hyperbolic 3-manifolds which extends to a homeomorphism of their conformal boundaries, is homotopic to a homeomorphism which lifts (and extends) to a quasiconformal homeomorphism of $\mathbf{H}^3 \cup S^2_\infty$. Thurston's Proposition 8.3.4 is a variation on Marden's Isomorphism theorem.

Proposition 8.3.4: *Let $N_1 = \mathbf{H}^n/\Gamma_1$ and $N_2 = \mathbf{H}^n/\Gamma_2$ be two convex cocompact hyperbolic n-manifolds and let M_1 and M_2 be strictly convex submanifolds of N_1 and N_2. If $\phi : M_1 \to M_2$ is a homotopy equivalence which is a homeomorphism from ∂M_1 to ∂M_2, then there exists a map $f : \mathbf{H}^n \cup S^{n-1}_\infty \to \mathbf{H}^n \cup S^{n-1}_\infty$ such that the restriction $\widehat{f}$ of f to S^{n-1}_∞ is quasiconformal, $f\Gamma_1 f^{-1} = \Gamma_2$, and the restriction of f to $\mathbf{H}^n$ is a quasi-isometry.*

In section 8.4, Thurston continues his study of geometrically finite hyperbolic 3-manifolds. Theorem 8.4.2 is Ahlfors' result, see [3], that the limit set $\Lambda(\Gamma)$ of a geometrically finite hyperbolic manifold $N = \mathbf{H}^n/\Gamma$ either has measure zero or is all of the sphere at infinity S^{n-1}_∞ and Γ acts ergodically on S^{n-1}_∞. Ahlfors' Measure Conjecture asserts that this is the case for all finitely generated Kleinian groups. In section 8.12, Thurston proves Ahlfors' conjecture for freely indecomposable geometrically tame Kleinian groups. Proposition 8.4.3 discusses three equivalent definitions of geometric finiteness. The various definitions of geometric finiteness are treated thoroughly by Bowditch [33].

2.2 Measured laminations and the boundary of the convex core

In section 8.5, Thurston introduces geodesic laminations and observes that the intrinsic metric on the boundary of the convex core is hyperbolic. This result is established in chapter 1 of Epstein-Marden [66] and by Rourke [140]. Later, Thurston will observe that the boundary of the convex core is an uncrumpled surface. Uncrumpled surfaces are now known as pleated surfaces. Geodesic laminations are treated in chapter 4 of our article [57] and in chapter 4 of Casson-Bleiler [60].

In section 8.6, Thurston introduces transverse measures on geodesic laminations. In particular, he develops the bending measure on the bending locus of the boundary of the convex core. The bending measure is discussed in section 1.11 of Epstein-Marden [66]. Measured laminations are discussed by Hatcher [75] and Penner-Harer [134]. The parallel theory of measured foliations is developed in great detail in the book by Fathi, Laudenbach and Poenaru [70]. The connection between measured laminations and measured foliations in made explicit by Levitt [97]. Hubbard and Masur [80] showed that measured foliations can themselves be naturally linked to the theory of quadratic differentials, see also Marden-Strebel [101]. One of the most spectacular applications of the theory of measured laminations was Kerckhoff's proof [87] of the Neilsen Realization Theorem.

Bonahon developed the theory of geodesic currents, which are a generalization of measured laminations, in [25] and [26]. This theory provides a beautiful and flexible conceptual

framework for the theory of measured laminations and was put to central use in Bonahon's proof [25] that finitely generated, freely indecomposable Kleinian groups are geometrically tame. Bonahon [26] also used geodesic currents to give a beautiful treatment of Thurston's compactification of Teichmüller space. Bonahon is currently preparing a research monograph [29] which covers geodesic laminations, measured laminations, train tracks and geodesic currents. It also describes Bonahon's more recent work on transverse cocycles and transverse Hölder distributions for geodesic laminations which provide powerful new tools for the study of deformation spaces of hyperbolic manifolds. As one application of these techniques Bonahon [27] has computed the derivative of the function with domain a deformation space of geometrically finite hyperbolic 3-manifolds given by considering the volumes of the convex cores. His formula is a generalization of Schläfli's formula for the variation of volumes of hyperbolic polyhedra. Bonahon's briefer survey paper [28] covers some of the same material; both the research monograph and the survey paper are highly recommended.

2.3 Quasifuchsian groups and bending

In section 8.7, Thurston begins his study of quasifuchsian groups. A finitely generated, torsion-free Kleinian group is said to *quasifuchsian* if its limit set is a Jordan curve and both components of its domain of discontinuity are invariant under the entire group. Thurston's definition of a quasifuchsian group is incomplete as it leaves out the condition on the domain of discontinuity. His definition allows Kleinian groups which uniformize twisted I-bundles over surfaces, as well as those which uniformize product I-bundles. Proposition 8.7.2 offers several equivalent definitions of quasifuchsian groups. We give a corrected version of Thurston Proposition 8.7.2 below:

Proposition 8.7.2: (Maskit [102]) *If* Γ *is a finitely generated, torsion-free Kleinian group, then the following conditions are equivalent:*

1. Γ *is quasifuchsian.*
2. *The domain of discontinuity* $\Omega(\Gamma)$ *of* Γ *has exactly two components, each of which is invariant under the entire group.*
3. Γ *is quasiconformally conjugate to a Fuchsian group, i.e. there exists a Fuchsian group* $\Theta \subset \mathrm{PSL}_2(\mathbf{R})$ *(such that its limit set* $\Lambda(\Theta) = \mathbf{R} \cup \infty$*) and a quasiconformal map* $\phi : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *such that* $\Gamma = \phi\Theta\phi^{-1}$.

This characterization is originally due to Maskit, see Theorem 2 in [102], although Thurston follows the alternate proof given by Marden in section 3 of [98].

Example 8.7.3 is the famous Mickey Mouse example, which is produced using the bending construction. Bending has been studied extensively by Apanasov [14] and Tetenov [13], Johnson and Millson [81], Kourouniotis [91] and others. Universal bounds on the bending lamination of a quasifuchsian group and hence on the bending deformation, are obtained by

Bridgeman [34, 35] (and generalized to other settings by Bridgeman-Canary [37]). These bounds are discussed in more detail in the addendum to Epstein-Marden [66] in this volume.

After the Mickey Mouse example, Thurston discusses simplicial hyperbolic surfaces, although he does not give them a name. A simplicial hyperbolic surface is a map, not necessarily an embedding, of a triangulated surface into a 3-manifold such that each face is mapped totally geodesically and the total angle around each vertex is at least 2π. The restriction on the vertices guarantees that the induced metric, usually singular, on the surface has curvature ≤ -1 in the sense of Alexandrov. Simplicial hyperbolic surfaces were used extensively by Bonahon [25] in his proof that freely indecomposable Kleinian groups are geometrically tame and they are discussed in detail in section 1.3 of [25].

Proposition 8.7.7 asserts that every complete geodesic lamination is realizable in a quasifuchsian hyperbolic 3-manifold. This statement is included in Theorem 5.3.11 in [57]. We will discuss realizability of laminations more fully when we come to sections 8.10 and 9.7.

2.4 Pleated surfaces and realizability of laminations

Section 8.8 of Thurston's notes concerns pleated surfaces, which are called uncrumpled surfaces in the notes. The results in this section form the basis of section 5 of our original article [57]. Pleated surfaces are also discussed in Thurston's articles on the Geometrization theorem [152, 154, 155].

Section 8.9 of Thurston's notes develops the theory of train tracks. Proposition 8.9.2 and Corollary 8.9.3 assert that any geodesic lamination on a surface may be well-approximated by a train track. Three-dimensional versions of these results play a key role in Bonahon's work and section 5 of his paper [25] discusses train track approximations to geodesic laminations in great detail. The general theory of train tracks is developed by Penner and Harer in [134].

In section 8.10, Thurston turns to the issue of realizability of laminations in 3-manifolds. We discuss this issue in detail in section 5.3 of [57]. One begins with an incompressible, type-preserving map $f : S \to N$ of a finite area hyperbolic surface S into a hyperbolic 3-manifold N. (An incompressible map $f : S \to N$ is said to be type-preserving if $f_*(g)$ is parabolic if and only if $g \in \pi_1(S)$ is parabolic where $f_* : \pi_1(S) \to \pi_1(N)$ is regarded as a map between the associated groups of covering tranformations.) One says that a geodesic lamination λ on S is *realizable* if there is a pleated surface $h : S \to N$ which maps λ into N in a totally geodesic manner. If a realization exists then the image of λ is unique (Proposition 8.10.2 in Thurston and Lemma 5.3.5 in [57].) The map from the space of pleated surfaces (which are homotopic to f) into the space of geodesic laminations (with the Thurston topology) given by taking a pleated surface to its pleating locus is continuous (Proposition 8.10.4 in Thurston and Lemma 5.3.2 in [57].) Propositions 8.10.5, 8.10.6 and 8.10.7 develop more basic properties of geodesic measured laminations, see the earlier references for details. Theorem 8.10.8 in Thurston's notes asserts that the set $\mathcal{R}_f$ of realizable laminations is open and dense in the set $GL(S)$ of all geodesic laminations on S, see Theorem 5.3.10 in [57] for details. Thurston's Corollary 8.10.9, which asserts that if N is geometrically finite then $\mathcal{R}_f = GL(S)$ unless N virtually fibres over the circle with fibre $f(S)$, is stated as Corollary 5.3.12 in [57].

We note that Thurston's Conjecture 8.10.10, which asserts that $f_*(\pi_1(S))$ is quasifuchsian if and only if $\mathcal{R}_f = GL(S)$, is a consequence of Bonahon's work [25], see the discussion after Proposition 9.7.1 and the discussion of Bonahon's work in section 4.

In related work, Brock [38] proved that the length function is continuous on the space of realizable laminations in $AH(S) \times ML(S)$ and extends to a continuous function on all of $AH(S) \times ML(S)$. Thurston claimed this result and used it in his proof [154] of the Geometrization theorem for 3-manifolds which fiber over the circle.

2.5 Relative compact cores and ends of hyperbolic 3-manifolds

It will be convenient to formalize the material in section 8.11 in the language of relative compact cores. If N is a hyperbolic 3-manifold, and we choose ϵ less than the Margulis constant (see Section 4.5 of Thurston [153] or Chapter D in Benedetti-Petronio [23] for example) we can define the ϵ-thin part of N to be the portion of N with injectivity radius at most ϵ. Each compact part of the ϵ-thin part will be a solid torus neighborhood of a geodesic, while each non-compact component will be the quotient of a horoball by a group of parabolic isometries (isomorphic to either $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$). We obtain N^0 from N by removing its "cusps", i.e. the non-compact components of its thin part. A relative compact core M for N is a compact 3-dimensional submanifold of N^0 whose inclusion into N is a homotopy equivalence which intersects each toroidal component of ∂N^0 in the entire torus and intersects each annular component of ∂N^0 in a single incompressible annulus. Bonahon [24], McCullough [106] and Kulkarni-Shalen [93] proved that every hyperbolic 3-manifold with finitely generated fundamental group admits a relative compact core.

Feighn-McCullough [71] and Kulkarni-Shalen [93] (see also Abikoff [1]) have used the relative compact core to give topological proofs of Bers' area inequality, which asserts that the area of the conformal boundary is bounded by the number of generators (see Bers [16]) and Sullivan's Finiteness Theorem, which asserts that the number of conjugacy classes of maximal parabolic subgroups of a Kleinian group is bounded by the number of generators (see Sullivan [148]). See section 7 of Marden [98] for a similar treatment of Bers' area inequalities in the setting of geometrically finite groups where Marden constructs an analogue of the relative compact core.

Much of the remainder of section 8 and section 9 are taken up with understanding the geometry and topology of ends of hyperbolic 3-manifolds. Ends of N^0 are in a one-to-one correspondence with components of $\partial M - \partial N^0$, see proposition 1.3 in [25], where M is a relative compact core for N. An end of N^0 is *geometrically finite* if it has a neighborhood which does not intersect the convex core. At the end of section 8.11, Thurston introduces the crucial notion of a simply degenerate end of a hyperbolic 3-manifold. If M is a relative compact core for N, then an end E of N^0 which has a neighborhood bounded by an incompressible component S of $\partial M - \partial N^0$ is said to be *simply degenerate* if there exists a sequence $\{\gamma_i\}$ of non-trivial simple closed curves on S whose geodesic representatives in N all lie in the component of $N^0 - M$ bounded by S and leave every compact subset of N. (Here we have given Bonahon's version of Thurston's definition, which is equivalent to Thurston's.)

A hyperbolic 3-manifold in which each component of $\partial M - \partial N^0$ is incompressible is said to be *geometrically tame* if each of its ends is either geometrically finite or simply degenerate.

We will say that the relative compact core M has *relatively incompressible* boundary if each component of $M - \partial N^0$ is incompressible. Thurston works almost entirely in the setting of hyperbolic 3-manifolds whose relative compact core has relatively incompressible boundary. If N has no cusps, the relative compact core has incompressible boundary if and only if $\pi_1(N)$ is freely indecomposable. In general, the relative compact core has relatively incompressible boundary if and only if there does not exist a non-trivial free decomposition of $\pi_1(N)$ such that every parabolic element is conjugate into one of the factors, see Proposition 1.2 in Bonahon or Lemma 5.2.1 in Canary-McCullough [58].

In section 4.1, we will explain how the definition of geometric tameness is extended to all hyperbolic 3-manifolds with finitely generated fundamental group.

2.6 Analytic consequences of tameness

In Section 8.12, Thurston proves a minimum principle for positive superharmonic functions on geometrically tame hyperbolic 3-manifolds.

Theorem 8.12.3: *If N is a geometrically tame hyperbolic 3-manifold (whose compact core has relatively incompressible boundary), then for every non-constant positive superharmonic (i.e. $\Delta h \leq 0$) function h on N,*

$$\inf_{C(N)} h = \inf_{\partial C(N)} h$$

where $C(N)$ denotes the convex core of N. In particular, if $C(N) = N$ (i.e. $L_\Gamma = S^2$) then there are no positive non-constant superharmonic functions on N.

As a corollary, he shows that Ahlfors' measure conjecture holds for geometrically tame hyperbolic 3-manifolds.

Corollary 8.12.4: *If $N = \mathbf{H}^3/\Gamma$ is a geometrically tame, 3-manifold (whose compact core has relatively incompressible boundary), then either L_Γ is all of S^2_∞ or it has measure zero. Moreover, if $L_\Gamma = S^2_\infty$ then Γ acts ergodically on S^2_∞.*

He also notes that one may combine his minimum principle with work of Sullivan [145] to show that the geodesic flow of a geometrically tame hyperbolic 3-manifold (whose compact core has relatively incompressible boundary) is ergodic if and only if its limit set is the entire Riemann sphere. These arguments are generalized to the setting of analytically tame hyperbolic 3-manifolds in [53], see also section 7 of Culler-Shalen [63] and Sullivan [147]. (A hyperbolic 3-manifold is *analytically tame* if its convex core may be exhausted by a nested sequence $\{C_i\}$ of compact submanifolds C_i such that there exists K and L such that ∂C_i has area $\leq K$ and the neighborhood of radius one of ∂C_i has volume $\leq L$.) In particular, Ahlfors' Measure conjecture is established for analytically tame hyperbolic 3-manifolds.

Sullivan [149] and Tukia [156] showed that limit sets of geometrically finite Kleinian groups have Hausdorff dimension less than 2, unless their quotient has finite volume in

which case the limit set is the entire sphere at infinity. Sullivan [147] provided the first examples of finitely generated Kleinian groups whose limit sets have measure zero but Hausdorff dimension 2 (see also Canary [52]). Bishop and Jones [22] later proved that the limit set of every finitely generated, geometrically infinite Kleinian group has Hausdorff dimension 2.

3 Chapter 9 of Thurston's notes

Chapter 9 is largely devoted to the study of limits of hyperbolic 3-manifolds. In section 9.1, the notion of geometric convergence of a sequence of Kleinian groups is discussed. In section 3 of [57] we prove the equivalence of several different notions of geometric convergence. In particular Thurston's Corollary 9.1.7 appears as Corollary 3.1.7 in [57].

3.1 Algebraic and geometric limits

Many of the most interesting results in chapter 9 concern the interplay between algebraic and geometric convergence of Kleinian groups. If a sequence $\{\rho_i : G \to \mathbf{PSL}_2(\mathbf{C})\}$ of discrete, faithful representations converges, in the compact-open topology, to $\rho : G \to \mathbf{PSL}_2(\mathbf{C})$, then we say that ρ is the *algebraic limit* of $\{\rho_i\}$. If G is not virtually abelian, then $\{\rho_i\}$ has a subsequence $\{\rho_j\}$ such that $\{\rho_j(G)\}$ converges geometrically to a Kleinian group $\widehat{\Gamma}$ which is called the *geometric limit* of $\{\rho_j(G)\}$ (Corollary 9.1.8 in Thurston and Proposition 3.8 in Jorgensen-Marden [82]). If ρ is the algebraic limit of $\{\rho_i\}$ and $\rho(G)$ is the geometric limit of $\{\rho_i(G)\}$, then we say that $\{\rho_i\}$ *converges strongly* to ρ.

Example 9.1.4, which is due to Jorgensen (see section 5 of Jorgensen-Marden [82]), is the most basic example of a sequence which converges algebraically but not strongly. In this example, the algebraic limit is an infinite cyclic group, while the geometric limit is a free abelian group of rank two. More complicated examples which contain this same phenomenon can be found in Marden [100], Kerckhoff-Thurston [88], Ohshika [126] and Thurston [154]. Brock [40] exhibited a sequence where the algebraic limit differs from the geometric limit, yet the geometric limit does not contain a free abelian subgroup of rank two. Anderson and Canary [7] exhibited examples where the (quotient of the) algebraic limit is topologically tame, but is not homeomorphic to any of its approximates. The most comprehensive reference on the foundations of the relationship between the algebraic and the geometric limit is Jorgensen-Marden [82].

3.2 Limits of quasifuchsian groups

In section 9.2, Thurston begins to study limits of quasifuchsian groups. This study was crucial in his original proof of the Geometrization theorem. If S is a compact surface (with negative Euler characteristic), a discrete, faithful representation $\tau : \pi_1(S) \to \mathbf{PSL}_2(\mathbf{C})$ is said to be *quasifuchsian* if $\tau(\pi_1(S))$ is quasifuchsian and $\tau(g)$ is parabolic if and only if g is a peripheral element of $\pi_1(S)$, i.e. if the curve representing g is freely homotopic into

a component of the boundary. Thurston's Theorem 9.2 asserts that any type-preserving algebraic limit of quasifuchsian representations is geometrically tame and is in fact a strong limit.

Theorem 9.2: *Let S be a compact surface with negative Euler characteristic. Suppose that $\{\rho_i : \pi_1(S) \to \mathbf{PSL}_2(\mathbf{C})\}$ is a sequence of quasifuchsian representations converging to ρ and that $\rho \circ \rho_i^{-1} : \rho_i(\pi_1(S)) \to \rho(\pi_1(S))$ is type-preserving for all i. Then, $N_\rho = \mathbf{H}^3/\rho(\pi_1(S))$ is geometrically tame and $\{\rho_i\}$ converges strongly to ρ.*

This result divides naturally into two pieces. Ohshika established that, in this setting, the convergence is strong in Corollary 6.1 of [130]. Evans established a generalization of this strong convergence result for surface groups as part of [67]. Generalizations of the fact that a strong, type-preserving limit of quasifuchsian groups is geometrically tame are obtained by Canary-Minsky [59] and by Ohshika [129] in the case that there are no parabolics and in the general situation by Evans [68].

3.3 The covering theorem

In the proof of theorem 9.2, Thurston develops the covering theorem which is a very important tool in the study of algebraic and geometric limits. It asserts that, with the exception of the cover of 3-manifold which fibres over the circle associated to the fibre, a simply degenerate end can only cover finite-to-one.

Theorem 9.2.2: *Let $\widehat{N}$ be a hyperbolic 3-manifold which covers another hyperbolic 3-manifold N by a local isometry $p : \widehat{N} \to N$. If $\widehat{E}$ is a simply degenerate end of $\widehat{N}^0$ then either*

a) $\widehat{E}$ has a neighborhood $\widehat{U}$ such that p is finite-to-one on $\widehat{U}$, or

b) N has finite volume and has a finite cover N' which fibers over the circle such that if N_S denotes the cover of N' associated to the fiber subgroup then $\widehat{N}$ is finitely covered by N_S. Moreover, if $\widehat{N} \neq N_S$, then $\widehat{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S.

In Thurston's covering theorem, the end $\widehat{E}$ is required to be associated to an incompressible surface in $\widehat{N}^0$ (as this is the setting in which Thurston has defined simply degenerate ends.) Canary generalized Thurston's covering theorem to the setting of simply degenerate ends of topologically tame hyperbolic 3-manifolds in [55], see also Lemma 2.2 in Ohshika [126]. For a survey of some of the remaining issues related to the covering theorem see [54].

3.4 An intersection number lemma

The key result in section 9.3 is Theorem 9.3.5, which is an intersection number lemma for geodesic laminations. Bonahon proves a version of this result as Proposition 3.4 in [25].

Theorem 9.3.5: (Bonahon [25]) *Let N be a hyperbolic 3-manifold with finitely generated fundamental group and let S be a properly embedded incompressible surface in N^0. There*

exists a constant $K \geq 0$ such that if α_1^* and α_2^* are two closed geodesics in N^0 of distance $\geq D$ from S, homotopic to 2 curves α_1 and α_2 in S by two homotopies which meet S only in α_i and arrive on the same side, and each geodesic is disjoint from the thin part of N, or is itself the core of a Margulis tube, then

$$i(\alpha_1, \alpha_2) \leq Ke^{-D} l(\alpha_1) l(\alpha_2) + 2$$

where i is intersection number in S, and l is length measured on S.

Thurston's version allows α_1^* and α_2^* to be measured geodesic laminations in N, but Thurston neglects to include the restriction on the geodesics.

One consequence of Theorem 9.3.5 is that a simply degenerate end admits a well-defined geodesic lamination, called the ending lamination. Let E be a simply degenerate end with a neighborhood bounded by an incompressible subsurface S of the boundary of a relative compact core for N^0. If $\{\alpha_i\}$ is a sequence of simple closed curves on S whose geodesic representatives $\{\alpha_i^*\}$ in N exit E, then the *ending lamination* $\epsilon(E)$ of E is the limit of $\{\alpha_i\}$ in $GL(S)$. (More formally, to ensure uniqueness, we must define $\epsilon(E)$ to be the maximal sublamination of $\lim \alpha_i$ which supports a measure.) Thurston uses Theorem 9.3.5 to show that $\epsilon(E)$ is well-defined, that every leaf of $\epsilon(E)$ is dense in $\epsilon(E)$ and that every simple closed curve in the complement of $\epsilon(E)$ is peripheral. These results are also established in Bonahon [24]. This discussion is generalized to the setting of topologically tame hyperbolic 3-manifolds in [53]. For a more thorough discussion of ending laminations and the Ending Lamination Conjecture see Minsky [114].

3.5 Topological tameness

Sections 9.4 and 9.5 of Thurston are devoted to proving that a geometrically tame hyperbolic 3-manifold is *topologically tame*, i.e. homeomorphic to the interior of a compact 3-manifold.

Theorem 9.4.1: *If N is a geometrically tame hyperbolic 3-manifold (whose relative compact core has relatively incompressible boundary), then N is topologically tame.*

Bonahon offers a simpler proof of Theorem 9.4.1 in [24], based on a result of Freedman, Hass and Scott [72]. An outline of this argument is also given in [25]. Canary [53] generalized the notion of a geometrically tame hyperbolic 3-manifold to the setting where the relative compact core need not have relatively incompressible boundary and proved that topologically tame hyperbolic 3-manifolds are geometrically tame. We will discuss this further in the next section.

Thurston's method of proof is based on a scheme for interpolating between any two pleated surfaces in a simply degenerate end with a family of negatively curved surfaces. This approach is discussed in section 5 of Ohshika's paper [130]. As part of this discussion, Thurston proves that the space $ML(S)$ of measured laminations has a piecewise integral linear structure and that the space $PL(S)$ of projective measured laminations has a piecewise integral projective structure, see Theorem 3.1.4 in Penner-Harer [134] for details.

In remarks at the end of section 9.5, Thurston describes two alternative approaches to this interpolation. The first approach, which makes use of simplicial hyperbolic surfaces and elementary moves on triangulations, is carried out by Canary in [55] and has also been used by Fan [69] and Evans [68]. The second approach makes use of the theory of harmonic maps. This approach was carried out by Minsky in [112] and was utilized in his proof of the ending lamination conjecture for geometrically tame hyperbolic 3-manifolds with freely indecomposable fundamental group and a lower bound on their injectivity radius [113].

3.6 Strong convergence

In section 9.6, Thurston generalizes Theorem 9.2 to the setting of hyperbolic manifolds whose relative compact cores have relatively incompressible boundary. Let M be a compact 3-manifold and let P be a collection of incompressible annuli and tori in ∂M. (One often explicitly requires that (M, P) be a pared 3-manifold, see section 4 of Morgan [119].) We say that $\rho : \pi_1(M) \to \mathbf{PSL}_2(\mathbf{C})$ *uniformizes* the pair (M, P) if there exists a relative compact core Q for $(N_\rho)^0$ and a homeomorphism of pairs $h : (M, P) \to (Q, Q \cap \partial(N_\rho)^0)$ such that $h_* = \rho$.

Theorem 9.6.1: *Let M be a compact 3-manifold and let P be a collection of incompressible annuli and tori in ∂M such that each component of $\partial M - P$ is incompressible. Suppose that $\{\rho_i\}$ is a sequence of geometrically tame uniformizations of (M, P) which converge to $\rho : \pi_1(M) \to \mathbf{PSL}_2(\mathbf{C})$ and that $\rho \circ \rho_i^{-1}$ is type-preserving for all i. Then,*

1. *$\{\rho_i\}$ converges strongly to ρ, and*
2. *ρ is a geometrically tame uniformization of (M, P).*

Thurston only sketches the proof of Theorem 9.6.1 in the case that there is no essential annulus in M with one boundary component in P. A generalization of the result in part (1), that $\{\rho_i\}$ converges strongly to the setting where M is allowed to have compressible boundary is given by Anderson and Canary [8, 9]. A generalization of part (1) which allows the sequence to be only weakly type-preserving is given by Evans [67]. (Kleineidam [89] has given a quite nice characterization of strong convergence from a different viewpoint.) A generalization of part (2), is given in the case where P is empty by Canary-Minsky [59] and Ohshika [129], and in the general setting by Evans [67].

3.7 Ending Laminations

The first result in section 9.7 sums up what has been learned about ending laminations. It can be derived from the work of Bonahon [24, 25] but is not explicitly stated in any of his papers. We say that an isomorphism $\tau : \Gamma \to \Theta$ is *weakly type-preserving* if whenever $\gamma \in \Gamma$ is parabolic, then $\tau(\gamma)$ is parabolic.

Proposition 9.7.1: *Let S be a finite area hyperbolic surface and let $\rho : \pi_1(S) \to \mathbf{PSL}_2(\mathbf{C})$ be a discrete, faithful, geometrically tame, weakly type-preserving representation. There exist*

two geodesic laminations λ_+ and λ_- on S, such that $\alpha \in GL(S)$ is realizable if and only if α contains no component of λ_+ or $\lambda-$.

In Proposition 9.7.1, one first constructs a relative compact core M for $(N_\rho)^0$. Then M is homeomorphic to $S \times [0,1]$ and $P = \partial M \cap \partial(N_\rho)^0$ is a collection of annuli which includes $\partial S \times [0,1]$. We let γ_+ and γ_- denote the core curves of the annuli in P which lie in $S \times \{0\}$ and $S \times \{1\}$. One obtains λ_+ by appending to γ_+ the ending laminations of any simply degenerate ends bounded by components of $S \times \{0\} - P$. One forms λ_- similarly.

The remainder of section 9.7 concerns train track coordinates for the space of measured laminations. We refer the reader again to sections 3.1 and 3.2 of Penner-Harer [134] for details. In particular, Thurston proves that $PL(S)$ is a sphere, see Theorem 3.1.4 in Penner-Harer [134] or Proposition 1.5 of Hatcher [75].

In section 9.9, Thurston surveys Sullivan's work [145] on ergodicity of geodesic flows on hyperbolic manifolds. Sullivan's work is also discussed in Ahlfors' book [4].

4 Selected Generalizations

In this section, we will briefly discuss a few of the most direct generalizations of the material in chapters 8 and 9 of Thurston's notes. The past few years has been a period of intense activity in the field and we will not have space to mention many important results. Other recent surveys of related material include Anderson [6], Brock-Bromberg [41], Canary [56] and Minsky [117, 118].

4.1 Tameness

Marden [98] conjectured that all hyperbolic 3-manifolds with finitely generated fundamental group are topologically tame. Marden's Tameness Conjecture has developed into a central goal of the field.

In a tour de force, Bonahon proved that every hyperbolic 3-manifold whose relative compact core has relatively incompressible boundary is geometrically tame, and hence topologically tame.

Theorem: (Bonahon [25]) *Suppose that N is a hyperbolic 3-manifold with finitely generated fundamental group and that M is a relative compact core for N^0. If each component of $\partial M - \partial N^0$ is incompressible, then N is geometrically tame.*

An immediate consequence of this result is that Ahlfors' Measure Conjecture is valid for all hyperbolic 3-manifolds whose relative compact cores have relatively incompressible boundary.

Subsequently, Canary [53] used Bonahon's work to prove that all topologically tame hyperbolic 3-manifolds are geometrically tame. In order to make sense of this result, one must first define geometric tameness for hyperbolic 3-manifolds whose relative compact core may not be relatively incompressible. For simplicity, we will assume that N has no cusps, for the full definition see [53]. We say that an end E of N is *simply degenerate* if it has a

neighborhood U which is homeomorphic to $S\times[0,\infty)$ (for some compact surface S) and there exists a sequence of simplicial hyperbolic surfaces $\{f_i : S \to U\}$, each of which is homotopic within U to $S \times \{0\}$, which leave every compact subset of U. A hyperbolic 3-manifold with finitely generated fundamental group is said to *geometrically tame* if all its ends are either geometrically finite or simply degenerate.

Theorem: (Canary [53]) *A hyperbolic 3-manifold with finitely generated fundamental group is topologically tame if and only if it is geometrically tame.*

Canary's result implies that topologically tame hyperbolic 3-manifolds are analytically tame so Thurston's Theorem 8.12.3 holds for topologically tame hyperbolic 3-manifolds. In particular, Ahlfors' Measure Conjecture holds.

Corollary: *If $N = \mathbf{H}^3/\Gamma$ is a topologically tame, 3-manifold, then either L_Γ is all of S^2_∞ or it has measure zero. Moreover, if $L_\Gamma = S^2_\infty$ then Γ acts ergodically on S^2_∞.*

There has been a steady progression in our understanding of tameness properties of limits of geometrically finite hyperbolic 3-manifolds due to many authors, including Canary-Minsky [59], Ohshika [129], and Evans [68]. The best results about limits combine work of Brock-Bromberg-Evans-Souto [43] and Brock-Souto [45].

Theorem: (Brock-Bromberg-Evans-Souto [43] and Brock-Souto [45]) *Any algebraic limit of geometrically finite hyperbolic 3-manifolds is topologically tame.*

Agol [2] and Calegari-Gabai [51] have recently given complete proofs of Marden's Tameness Conjecture.

4.2 Spaces of geometrically finite hyperbolic 3-manifolds

The parameterization of the space of geometrically finite hyperbolic structures on a fixed compact 3-manifold has been well-understood since the 1970s. Roughly, a geometrically finite hyperbolic 3-manifold is known to be determined by its topological type and the conformal structure on its conformal boundary. This parameterization combines work of Ahlfors [5], Bers [18], Kra [92], Marden [98] and Maskit [103]. For complete discussions of this parameterization see Bers [19], section 6 of Marden [99], or section 7 of Canary-McCullouch [58].

One might hope that one could also parameterize geometrically finite hyperbolic 3-manifolds by internal geometric data, e.g. the bending lamination. Bonahon and Otal [32] characterize exactly which measured laminations arise as the bending lamination of the convex core of a geometrically finite hyperbolic 3-manifold whose relative compact core has relatively incompressible boundary. Lecuire [95] has extended their result to the general case. Keen and Series, see [85] for example, have done an extensive analysis of the "pleating rays" (i.e. lines where the support of the bending lamination is constant) in a number of concrete situations. It is conjectured that the topological type of the convex core and the bending lamination determines a geometrically finite hyperbolic 3-manifold, but Bonahon-Otal [32] and Lecuire [95] only establish this for finite-leaved laminations. Series [143] has recently

established this conjecture for punctured torus groups. For more general surfaces, Bonahon [30] proved the conjecture for quasifuchsian groups lying in a neighborhood of the set of Fuchsian groups.

Similarly, it is conjectured that the topological type of the convex core and the conformal structure on the boundary of the convex core determine a geometrically finite hyperbolic 3-manifold. It is known that every possible conformal structure arises, but the uniqueness remains unknown. If the boundary of the convex core has incompressible boundary, then the existence follows immediately from the continuity of the structure on the conformal boundary, see Keen-Series [86], and Sullivan's theorem, see Epstein-Marden [66]. If the boundary of the convex core is compressible, one may replace the use of Sullivan's theorem with the results of Bridgeman-Canary [36]. See Labourie [94] for a generalization of the existence result.

Scannell and Wolf [141] established that a quasifuchsian hyperbolic 3-manifold is determined by the conformal structure on one boundary component and the bending lamination on the associated component of the boundary of the convex core. McMullen [109] had previously established this fact for quasifuchsian once-punctured torus groups.

Bers [17], Sullivan [149] and Thurston [151] conjectured that every hyperbolic 3-manifold with finitely generated fundamental group arises as the (algebraic) limit of a sequence of geometrically finite hyperbolic 3-manifolds. Bromberg [47] and Brock-Bromberg [42] proved that all hyperbolic 3-manifolds with freely indecomposable fundamental group and no cusps arise as limits of geometrically finite hyperbolic 3-manifolds.

4.3 Thurston's Ending Lamination Conjecture

Another subject which is hinted at, although not addressed directly, in chapters 8 and 9, is Thurston's Ending Lamination Conjecture, which Thurston first explicitly states in [151]. Thurston conjectured that a hyperbolic 3-manifold is determined by the topological type of its relative compact core, the conformal structure at infinity of each of its geometrically finite ends and the ending laminations of its simply degenerate ends. For a discussion of the background of this conjecture see Minsky [114, 118].

Minsky [113] established Thurston's Ending Lamination Conjecture for hyperbolic 3-manifolds with a lower bound on their injectivity radius and freely indecomposable fundamental group. Ohshika [127] generalized Minsky's proof to the setting of topologically tame hyperbolic 3-manifolds with a lower bound on their injectivity radius. Minsky [115] subsequently established Thurston's conjecture for punctured torus groups, i.e. weakly type-preserving representations of the fundamental group of a finite area punctured torus, see Minsky [115]. Ohshika [125] used results of Thurston [154, 155] to give a complete characterization of which laminations can arise as the ending invariants of a hyperbolic 3-manifold whose relative compact core has relatively incompressible boundary.

Brock, Canary and Minsky [116, 44] established Thurston's Ending Lamination Conjecture for hyperbolic 3-manifolds with freely indecomposable fundamental group (and more generally for hyperbolic 3-manifolds whose relative compact core has relatively incompress-

ible boundary.) In combination with work of Ohshika [125], this establishes the Bers-Sullivan-Thurston Density Conjecture for hyperbolic 3-manifolds whose relative compact core has relatively incompressible boundary. See Minsky [118] for a survey of this work.

Brock, Canary and Minsky have announced a proof of Thurston's Ending Lamination Conjecture for topologically tame hyperbolic 3-manifolds. In combination with the recent resolution of Marden's Tameness Conjecture, see Agol [2] and Calegari-Gabai [51], this gives a complete resolution of Thurston's Ending Lamination Conjecture. One may combine work of Ohshika [125], Kleineidam-Souto [90] and Lecuire [96] with the resolution of Thurston's Ending Lamination Conjecture to give a full proof of the Bers-Sullivan-Thurston Density Conjecture.

The resolution of Thurston's Ending Lamination Conjecture gives a complete classification of hyperbolic 3-manifolds with finitely generated fundamental group. One might hope that it would also give a topological parameterization of the space $AH(M)$ of hyperbolic 3-manifolds homotopy equivalent to a fixed compact 3-manifold M. However, both the topological type (see Anderson-Canary [7]) and the ending invariants themselves (see Brock [39]) vary discontinuously, so one does not immediately obtain such a parameterization. Moreover, Holt [78, 79] showed that there are points in the closures of arbitrarily many components of the interior of $AH(M)$, i.e. arbitrarily many components can "bump" at a single point. McMullen [109] and Bromberg-Holt [49] showed that individual components of the interior of $AH(M)$ often "self-bump," i.e. there is a point in the closure of the component such that the intersection of any small enough neighborhood of the point with the component is disconnected. Most recently, Bromberg [48] has shown that the space of punctured torus groups is not locally connected. Thus, any parameterization of $AH(M)$ must be rather complicated.

To finish on a positive note, the work of Anderson, Canary and McCullough [10] may be combined with the proof of Thurston's Ending Lamination Conjecture to give a complete enumeration of the components of $AH(M)$ whenever M has incompressible boundary. This enumeration can be expressed entirely in terms of topological data.

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