# Approximation by maximal cusps in boundaries of deformation spaces of Kleinian groups

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**Abstract**

Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold with non-empty boundary. Let $CC_0(M)$ denote the space of convex cocompact Kleinian groups uniformizing $M$. We show that any Kleinian group in the boundary of $CC_0(M)$ whose limit set is the whole sphere can be approximated by maximal cusps. Density of maximal cusps on the boundary of Schottky space is derived as a corollary. We further show that maximal cusps are dense in the boundary of the quasiconformal deformation space of any geometrically finite hyperbolic 3-manifold with connected conformal boundary.

## 1 Introduction

Let $M$ be a compact, hyperbolizable 3-manifold whose (non-empty) boundary consists of surfaces of genus at least two. The space $CC_0(M)$ of convex cocompact uniformizations of $M$ is a component of the interior of the space $AH(\pi_1(M))$ of all marked hyperbolic 3-manifolds homotopy equivalent to $M$. We show that if a hyperbolic 3-manifold in the boundary of $CC_0(M)$ has empty conformal boundary, i.e. all of its ends are geometrically infinite, then it may be approximated by maximal cusps. Recall that maximal cusps are geometrically finite hyperbolic 3-manifolds such that every component of their conformal boundary is a thrice-punctured sphere. As a corollary we show that maximal cusps are dense in the boundary of $CC_0(M)$ if the boundary of $M$ is connected.

We view this result as part of a family of recent results which study the topology of $AH(\pi_1(M))$. Recall that each component of the interior of $AH(\pi_1(M))$ can be identified with $CC_0(M')$ where $M'$ is homotopy equivalent to $M$. It is conjectured that $AH(\pi_1(M))$ is the closure of its interior. Thurston's Ending Lamination Conjecture provides a conjectural classification of the manifolds in $AH(\pi_1(M))$. In this classification, geometrically finite hyperbolic 3-manifolds correspond to "rational points" in the boundary of $CC_0(M)$. (Each

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ending invariant of a geometrically finite hyperbolic 3-manifold is either a point in a Teichmüller space or a finite leaved geodesic lamination. The finite leaved laminations are the "rational points" in the space of geodesic laminations.) So one may think of our corollary as saying that "rational points" are dense in the boundary of $CC_0(M)$ whenever the boundary of $M$ is connected. Our main result can then be thought of as asserting that, in general, the "most irrational" points in $\partial CC_0(M)$ can be approximated by "rational points."

McMullen [Mc 1] was the first to study the density of "rational points" in the boundary of deformation spaces of Kleinian groups. He showed that "maximal cusps" are dense in the boundary of any Bers slice of quasifuchsian space. Recall that if $S$ is a closed surface then quasifuchsian space $QF(S)$ is the space of convex cocompact uniformizations of $S \times I$. A Bers slice of $QF(S)$ consists of convex cocompact uniformizations of $S \times I$ such that the component of the conformal boundary corresponding to $S \times \{0\}$ has a fixed conformal structure. In his setting a "maximal cusp" is a geometrically finite manifold in the boundary of $QF(S)$ whose conformal boundary consists of one copy of $S$ and a collection of thrice-punctured spheres. He later claimed that maximal cusps, in the sense of this paper, are dense in the boundary of Schottky space, i.e. the space of convex cocompact uniformizations of a handlebody. This result was used by Culler, Shalen and their coauthors in a series of papers which studied the relationship between the topology of a hyperbolic 3-manifold and its volume. We make central use of the analytical machinery developed by McMullen [Mc 1].

We now develop the notation needed to state our results precisely. Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold with non-empty boundary. If $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ is any discrete, faithful representation we let $\Omega(\rho)$ denote the domain of discontinuity of the action of $\rho(\pi_1(M))$ on the Riemann sphere $\widehat{\mathbf{C}}$. Then

$$\bar{N}_\rho = (\mathbf{H}^3 \cup \Omega(\rho))/\rho(\pi_1(M))$$

is a 3-manifold with boundary. The interior $N_\rho = \mathbf{H}^3/\rho(\pi_1(M))$ of $\bar{N}_\rho$ inherits the structure of a hyperbolic 3-manifold, while the boundary of $\bar{N}_\rho$, which is denoted $\partial_c N_\rho$ and called the *conformal boundary*, has a natural conformal structure induced from that on the sphere at infinity. We will identify $\pi_1(\bar{N}_\rho)$ with the subgroup $\rho(\pi_1(M))$ of $\mathrm{PSL}_2(\mathbf{C})$.

A discrete, faithful representation $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ is called a *convex cocompact uniformization of* $M$ if there exists an orientation-preserving homeomorphism $h\colon M \to \bar{N}_\rho$ such that the two isomorphisms $h_*\colon \pi_1(M) \to \pi_1(\bar{N}_\rho)$ and $\rho : \pi_1(M) \to \rho(\pi_1(M)) = \pi_1(\bar{N}_\rho)$ differ by inner automorphisms. In this situation, if we view $h_*$ as a representation of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbf{C})$ then $\rho$ and $h_*$ are conjugate representations. We shall indicate this by writing $[h_*] = [\rho]$, and in general we shall use the notation $[\rho]$ to denote the conjugacy class of a representation $\rho$. Note that if $\partial M$ contains no tori, then Thurston's Uniformization Theorem implies that there exists a convex cocompact uniformization of $M$.

Let $CC_0(M)$ denote the set of conjugacy classes of convex cocompact uniformizations of $M$. The space $CC_0(M)$ naturally sits inside the set $AH(\pi_1(M))$ of conjugacy classes of discrete faithful representations of $\pi_1(M))$ into $\mathrm{PSL}_2(\mathbf{C})$. (We give $AH(\pi_1(M))$ the quotient topology induced by the compact-open topology on the space of discrete faithful representations.) Marden [Mar 1] showed that $CC_0(M)$ is an open subset of $AH(\pi_1(M))$. Bers, Kra and Maskit (see [Bers 2]) showed that $CC_0(M)$ may be parameterized as the quotient $\mathcal{T}(\partial M)/\operatorname{Mod}_0(M)$ where $\mathcal{T}(\partial M)$ is the Teichmüller space of all (marked) conformal structures on $M$ and $\operatorname{Mod}_0(M)$ is the group of all isotopy classes of homeomorphisms of $M$ which are homotopic to the identity.

A collection $C$ of disjoint simple closed curves in a surface $S$ is called a *pants decomposition* of $S$ if each component of $S - C$ is an open pair of pants. A discrete, faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ is called a *maximally cusped uniformization of* $M$ if there exists a pants decomposition $C$ of $\partial M$ and an orientation-preserving homeomorphism $h : M - C \to \bar{N}_\rho$ such that $[h_*] = [\rho]$. The conjugacy class of a maximally cusped uniformization of $M$ will be called a *maximal cusp* for $M$. Every maximally cusped uniformization of $M$ lies in the boundary of $CC_0(M)$ (see [Ke-Mas-Se, Theorem III] and [Oh, Theorem 5.1].)

The following theorem is the main result of the paper.

**Theorem 6.1** (Approximations by maximal cusps) *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold whose boundary is non-empty and contains no tori. If $[\rho]$ is an element of $\partial CC_0(M)$ such that $\Omega(\rho) = \emptyset$, then $[\rho]$ is the limit of a sequence of maximal cusps in $\partial CC_0(M)$.*

If $H_g$ is a handlebody of genus $g$, then $CC_0(H_g)$ is known as *Schottky space.* We may combine our main theorem with Marden's observation that there is a dense subset in the boundary of Schottky space consisting of conjugacy classes of representations with empty domain of discontinuity to obtain the following immediate corollary.

**Corollary 15.1** *If $H_g$ is a handlebody of genus $g \geq 2$, then maximal cusps are dense in the boundary of the Schottky space $CC_0(H_g)$.*

One may combine the main theorem with work of Anderson, Canary, Kapovich, Minsky and Sullivan to obtain the following generalization of Corollary 15.1.

**Corollary 15.3** *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold with (non-empty) connected boundary which is not a torus. Then maximal cusps are dense in the boundary of $CC_0(M)$.*

In a final section, we notice that the same techniques may be used to obtain an analogue of our main theorem in the setting of geometrically finite uniformizations of pared 3-manifolds. In particular, we will generalize our results to the setting where $M$ is allowed to have toroidal boundary components. We then combine this result with work of Anderson, Canary, Evans, Kapovich and Sullivan, to show that maximal cusps are dense in the boundary of the quasiconformal deformation space of any geometrically finite hyperbolic 3-manifold with connected conformal boundary.

The outline of a proof of Corollary 15.1 was provided to the authors of the present paper by Curt McMullen. The result was quoted as Theorem 8.9 of [CS2] and was used in the proof of Theorem 8.2 of [CS2]. The latter theorem was also quoted and used on page 23 of [CHS]. Likewise, Corollary 15.1 was quoted in the discussion beginning Section 5 of [ACCS] and was used in the proof of Theorem 5.2 of [ACCS]. These applications of Corollary 15.1 were crucial to the proofs of the main results of [CS2], [ACCS] and [CHS], and therefore to the results of the subsequent papers [CS3], [CS4], [CS5], [CS6]. In these papers, the authors obtain lower bounds for the volume of a hyperbolic 3-manifold under a variety of topological restrictions. No proof of Corollary 15.1 has appeared to date, and it is hoped that the present paper will fill the resulting gap in the literature.

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## 2 Background Material and Outline

In this section, we survey some of the basic material from Teichmüller theory and the deformation theory of Kleinian groups which will be used throughout the paper. We also explain the construction of a metric on the space $AH(\pi_1(M))$ of conjugacy classes of discrete faithful representations which will be used in the proof. We finish by giving a brief outline of the argument.

We recommend the books of Gardiner [Ga] and Imayoshi-Taniguchi [ImTa] as sources for Teichmüller theory and the papers by Bers [Bers 2] and Canary-McCullough [CaMc] as references for the deformation theory of Kleinian groups.

### 2.1 Beltrami differentials and quadratic differentials

Let $X$ be a finite type Riemann surface. A *Beltrami differential* $\mu$ is a differential of type $(-1,1)$ on $X$, i.e. it is given in a local coordinate $z$ as $f(z)\frac{d\bar{z}}{dz}$, where $f$ is a measurable function. If $w(z)$ is another local coordinate then $\mu$ is written as $g(w)\frac{d\bar{w}}{dw}$ where $g(w(z)) = f(z)\overline{\frac{dw}{dz}}/\frac{dw}{dz}$. The function $f$ depends on the choice of coordinate, but the modulus $|\mu|(z) = |f(z)|$ of $f$ is invariant under change of coordinates and hence is a globally defined real-valued measurable function on $X$. Let $B(X)$ denote the space of bounded Beltrami differentials on $X$ with $L^\infty$-norm

$$||\mu|| = \sup_{x\in X} |\mu|(x)$$

and let $B_1(X) \subset B(X)$ denote the open unit ball in $B(X)$.

The Teichmüller space $\mathcal{T}(X)$ of all marked Riemann surfaces which are quasi-conformally homeomorphic to $X$ can be identified with $B_1(X)/\mathcal{Q}_0(X)$, where $\mathcal{Q}_0(X)$ denotes the group of all quasiconformal self-homeomorphisms of $X$ that are homotopic to the identity. The marked Riemann surface $X$ is the image of the origin under this quotient map, and will be referred to as the basepoint of $\mathcal{T}(X)$. The Teichmüller space $\mathcal{T}(X)$ can also be thought of as the space of pairs $(Y, f)$ where $Y$ is a Riemann surface and $f : X \to Y$ is a quasiconformal homeomorphism, and where two pairs $(Y_1, f_1)$ and $(Y_2, f_2)$ are equivalent if there exists a conformal map $g : Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$. The pair $(Y, f)$ can be identified with the equivalence class of the Beltrami differential $(f_{\bar{z}}/f_z)\frac{d\bar{z}}{dz}$ of $f$.

Let $B_0(X)$ denote the the orbit of $0 \in B_1(X)$ under $\mathcal{Q}_0(X)$. Elements of $B_0(X)$ are called *trivial Beltrami differentials*, since they are the Beltrami differentials of quasiconformal homeomorphisms in $\mathcal{Q}_0(X)$. We let $\Phi : B_1(X) \to B_1(X)/\mathcal{Q}_0(X)$ be the projection map. In this situation, the tangent space to the image of the origin in the quotient $\mathcal{T}(X) = B_1(X)/\mathcal{Q}_0(X)$ can be identified with the vector space quotient of the tangent space of $B_1(X)$ at 0 by the tangent space to $B_0(X)$ at 0.

One may use quadratic differentials to define a norm on the tangent space of $\mathcal{T}(X)$. A quadratic differential $\phi$ is given locally as $f(z)dz^2$. If $w(z)$ is another local coordinate then $\phi$ can be written as $g(w)dw^2$ where $g(w(z)) = f(z)(\frac{dw}{dz})^2$. The modulus $|\phi|$ of a quadratic differential is a non-negative real-valued 2-form on the Riemann surface $X$ and thus has a well defined integral over $X$. We obtain a finite dimensional normed linear space by defining $Q(X)$ to be space of holomorphic quadratic differentials $\phi$ on $X$ such that

$$||\phi|| = \int_X |\phi| < \infty.$$

Suppose that $\mu$ is a Beltrami differential on $X$ and that $\phi$ is a quadratic differential on $X$ given respectively in a local coordinate $z$ as $\mu = f(z)\frac{d\bar{z}}{dz}$ and $\phi = g(z)dz^2$. The product of $\mu$ and $\phi$, which is given in the local coordinate $z$ as $\phi\mu = f(z)g(z)dzd\bar{z}$, is a complex valued 2-form on $X$ and hence has a well-defined integral over $X$. We may therefore define a real-valued bilinear pairing between $B(X)$ and $Q(X)$ by the formula

$$\langle \phi, \mu \rangle = \mathrm{Re} \int_X \phi\mu. \tag{1}$$

Let $N(X)$ denote the set of all $\mu \in B(X)$ such that $\langle \phi, \mu \rangle = 0$ for all $\phi \in Q(X)$. (These are called *infinitesimally trivial Beltrami differentials.*) It can be shown that $N(X)$ is equal to the tangent space of $B_0(X)$ at the origin and that the tangent space to the Teichmüller space $\mathcal{T}(X)$ at the base point can be identified with the quotient $B(X)/N(X)$. We abuse notation by identifying tangent vectors at the origin of $B_1(X)$ with elements of the linear space $B(X)$. Associated to such a tangent vector $\mu \in B(X)$ at the origin of $B_1(X)$ there is then a tangent vector $D\Phi(\mu)$ to $\mathcal{T}(X)$ at the base point of the Teichmüller space. The Teichmüller metric is a Finsler metric with infinitesimal form

$$||D\Phi(\mu)|| = \sup\{\langle \phi, \mu \rangle \mid \phi \in Q(X),\ ||\phi|| = 1\}. \tag{2}$$

If $Y$ is a marked Riemann surface which is quasiconformally homeomorphic to $X$, we may view $Y$ as a point of $\mathcal{T}(X)$. There is a canonical identification of the tangent space of $\mathcal{T}(X)$ at $Y$ with the tangent space of $\mathcal{T}(Y)$ at its basepoint. This gives rise to a norm on the tangent space at any point of $\mathcal{T}(X)$ which is the infinitesimal form of the Teichmüller metric on $\mathcal{T}(X)$.

## 2.2 Controlled pinching

It will be important for us to know that one can pinch the length of a curve of length less than L in half in a controlled manner. In particular, the pinching is accomplished by a bounded length deformation in Teichmuller space such that the tangent vector the path at each point is represented by a unit norm Beltrami differential supported on the $2L$-thin part of the surface. Although this fact is well-known in Teichmüller theory we will provide an outline of the proof.

If $X$ is a finite type Riemann surface which is not homeomorphic to a sphere, a torus or an annulus, then the conformal structure on $X$ is compatible with a unique hyperbolic metric, called the *Poincaré metric*. If $x$ is a point of $X$ then we define $\mathrm{inj}_X(x)$ to be half the length of the shortest non-trivial loop through $x$, measured in the Poincaré metric. The *L-thin part* of $X$ is the set of points on which $\mathrm{inj}_X(x) \leq L$. We say that $x$ is in the $L$-thin part associated to a curve $\gamma$ on $X$, if there exists a non-trivial loop based at $x$ which is homotopic to $\gamma$ and has length at most $2L$.

If $\gamma$ is a simple closed curve on a finite area hyperbolic surface $X$ we will write $l_X(\gamma)$ for the length of the closed geodesic in the homotopy class of $\gamma$.

We remind the reader that the Euclidean annulus $A(R) = \{z : 1 < |z| < R\}$ has conformal *modulus* $\mathrm{mod}(A) = \log R$. Notice that $A(R)$ is conformally isomorphic to a right cylinder of height $\mathrm{mod}(A)/2\pi$ and of circumference 1. Since any annulus in a Riemann surface is conformal to $A(R)$ for some (unique) $R$, any such annulus has a well-defined modulus.

If $\beta : [0, B] \to \mathcal{T}(X)$ is a differentiable path and $\beta(t) = (X_t, g_t)$, then for each $t \in [0, B]$ there is a map $\Phi_t : B_1(X_t) \to \mathcal{T}(X)$ such that $\Phi_t(0) = \beta(t)$. The tangent vector $\beta'(t)$ then lives in $D\Phi_t(B(X_t))$ which may be identified with the tangent space to $\mathcal{T}(X)$ at $\beta(t)$.

**Lemma 2.1** *Let $L_0 > 0$ be given, let $X$ be a finite area hyperbolic surface, and let $\gamma$ be a simple closed geodesic on $X$ of length $L \leq L_0$. There exists a positive number $B$ depending only on $L_0$ and a differentiable path $\beta : [0, B] \to \mathcal{T}(X)$, with $\beta(t) = (X_t, g_t)$, such that the following conditions hold:*

1. *$\beta(0) = (X, id)$,*
2. *for all $t \in [0, B]$ we have $l_{X_t}(g_t(\gamma)) \leq L$,*
3. *$\beta'(t) = D\Phi_t(\mu_t)$, $||\mu_t|| \leq 1$ and the support of $\mu_t$ is contained in the $2L$-thin part associated to the curve $g_t(\gamma)$, and*
4. *$l_{X_B}(g_B(\gamma)) \leq \frac{L}{2}$.*

Recall that $||\mu_t||$ denotes the $L^\infty$-norm of $|\mu_t|$. Thus, applying equation (2) in section 2.1, we see that $||\mu_t|| \leq 1$ implies that $||\beta'(t)|| \leq 1$, so $\beta([0, B])$ has length at most $B$ in $\mathcal{T}(X)$.

*Proof of 2.1:* One may argue as in the proof of Proposition 2 in Maskit [Mas 2] to show that there is a constant $a > 0$ (depending only on $L_0$) such that if $X$ is a finite area hyperbolic surface and $\gamma$ is a geodesic of length $L \leq L_0$, then there is an annulus $A$ contained within the $2L$-thin part associated to $\gamma$ such that $\gamma$ is a core curve of $A$ and $\mathrm{mod}(A) \geq \frac{a}{L}$. Moreover, Proposition 1 of [Mas 2] shows that if $A'$ is an incompressible annulus in a finite area hyperbolic surface with modulus at least $\frac{4\pi^2}{L}$, then the geodesic homotopic to the core curve of $A'$ has length at most $\frac{L}{2}$. Set

$$B = \frac{1}{2}\log\left(\frac{4\pi^2}{a}\right).$$

Let $A$ be the annulus provided by the previous paragraph and let $m = \mathrm{mod}(A)$. We may conformally identify $A$ with the planar annulus

$$\{1 < |z| < e^m\}$$

and let

$$A_t = \{1 < |z| < e^{me^{2t}}\}.$$

Let $f_t : A \to A_t$ be the Teichmüller map

$$f_t(z) = z|z|^{e^{2t}-1}$$

with associated Beltrami differential $\mu_t$. The resulting path $\widehat{\beta} : [0, \infty) \to \mathcal{T}(A)$ is a unit speed geodesic in $\mathcal{T}(A)$ and each tangent vector $\widehat{\beta}'(t)$ is represented by a unit norm Beltrami differential supported on $A_t$.

Extend each $\mu_t$ to a Beltrami differential $\widehat{\mu}_t$ on $X$ by setting $\widehat{\mu}_t$ equal to 0 on $X - A$. Then one may use the Measurable Riemann Mapping Theorem to produce a path of quasiconformal maps $\{g_t : X \to X_t\}$ and hence a path $\beta : [0, B] \to \mathcal{T}(X)$. For each $t$ in the interval $[0, B]$ the annulus $g_t(A)$, is conformally equivalent to $A_t$ and the tangent vector $\beta'(t)$ is represented by a unit-norm Beltrami differential supported on $g_t(A)$. Moreover, $g_B(A)$ has modulus at least $\frac{4\pi^2}{L}$, so $l_{X_B}(g_t(\gamma)) \leq \frac{L}{2}$ as required.

It remains to check that for all $t \in [0, B]$ we have that $l_{X_t}(g_t(\gamma)) \leq L$ and that $g_t(A)$ is contained in the $2L$-thin part of $X_t$. Notice that $X_t$ is conformally equivalent to the surface obtained by cutting $X$ along $\gamma$ and inserting a Euclidean annulus $E_t$ whose core geodesic has length $L$. The conformal equivalence maps the annulus $g_t(A)$ to the annulus $A_t'$ which is obtained by cutting $A$ along $\gamma$ and inserting $E_t$.

It follows from Proposition 2.2 of Tanigawa [Tan] (see also the discussion preceding Theorem 3.1 of McMullen [Mc 2]) that the hyperbolic metric on $X_t$ is (pointwise) bounded from above by the singular metric which agrees with the hyperbolic metric on $X$ and with the Euclidean metric on $E_t$. Since $A$ is contained in the $2L$-thin part of $X$ associated to $\gamma$, we see immediately $g_t(A)$ is contained in the $2L$-thin part of $X_t$ associated to $g_t(\gamma)$ and that $l_{X_t}(g_t(\gamma)) \leq L$.

$\boxed{2.1}$

## 2.3 The deformation space $CC_0(M)$

Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold with non-empty boundary and no toroidal boundary components. Let $[\rho_0]$ denote a fixed conjugacy class in $CC_0(M)$. Then there exists a homeomorphism $h : M \to \bar{N}_{\rho_0}$ such that $[h_*] = [\rho_0]$ and $\partial_c N_{\rho_0}$ is a Riemann surface homeomorphic to $\partial M$. If $[\rho]$ is another conjugacy class in $CC_0(M)$ then Marden's Isomorphism theorem [Mar 1] implies that there exists a quasiconformal map $\tilde{\phi} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $\tilde{\phi}\rho_0(g)\tilde{\phi}^{-1} = \rho(g)$ for all $g \in \pi_1(M)$. The map $\tilde{\phi}$ descends to a quasiconformal map $\phi : \partial_c N_{\rho_0} \to \partial_c N_\rho$ and $\phi$ extends to a homeomorphism $\bar{\phi} : \bar{N}_{\rho_0} \to \bar{N}_\rho$ such that $[(\bar{\phi} \circ h)_*] = [\rho]$. In particular, $(\partial_c N_\rho, \phi)$ may be thought of as a point in the Teichmüller space $\mathcal{T}(\partial_c N_{\rho_0})$ of all (marked) Riemann surfaces homeomorphic to $\partial_c N_{\rho_0}$. However, if we precompose $\phi$ by a quasiconformal self-map $\psi$ of $\partial_c N_{\rho_0}$ which extends to a homeomorphism of $\bar{N}_{\rho_0}$ that is homotopic to the identity, then $(\partial_c N_\rho, \phi \circ \psi)$ is another point in $\mathcal{T}(\partial_c N_{\rho_0})$ which is naturally associated to $\rho$. Using work of Bers, Kra and Maskit (see Bers [Bers 2] or [CaMc]) we may identify $CC_0(M)$ with $\mathcal{T}(\partial_c N_{\rho_0})/\operatorname{Mod}_0(\rho_0)$, where $\operatorname{Mod}_0(\rho_0)$ denotes the group of isotopy classes of quasiconformal automorphisms of $\partial_c N_{\rho_0}$ which extend to maps of $\bar{N}_{\rho_0}$ that are homotopic to the identity. Maskit [Mas 1] showed that $\operatorname{Mod}_0(\rho_0)$ acts freely and properly discontinuously on $\mathcal{T}(\partial_c N_{\rho_0})$.

Since $\partial_c N_{\rho_0}$ is homeomorphic to $\partial M$ we may identify $\mathcal{T}(\partial_c N_{\rho_0})$ with $\mathcal{T}(\partial M)$ and $\operatorname{Mod}_0(\rho_0)$ with the group $\operatorname{Mod}_0(M)$ of isotopy classes of homeomorphisms of $M$ that are homotopic to the identity. The space $\mathcal{T}(\partial M)$ may be identified with the set of equivalence classes of pairs $(Y, f)$, where $Y$ is a Riemann surface, $f : \partial M \to Y$ is an orientation-preserving homeomorphism, and two pairs $(Y_1, f_1)$ and $(Y_2, f_2)$ are equivalent if there exists a conformal map $g : Y_1 \to Y_2$ that is homotopic to $f_2 \circ f_1^{-1}$. For the remainder of the paper, we will identify $CC_0(M)$ with $\mathcal{T}(\partial M)/\operatorname{Mod}_0(M)$ and let $q_M : \mathcal{T}(\partial M) \to CC_0(M)$ denote the quotient map. With this identification, if $(Y, f) \in \mathcal{T}(\partial M)$ and $q_M(Y, f) = [\rho]$, then one may identify $\partial_c N_\rho$ with the Riemann surface $Y$ and $f$ extends to a homeomorphism $\bar{f} : M \to \bar{N}_\rho$ such that $[\bar{f}_*] = [\rho]$.

If $a$ is any element of $\pi_1(M)$, then there is a natural map $\Upsilon_a : CC_0(M) \to CC_0(S^1 \times D^2)$ given by $\Upsilon_a([\rho]) = [\rho_a]$ where $\rho_a$ denotes the restriction of $\rho$ to the cyclic subgroup $\langle a \rangle$ of $\pi_1(M)$ generated by $a$. We can identify $CC_0(S^1 \times D^2)$ with $\mathcal{T}(T^2)/\operatorname{Mod}_0(S^1 \times D^2)$. Explicitly, $\mathcal{T}(T^2)$ can be identified with $\mathbf{H}^2$ so that $\operatorname{Mod}_0(S^1 \times D^2)$ is generated by $z \mapsto z + 1$. Let $q_T : \mathcal{T}(T^2) \to CC_0(S^1 \times D^2)$ denote the quotient map. Notice that, since $\mathcal{T}(\partial M)$ is simply connected, $\Upsilon_a$ lifts to a map $\tilde{\Upsilon}_a : \mathcal{T}(\partial M) \to \mathcal{T}(T^2)$.

Since $\operatorname{Mod}_0(M)$ and $\operatorname{Mod}_0(S^1 \times D^2)$ act freely, properly discontinuously and by isometries (of the Teichmüller metrics) on $\mathcal{T}(\partial M)$ and $\mathcal{T}(T^2)$, both $CC_0(M)$ and $CC_0(S^1 \times D^2)$ inherit the structure of a smooth manifold with a quotient Teichmüller metric. It follows that if $q_M(Y, f) = [\rho]$ then we have an identification between the tangent spaces $T_{[\rho]}(CC_0(M))$ and $T_{(Y,f)}(\mathcal{T}(\partial M))$ and between the tangent spaces $T_{\Upsilon_a([\rho])}(CC_0(S^1 \times D^2))$ and $T_{\tilde{\Upsilon}_a(Y,f)}(\mathcal{T}(T^2))$. Moreover, we have a projection map $\overline{\Phi} = q_M \circ \Phi$ from $B_1(Y) \to CC_0(M)$ whose derivative, at the basepoint, agrees with $D\Phi$ once we have identified $T_{[\rho]}(CC_0(M))$ with $T_{(Y,f)}(\mathcal{T}(\partial M))$.

If we set $\Gamma = \rho(\pi_1(M))$, then we may identify $B_1(Y)$ with the space $B_1(\Omega(\Gamma), \Gamma)$ of $\Gamma$-

invariant Beltrami differentials on $\Omega(\Gamma)$. (A Beltrami differential $\tilde{\mu}$ on $\Omega(\Gamma)$ is said to be $\Gamma$-invariant if $\tilde{\mu}(\gamma(z)) = \tilde{\mu}(z)\left(\gamma'(z)\ /\ \overline{\gamma'(z)}\right)$ for all $\gamma \in \Gamma$, in which case $\tilde{\mu}$ is a lift of a Beltrami differential $\mu$ on $Y$.) Similarly, we may identify $B_1(\widetilde{\Upsilon}_a(Y))$ with $B_1(\Omega(\langle\rho(a)\rangle), \langle\rho(a)\rangle)$. The map $\widetilde{\Upsilon}_a$ lifts again to a map

$$\widehat{\Upsilon}_a : B_1(\Omega(\Gamma), \Gamma) \to B_1(\Omega(\langle\rho(a)\rangle), \langle\rho(a)\rangle)$$

which is simply the inclusion map. (Formally, if $\mu \in B_1(\Omega(\Gamma), \Gamma)$, then $\widehat{\Upsilon}_a(\mu)$ is obtained from $\mu$ by setting $\widehat{\Upsilon}_a(\mu)$ equal to 0 at any point in $\Omega(\Gamma) - \Omega(\langle\rho(a)\rangle$.) We thus have the following commutative diagram:

$$\begin{array}{ccc}
B_1(Y) & \xrightarrow{\widehat{\Upsilon}_a} & B_1(\widetilde{\Upsilon}_a(Y)) \\
\Phi \downarrow & & \Phi \downarrow \\
\mathcal{T}(\partial M) & \xrightarrow{\widetilde{\Upsilon}_a} & \mathcal{T}(T^2) \\
q_M \downarrow & & q_T \downarrow \\
CC_0(M) & \xrightarrow{\Upsilon_a} & CC_0(S^1 \times D^2)
\end{array}$$

Taking derivatives we obtain:

$$\begin{array}{ccc}
B(Y) & \xrightarrow{D\widehat{\Upsilon}_a} & B(\widetilde{\Upsilon}_a(Y)) \\
\mathrm{D}\Phi \downarrow & & \mathrm{D}\Phi \downarrow \\
T_{(Y,f)}(\mathcal{T}(\partial M)) & \xrightarrow{D\widetilde{\Upsilon}_a} & T_{\widetilde{\Upsilon}_a(Y,f)}(\mathcal{T}(T^2)) \\
\mathrm{D}q_M \downarrow & & \mathrm{D}q_T \downarrow \\
T_{[\rho]}(CC_0(M)) & \xrightarrow{D\Upsilon_a} & T_{\Upsilon_a([\rho])}(CC_0(S^1 \times D^2))
\end{array} .$$

Notice that $D\widehat{\Upsilon}_a : B(Y) \to B(\widetilde{\Upsilon}_a(Y))$ is simply the inclusion map from $B(\Omega(\Gamma), \Gamma)$ to $B(\Omega(\langle\rho(a)\rangle), \langle\rho(a)\rangle)$, since it is the derivative of the inclusion map. In particular, if $D\Phi(\mu) \in T_Y(\mathcal{T}(\partial M))$ and $\tilde{\mu}$ is a lift of $\mu$ to $\Omega(\Gamma)$, then $\tilde{\mu}$ is also the lift of a representative of $D\widetilde{\Upsilon}_a(D\Phi(\mu))$ to $\Omega(\langle\rho(a)\rangle)$.

## 2.4 Pinchable collections of curves

If $M$ is a compact, oriented, atoroidal, irreducible 3-manifold with no toroidal boundary components, then a disjoint collection of non-parallel simple closed curves $C$ in $\partial M$ is said to be *pinchable* if every curve in $C$ is homotopically non-trivial in $M$ and if there is no essential annulus in $M$ both of whose boundary components are homotopic in $\partial M$ to curves in $C$.

It is a consequence of Thurston's uniformization theorem (see Morgan [Mo]) that if $C$ is any pinchable pants decomposition of $\partial M$, then there exists a maximal cusp $[\rho]$ such that $M - C$ is homeomorphic to $\bar{N}_\rho$ by an orientation-preserving homeomorphism $h : M - C \to N_\rho$ with $[h_*] = [\rho]$. Keen, Maskit and Series [Ke-Mas-Se] established that the maximal cusp $[\rho]$ is uniquely determined by the free homotopy classes (in $M$) of the curves in $C$.

If $\rho \in CC_0(M)$ and $C = \{c_1, \ldots, c_m\}$ is a pinchable collection of curves in $\partial_c N_\rho$ then each element of $C$ is associated to a conjugacy class of hyperbolic elements of $\rho(\pi_1(M))$, since each element of $C$ is homotopically non-trivial in $\bar{N}_\rho$ and every non-trivial element of $\rho(\pi_1(M))$ is hyperbolic. The conjugacy classes are distinct, since otherwise there would be an essential annulus in $\bar{N}_\rho$ joining distinct elements of $C$. Each element in the conjugacy class determined by an element of $C$ is primitive, since otherwise there would be an essential annulus in $\bar{N}_\rho$ with both boundary components parallel to a single component of $C$.

## 2.5 A metric on $AH(\pi_1(M))$

Proposition 2.2 below assures us that we may find a finite collection of elements of $\pi_1(M)$ whose squared traces give rise to a proper embedding of $AH(\pi_1(M))$ into $\mathbf{C}^m$ for some $m$. We may then use this embedding to construct a metric on $AH(\pi_1(M))$.

Let $G$ be a finitely generated group. If we let $\tau_g(\rho)$ denote the square of the trace of $\rho(g)$, then $\tau_g$ is a well defined continuous function on $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$. (Notice that although the trace of an element of $\mathrm{PSL}_2(\mathbf{C})$ is not well defined, its square is well defined.) Since $\tau_g$ is invariant under conjugation, it descends to a continuous function $\bar{\tau}_g : AH(G) \to \mathbf{C}$.

**Proposition 2.2** *Let $M$ be a compact, orientable, irreducible, atoroidal 3-manifold whose boundary has a non-torus component. Suppose that $F$ is a finite set of primitive conjugacy classes in $\pi_1(M)$. Then there exists a finite set $\{a_1, \ldots, a_m\}$ of primitive elements of $\pi_1(M)$ such that*

1. *no $a_i^{\pm 1}$ belongs to a conjugacy class in $F$;*
2. *if $[\rho_1], [\rho_2] \in AH(\pi_1(M))$ and $\bar{\tau}_{a_i}([\rho_1]) = \bar{\tau}_{a_i}([\rho_2])$ for all $i = 1, \ldots, m$, then $[\rho_1] = [\rho_2]$; and*
3. *given any $K > 0$, the set*
$$\{[\rho] \in AH(\pi_1(M)) | \sum_{i=1}^{m} |\bar{\tau}_{a_i}(\rho)| \leq K\}$$
*is compact.*

If $\mathcal{A} = \{a_1, \ldots, a_m\}$ is a collection of primitive elements of $\pi_1(M)$ which satisfies conditions (2) and (3) of Proposition 2.2 then we call $\mathcal{A}$ an *allowable collection of test elements.* Then $\tau : AH(\pi_1(M)) \to \mathbf{C}^m$ given by $\tau(\rho) = (\tau_{a_i}(\rho))$ is a proper embedding of $AH(\pi_1(M))$ into $C^m$. We let $d_{\mathcal{A}}$ be the metric on $AH(\pi_1(M))$ which it inherits as a subset of $\mathbf{C}^m$. Explicitly,
$$d_{\mathcal{A}}([\rho_1], [\rho_2]) = \sqrt{\sum_{i=1}^{m} |\bar{\tau}_{a_i}([\rho_1]) - \bar{\tau}_{a_i}([\rho_2])|^2}.$$

The following three lemmas will be needed for the proof of Proposition 2.2. Our first lemma will be used to obtain Property 1.

**Lemma 2.3** *Let $G$ be a finitely generated group which admits a homomorphism onto $\mathbf{Z}\oplus\mathbf{Z}$. Let $F$ be a finite set of conjugacy classes in $G$. Then there is a finite set $\{h_1,\dots,h_n\}$ of generators of the group $G$ with the property that no power of any element of a conjugacy class in $F$ can be written as a non-empty positive word in $h_1,\dots,h_n$.*

*Proof of 2.3:* We may assume without loss of generality that $F^{-1}=F$. We fix a generating set $\{g_1,\dots,g_n\}$ for $G$ and a surjective homomorphism $\phi: G\to\mathbf{Z}\oplus\mathbf{Z}$. We set $v_i=\phi(g_i)\in\mathbf{Z}\oplus\mathbf{Z}$ for $i=1,\dots,n$. Since the homomorphism $\phi$ is surjective, the elements $v_1,\dots,v_n$ generate $\mathbf{Z}\oplus\mathbf{Z}$. If we regard $\mathbf{Z}\oplus\mathbf{Z}$ as a lattice in $\mathbf{R}^2$, it follows that two of the vectors $v_i$, which after re-indexing we may take to be $v_1$ and $v_2$, are linearly independent in $\mathbf{R}^2$. Hence if $L_j$ denotes the linear subspace of $\mathbf{R}^2$ spanned by $v_1+jv_2$ for each $j\geq 0$, the $L_j$ are all 1-dimensional and pairwise distinct.

We fix a finite set $S\subset\mathbf{Z}\oplus\mathbf{Z}-\{0\}\subset\mathbf{R}^2-\{0\}$ such that $\phi$ maps every conjugacy class in $F$ to an element of $S\cup\{0\}$. Since the $L_i$ are distinct 1-dimensional subspaces of $\mathbf{R}^2$ and $0\notin S$, there is an integer $m\geq 0$ such that $L_m\cap S=\emptyset$. We set $L=L_m$. We also set $d_1=g_1g_2^m$, and observe that $\{d_1,g_2,\dots,g_n\}$ is a generating set for $G$, and that $w_1=\phi(d_1)=v_1+mv_2$ is a non-zero vector in $L$. Let $r\subset L$ denote the open ray from the origin which contains $w_1$. Since the finite set $S$ is disjoint from $L$, there is an open neighborhood $V$ of $r$ in $\mathbf{R}^2$, whose frontier is the union of two rays from the origin, such that $S\cap V=\emptyset$. Note that $0\notin V$.

For $i=2,\dots,n$ and for each $k\geq 0$, let us set $d_i^{(k)}=g_id_1^k$, and $w_i^{(k)}=\phi(d_i^{(k)})=v_i+kw_1$. For each $k\geq 0$ the set $\{d_1,d_2^{(k)},\dots,d_n^{(k)}\}$ generates $G$. We have

$$\frac{w_i^{(k)}}{k}=\frac{v_i}{k}+w_1\to w_1\in r$$

as $k\to\infty$, and since $V$ is invariant under positive dilatations it follows that for large enough $k$ we have $w_i^{(k)}\in V$ for $i=2,\dots,n$. Fixing such a $k$, we claim that the generating set $\{d_1,d_2^{(k)},\dots,d_n^{(k)}\}$ has the property stated in the lemma, that no power of any element of a conjugacy class in $F$ can be written as a non-empty positive word in $d_1,d_2^{(k)},\dots,d_n^{(k)}$.

Indeed, suppose to the contrary that for some element $a\in G$ whose conjugacy class belongs to $F$, some power of $a$, say $a^h$ with $h\in\mathbf{Z}$, can be written as a non-empty positive word in $d_1,d_2^{(k)},\dots,d_n^{(k)}$. Since $F^{-1}=F$ we may take $h$ to be non-negative. Set $z=\phi(a)$. Then $hz=\phi(a^h)$ is a linear combination, with strictly positive coefficients, of some non-empty subset of $\{w_1,w_2^{(k)},\dots,w_n^{(k)}\}\subset V$. As $V$ is clearly invariant under positive linear combinations it follows that $hz\in V$ and hence that $z\in V$. But since the conjugacy class of $a$ belongs to $F$ we have $z\in S\cup\{0\}$, and the latter set is disjoint from $V$.

$\boxed{2.3}$

In the remainder of the section we will make use of the theory of the $\mathrm{SL}_2(\mathbf{C})$-character variety of $G$, which is presented in Section 1 of [CS1]. Let $R(G)=\mathrm{Hom}(G,\mathrm{SL}_2(\mathbf{C}))$. For each $g\in G$ let $t_g: R(G)\to\mathbf{C}$ be the function defined by $t_g(\rho)=\operatorname{trace}\rho(g)$. Let $T$ denote the ring generated by all functions of the form $t_g$ for $g\in G$. The discussion given in [CS1] depends on a finite subset $W \subset G$ such that the functions $t_g$ for $g \in W$ generate $T$; in what follows we shall fix a generating set $\{h_1, \ldots, h_n\}$ for $G$, and take $W$ to consist of all elements of the form $h_{i_1} \cdots h_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$. It follows from Proposition 4.4.2 of [Sh] that $W$ has the required property. According to the definition given in [CS1], the $\mathrm{SL}_2(\mathbf{C})$-character variety of $G$, denoted by $X(G)$, is the set $t(R(G)) \subset \mathbf{C}^W$, where $t : R(G) \to \mathbf{C}^W$ is the map defined by $t(\rho) = (t_g(\rho))_{g \in W}$. Corollary 1.4.5 of [CS1] asserts that $X(G)$ is a closed affine algebraic subset of $\mathbf{C}^W$. Moreover (see Proposition 1.5.2 in [CS1]) if $\rho_1, \rho_2 \in R(G)$ are irreducible, then $t(\rho_1) = t(\rho_2)$ if and only if $\rho_1$ and $\rho_2$ are conjugate.

The following result is standard and we will omit its elementary proof.

**Lemma 2.4** *Let $G$ be a finitely generated group, and let $(\rho_i)_{i \ge 0}$ be a sequence of representations in $R(G)$ such that $(t(\rho_i))_{i \ge 0}$ converges in $X(G)$ to $t(\rho)$, where $\rho \in R(G)$ is an irreducible representation. Then there is a sequence of representations $(\rho_i')_{i \ge 0}$ in $R(G)$ such that $\rho_i'$ is conjugate to $\rho_i$ for each $i \ge 0$, and $(\rho_i')_{i \ge 0}$ converges to $\rho$ in $R(G)$.*

The next lemma will allow us to obtain properties 2 and 3 of Proposition 2.2. We recall that a representation into $\mathrm{PSL}_2(\mathbf{C})$ or $\mathrm{SL}_2(\mathbf{C})$ is *reducible* if it is conjugate to a representation whose image lies entirely in the subgroup of upper triangular matrices, otherwise the representation is called *irreducible.*

**Lemma 2.5** *Let $G$ be a finitely generated, non-abelian, torsion-free group, let $\{h_1, \ldots, h_n\}$ be a generating set for $G$, and let $Q$ denote the set of all elements of $G$ that may be written as positive words of length at most $n+1$ in the $h_i$. If $\rho_1$ and $\rho_2$ are irreducible representations of $G$ in $\mathrm{PSL}_2(\mathbf{C})$ such that $\tau_g(\rho_1) = \tau_g(\rho_2) \neq 0$ for every $g \in Q$, then $\rho_1$ and $\rho_2$ are conjugate. Furthermore, if $K > 0$ then*

$$\mathcal{K} = \{[\rho] \in AH(G) \mid \sum_{g \in Q} |\bar{\tau}_g([\rho])| \le K\}$$

*is a compact subset of $AH(G)$.*

*Proof of 2.5:* To prove the first assertion of the lemma, we consider the free group $F_n$ on generators $x_1, \ldots, x_n$ and the homomorphism $h : x_i \mapsto h_i$ from $F_n$ to $G$. It is clearly enough to prove that the conclusion holds when $G$, $\rho_1$ and $\rho_2$ are replaced by $F_n$, $\rho_1 \circ h$ and $\rho_2 \circ h$; hence we may assume without loss of generality that $G = F_n$ is free on the generators $h_1, \ldots, h_n$.

For $i = 1, \ldots, n$ we choose matrices $\tilde{A}_i^{(1)}, \tilde{A}_i^{(2)} \in \mathrm{SL}_2(\mathbf{C})$ which map to $\rho_1(h_i)$ and $\rho_2(h_i)$ under $\pi$. Since $\tau_{h_i}(\rho_1) = \tau_{h_2}(\rho_2)$, we may choose the $A_i^{(j)}$ in such a way that $\operatorname{trace} A_i^{(1)} = \operatorname{trace} A_i^{(2)}$ for $i = 1, \ldots n$. For $j = 1, 2$ we define a representation $\tilde{\rho}_j : F_n \to \mathrm{SL}_2(\mathbf{C})$ by $\tilde{\rho}_j(h_i) = A_i$, so that $\pi \circ \tilde{\rho}_j = \rho_j$.

We claim that $t(\tilde{\rho}_1) = t(\tilde{\rho}_2)$. By definition this means that $\operatorname{trace} \tilde{\rho}_1(g) = \operatorname{trace} \tilde{\rho}_2(g)$ for every $g \in W \subset Q$. Any element $g \in W$ may by definition be written in the form $h_{i_1} \cdots h_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$, and we prove the required equality by induction on

$k \geq 1$. For $k = 1$ the equality follows from our choice of the $A_i^{(j)}$. For $k > 1$ we may write $g = ab$ where $a = h_{i_1} \cdots h_{i_{k-1}}$ and $b = h_{i_k}$, so that by the induction hypothesis we have $\operatorname{trace} \tilde\rho_1(a) = \operatorname{trace} \tilde\rho_2(a) = \alpha$, say, and by the base case $\operatorname{trace} \tilde\rho_1(b) = \operatorname{trace} \tilde\rho_2(b) = \beta$, say. Since $g \in W \subset Q$, the hypothesis gives $\tau_g(\rho_1) = \tau_g(\rho_2)$. Hence $\operatorname{trace} \tilde\rho_1(g) = \pm \operatorname{trace} \tilde\rho_2(g)$. To complete the induction we will assume that $\operatorname{trace} \tilde\rho_1(g) = -\operatorname{trace} \tilde\rho_2(g)$ and obtain a contradiction. Set $\gamma = \operatorname{trace} \tilde\rho_1(g)$, so that $\operatorname{trace} \tilde\rho_2(g) = -\gamma$. From the identity

$$(\operatorname{trace} Y)(\operatorname{trace} Z) = \operatorname{trace} YZ + \operatorname{trace} YZ^{-1},$$

which holds for all $Y, Z \in \mathrm{SL}_2(\mathbf{C})$ (see [CS1], proof of Proposition 1.4.1), by setting $Y = \tilde\rho_j(g)$ and $Z = \tilde\rho^j(b)$ for $j = 1, 2$, we obtain

$$\operatorname{trace} \tilde\rho_j(gb) = (\operatorname{trace} \tilde\rho_j(g))(\operatorname{trace} \tilde\rho_j(b)) - \operatorname{trace} \tilde\rho_j(a),$$

so that $\operatorname{trace} \tilde\rho_1(gb) = \gamma\beta - \alpha$ and $\operatorname{trace} \tilde\rho_2(gb) = -(\gamma\beta + \alpha)$. But we have $gb \in Q$, so that $\tau_{gb}(\rho_1) = \tau_{gb}(\rho_2)$ and hence $\operatorname{trace} \tilde\rho_1(gb) = \pm \operatorname{trace} \tilde\rho_2(gb)$; that is,

$$\gamma\beta - \alpha = \pm(\gamma\beta + \alpha).$$

But this last equality is possible only if one of the (complex) numbers $\alpha$, $\beta$ or $\gamma$ is 0; and since $a$, $b$ and $g$ belong to $W \subset Q$, the hypothesis implies that the numbers $\tau_a(\rho_1) = \alpha^2$, $\tau_b(\rho_1) = \beta^2$ and $\tau_g(\rho_1) = \gamma^2$ are nonzero. This is the desired contradiction, and the claim is proved.

Since $\rho_j : G \to \mathrm{PSL}_2(\mathbf{C})$ is assumed to be irreducible for $j = 1, 2$ it follows that $\tilde\rho_j : G \to \mathrm{PSL}_2(\mathbf{C})$ is irreducible. Since $t(\tilde\rho_1) = t(\tilde\rho_2)$, it now follows that $\tilde\rho_1$ and $\tilde\rho_2$ are conjugate. This immediately implies that $\rho_1$ and $\rho_2$ are conjugate. This proves the first assertion in the free case. (An alternate proof of the first assertion is provided by Lemma 3.1 of [BZ].)

To establish the second assertion, we need to show that any sequence $[\rho_i]$ of points of $\mathcal{K}$ has a convergent subsequence. We construct, for each $i$, a representation $\tilde\rho_i : F_n \to \mathrm{SL}_2(\mathbf{C})$ such that $\pi \circ \tilde\rho_i = \rho_i \circ h$. For each $i \geq 0$ and each $g \in Q$, the hypothesis guarantees that $|t_g(\tilde\rho_i)| = \sqrt{|\tau_g(\rho_i)|}$ is bounded as $i \to \infty$. As this applies in particular when $g \in W$, it follows that in the sequence $(t(\tilde\rho_i)_{i \geq 0})$ of points of $X(F_n)$, all the coordinates are bounded. Hence after passing to a subsequence we may assume that $(t(\tilde\rho_i))$ converges to a point $\chi = t(\tilde\rho)$ for some $\tilde\rho \in R(F_n)$. Since $G$ is isomorphic to the torsion-free, non-abelian, discrete subgroup $\rho_0(G)$ of $\mathrm{PSL}_2(\mathbf{C})$, there exist elements $a$ and $b$ of $G$ which generate a free subgroup of rank 2. If $\tilde\rho$ were reducible, then we would have $t_a(\tilde\rho) = t_{aba^{-1}b^{-1}}(\tilde\rho) = 2$. But, since $\mathcal{K} \subset AH(G)$, Jørgensen's inequality (see [Jo, Lemma 1]) guarantees that $|t_a(\tilde\rho_i)^2 - 4| + |t_{aba^{-1}b^{-1}}(\tilde\rho_i) - 2| \geq 1$ for all $i$. Hence $\tilde\rho$ must be irreducible.

It now follows from Lemma 2.4 that the $\tilde\rho_i$ are conjugate in $\mathrm{SL}_2(\mathbf{C})$ to representations $\tilde\rho_i'$ such that $\tilde\rho_i' \to \tilde\rho$. If $P : \mathrm{SL}_2(\mathbf{C}) \to \mathrm{PSL}_2(\mathbf{C})$ denotes the natural homomorphism, it follows that $\rho_i' = P \circ \tilde\rho_i'$ converges to $\rho = P \circ \tilde\rho_i$, and hence that $[\rho_i] = [\rho_i']$ converges to $[\rho]$. But, $\rho$ is a discrete faithful representation, since the discrete faithful representations form a closed subset of $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ (see [Jo, Theorem 1].)

$\boxed{2.5}$

*Proof of Proposition 2.2:* Since $M$ is a compact orientable 3-manifold with a non-torus boundary component, it follows from Poincaré-Lefschetz duality that the first betti number of $M$ is at least 2. Hence there is a surjective homomorphism $\phi : \pi_1(M) \to \mathbf{Z} \oplus \mathbf{Z}$. Moreover, $\pi_1(M)$ is finitely generated and non-abelian. By Lemma 2.3 it follows that there is a finite set $\{h_1, \ldots, h_n\}$ of generators of the group $G$ with the property that no power of any element of a conjugacy class in $F$ can be written as a non-empty positive word in $h_1, \ldots, h_n$.

Now set $m = \sum_{i=1}^{n+1} n^i$, and let $Q = \{b_1, \ldots, b_m\}$ be the nontrivial elements of $\pi_1(M)$ which can be written as positive words of length at most $n+1$ in the $h_i$. Since $\pi_1(M)$ admits a discrete faithful representation in $\mathrm{PSL}_2(\mathbf{C})$, the centralizer of the nontrivial element $b_i \in \pi_1(M)$ is a free abelian group of rank at most 2, for $i = 1, \ldots, m$. Hence we may write $b_i$ as a positive power $a_i^{d_i}$ of some primitive element of $\pi_1(M)$. It follows from our choice of $\{h_1, \ldots, h_n\}$ that no $a_i^{\pm 1}$ belongs to a conjugacy class in $F$. This is conclusion 1 of the proposition. To prove conclusion 2, suppose that $[\rho_1], [\rho_2] \in AH(\pi_1(M))$ satisfy $\bar{\tau}_{a_i}([\rho_1]) = \bar{\tau}_{a_i}([\rho_2])$ for all $i = 1, \ldots, m$. Since $\pi_1(M)$ is non-abelian and torsion-free and $\rho_i$ is discrete and faithful, for $i = 1, 2$, $\rho_i(\pi_1(M))$ is not solvable, $\rho_i$ is irreducible and $\tau_g(\rho_i) \neq 0$ for any $g \in \pi_1(M)$. If $b_i \in Q$, then $\bar{\tau}_{a_i}([\rho_1]) = \bar{\tau}_{a_i}([\rho_2])$, which implies, since $b_i = a_i^{d_i}$, that $\bar{\tau}_g([\rho_1]) = \bar{\tau}_g([\rho_2])$. Hence by the first assertion of Lemma 2.5 we have $[\rho_1] = [\rho_2]$.

To prove conclusion 3, we apply the second assertion of Lemma 2.5. Let

$$\mathcal{K}' = \{[\rho] \in AH(\pi_1(M)) | \sum_{i=1}^{m} |\bar{\tau}_{a_i}([\rho])| \ \leq K\}.$$

It is easy to check that if $A \in \mathrm{PSL}_2(\mathbf{C})$, then $|tr(A^d)| \leq (|tr(A)| + 1)^d + 1$ for any positive integer $d$, so $\mathcal{K}'$ is a closed subset of

$$\mathcal{K} = \left\{ [\rho] \in AH(\pi_1(M)) | \sum_{i=1}^{m} |\bar{\tau}_{b_i}([\rho])| \leq m \left( (\sqrt{K} + 1)^D + 1 \right)^2 \right\}$$

where $D = \max\{d_1, \ldots, d_m\}$. But Lemma 2.5 implies that $\mathcal{K}$ is compact, so $\mathcal{K}'$ is also compact.

$\boxed{2.2}$

## 2.6 An outline of the argument

In order to provide an outline of the argument we fix a point $[\rho]$ on the boundary of $CC_0(M)$ such that $\Omega(\rho) = \emptyset$. Let $([\bar{\rho}_n])$ be a sequence in $CC_0(M)$ converging to $[\rho]$. We must find a sequence $([\widehat{\rho}_n])$ of maximal cusps in $\partial CC_0(M)$ which also converges to $[\rho]$.

The following observation of Bers allows us to conclude that $\partial_c N_{\bar{\rho}_n}$ has a pants decomposition of uniformly bounded length.

**Proposition 2.6** *(Bers) Given any $A > 0$, there exists a constant $k_1$ such that any hyperbolic surface of area at most $A$ has a pants decomposition of length at most $k_1$.*

In the proof of Proposition 4.1 we use Proposition 2.6 together with Lemma 2.1 and Sullivan's rigidity theorem to produce a new sequence $([\rho_n])$ in $CC_0(M)$ converging to $[\rho]$ such that if $C_n$ is the shortest pants decomposition of $\partial_c N_{\rho_n}$, then the length $l(C_n)$ of $C_n$ in $\partial_c N_{\rho_n}$, converges to 0. Proposition 3.1 will allow us to conclude that, for all large enough $n$, $C_n$ is pinchable.

Proposition 2.2 allows us to choose an allowable collection $\mathcal{A} = \{a_1, \ldots, a_m\}$ of test elements such that no element of $\mathcal{A}$ is taken to a parabolic element by $\rho$.

Our main local estimate, Theorem 14.1, asserts that if $\rho \in CC_0(M)$ and $\mu$ is a unit norm Beltrami differential supported on the portion of the $2L$-thin part of $\partial_c N_\rho$ associated to a pinchable collection of simple closed geodesics, then $D\Upsilon_a(D\bar{\Phi}(\mu))$ has length $O(L)$ (assuming that $L$ is sufficiently close to 0 and that $\rho(a)$ has moderate real translation length.) By iterative application of Lemma 2.1 we may produce, for each $n$, an infinite path $\beta_n\colon [0,\infty) \to \mathcal{T}(\partial M)$ which begins at a lift of $\rho_n$ to $\mathcal{T}(\partial M)$ and which pinches the length of $C_n$ to 0. For each $t$ the tangent vector $\beta_n'(t)$ is represented by a Beltrami differential supported on the appropriate thin part. If we apply the estimate coming from Theorem 14.1 we see that $\Upsilon_{a_i}(\beta_n)$ has length $O(l(C_n))$ in $CC_0(S^1 \times D^2)$ for each $a_i \in \mathcal{A}$. It is then easily checked that $q_M(\beta_n)$ has length $O(l(C_n))$ in the $d_{\mathcal{A}}$-metric on $AH(\pi_1(M))$ and hence accumulates at some conjugacy class $[\widehat{\rho}_n]$. Since the homotopy class of any component of $C_n$ is mapped to a parabolic by $\widehat{\rho}_n$, one may apply work of Keen, Maskit and Series [Ke-Mas-Se], to see that $[\widehat{\rho}_n]$ is a maximal cusp. The estimates also give that $([\widehat{\rho}_n])$ converges to $[\rho]$.

In section 6 we will assemble the proof, assuming Theorem 14.1. Most of the remainder of the paper will be devoted to the analytical proof of this local estimate. An outline of the ideas underlying the analytical arguments is given in section 7. Extensions and corollaries of the Main Theorem appear in sections 15 and 16.

Our rough outline of argument is similar to the outline of proof of McMullen's result in [Mc 1], although there are several key differences. Most importantly, McMullen bounds the rate of change of the Schwarzian derivative during the pinching. Since there is no analogue of the Schwarzian derivative in our setting, we instead bound the change of the complex lengths of a collection of test elements. We do so by bounding the induced deformation in the Teichmüller space of the torus associated to each test element. The proof of Theorem 14.1, which is used to obtain the bounds, uses much of the analytical machinery developed in [Mc 1] in combination with a number of estimates obtained by applying the Margulis lemma to the subgroup generated by a test element and a pinching element. This change also necessitated the development, in section 2.5, of an explicit metric on $AH(\pi_1(M))$ associated to a well-chosen collection of test-elements. Furthermore, in McMullen's setting any pants decomposition is pinchable, which is not the case in our situation. Proposition 3.1 was developed to check that our chosen pants decompositions are indeed pinchable.

## 3 Bounded pants are eventually pinchable

In this section we will see that if a sequence $([\rho_n])$ in $CC_0(M)$ converges to $[\rho]$, where $\Omega(\rho)=\emptyset$, and if $C_n$ is a bounded length collection of curves in the conformal boundary of $N_{\rho_n}$, then $C_n$ is pinchable for all large enough $n$.

**Proposition 3.1** *Let $M$ be a compact, oriented, atoroidal, irreducible 3-manifold whose boundary is non-empty and contains no tori. Let $(\rho_n)$ be a sequence of convex cocompact uniformizations of $M$ converging to $\rho:\pi_1(M)\to \mathrm{PSL}_2(\mathbf{C})$, such that $\Omega(\rho)=\emptyset$. If $K>0$ and, for each $n$, $C_n$ is a collection of disjoint simple closed geodesics on $\partial_c N_{\rho_n}$ of total length at most $K$, then $C_n$ is pinchable for all sufficiently large $n$.*

In the proof, we will need to use a few facts about the convex core of a hyperbolic 3-manifold. The convex core $C(N)$ of a hyperbolic 3-manifold $N=\mathbf{H}^3/\Gamma$ is the smallest convex submanifold whose inclusion is a homotopy equivalence. More concretely, if $\Lambda(\Gamma)=\widehat{\mathbf{C}}-\Omega(\Gamma)$ is the limit set of $\Gamma$ then $C(N)$ is the quotient of the convex hull $CH(\Lambda(\Gamma))$ under the action of $\Gamma$. The hyperbolic metric on $N$ induces an intrinsic metric on the 2-manifold $\partial C(N)$ which is itself hyperbolic. The nearest point retraction $r:N\to C(N)$ sends a point of $N$ to the (unique) point nearest to it in the convex core. There is a continuous extension of $r$ to a map $\bar r: N\cup\partial_c N\to C(N)$. If $\Gamma$ does not preserve a circle in $\widehat{\mathbf{C}}$, then $\bar r$ is homotopic to a homeomorphism between $\bar N$ and $C(N)$. (See Epstein-Marden [Ep-Mar] for more details on the convex core and the nearest point retraction.)

Canary [Ca] showed that curves of "moderate" length in the conformal boundary also have "moderate" length in the boundary of the convex core, with respect to its intrinsic metric.

**Theorem 3.2** *Let $N$ be a hyperbolic 3-manifold and let $\gamma$ be a closed geodesic of length $L$ in $\partial_c N$, then*

$$l_{\partial C(N)}(r(\gamma)^*)\le 45Le^{\frac{L}{2}}$$

*where $r(\gamma)^*$ denotes the geodesic in the intrinsic metric on $\partial C(N)$ in the homotopy class of $r(\gamma)$.*

*Proof of Proposition 3.1:* Let $\{g_1,\dots,g_k\}$ be a set of generators for $\pi_1(M)$. Fix a point $x_0\in\mathbf{H}^3$. Since $(\rho_n)$ converges, there exists a uniform upper bound $S$ on $d(x_0,\rho_n(g_i)(x_0))$ for all $i$ and $n$. Moreover, there exists $\delta>0$ such that if $d(x_0,x)\le S$, then $d(x,\gamma(x))\ge 2\delta$ for any $\gamma\in\rho_n(\pi_1(M)-\{id\})$ and any $n$. In particular, if $\gamma_{i,n}$ is the image in $N_n=N_{\rho_n}$ of the geodesic joining $x_0$ to $\rho_n(g_i)(x_0)$, then $\mathrm{inj}_{N_n}(y)\ge\delta$ at any point $y$ of $\gamma_{i,n}$.

Let $\bar r_n:\bar N_n\to C(N_n)$ be the nearest point retraction. Theorem 3.2 implies that $\bar r_n(C_n)$ is homotopic, in $\partial C(N)$, to a collection $C_n'$ of curves in $\partial C(N)$ of length at most $K'=45Ke^{K/2}$. We notice that for all large enough $n$, $\rho_n(\pi_1(M))$ does not preserve a circle, since otherwise $\rho(\pi_1(M))$ would preserve a circle. This would imply that $\Lambda(\rho)$ is contained in a circle, which would contradict our assumption that $\Omega(\rho)=\emptyset$. Therefore, we may assume that $\bar r_n$ is homotopic to a homeomorphism for all $n$.

If the theorem fails, we can pass to a subsequence, again called $(\rho_n)$, such that $C_n$ is not pinchable for any $n$. Therefore, there exists, for all $n$, a surface $B_n$ which is either a compressing disk or an immersed essential annulus with boundary contained in $C_n$. So, there exists a surface $B'_n$ in $C(N_n)$, which is properly homotopic, in $C(N_n)$, to $\bar{r}_n(B_n)$ and is either a compressing disk or an immersed essential annulus with boundary contained in $C'_n$. In particular, the boundary of $B'_n$ has length at most $2K'$.

We claim that each surface $B'_n$ is homotopic, rel boundary, to a surface $Y_n$ with the following property: if $x$ is a point of $Y_n$ and if the injectivity radius $\mathrm{inj}_{N_n}(x)$ is greater than $\delta$, then the distance in $N_n$ from $x$ to $\partial Y_n$ is less than $k(\delta)$ for some uniform constant $k(\delta)$.

To construct $Y_n$ we first subdivide $\partial B'_n$ into subarcs of length less than 1. If $B'_n$ is a disk we arrange that there are at least three subarcs in the subdivision, and if $B'_n$ is an annulus then we arrange that there be at least one subdivision point on each boundary component. We may then extend this subdivision of $\partial B'_n$ to a topological triangulation of $B'_n$ with all vertices on the boundary. That is, we have a collection $\mathcal{E}$ of arcs in $B'_n$, which are either elements of the subdivision of $\partial B'_n$ or properly embedded arcs whose endpoints are also endpoints of subdivision arcs; furthermore the closure of each component of $B'_n - \mathcal{E}$ is a disk whose boundary consists of three arcs of $\mathcal{E}$. In the case that $B'_n$ is an annulus we may easily arrange that each arc of $\mathcal{E}$ which is not contained in $\partial B'_n$ has an endpoint on each boundary component. For each arc $e \in \mathcal{E}$ let $e'$ be the geodesic arc which is homotopic rel endpoints to $e$. For each component $t$ of $B'_n - \mathcal{E}$, if we denote the arcs that comprise $\partial \bar{t}$ by $e_1$, $e_2$ and $e_3$, then we define $t'$ to be the geodesic triangle bounded by $e'_1$, $e'_2$ and $e'_3$. The union of the triangles $t'$ is an immersed hyperbolic surface $X_n$. Since the area of a hyperbolic triangle is at most the length of any of its sides, $X_n$ has area at most $2K'$.

The surface $B'_n$ is homotopic to $X_n$ by a homotopy $H_n$ such that the track of any point on $\partial B'_n$ has length less than 1. Let $Y_n$ be the union of $X_n$ with the annulus (or annuli) which is the image of $\partial B'_n \times [0,1]$ under the homotopy $H_n$. Clearly $Y_n$ is homotopic rel boundary to $B'_n$.

If $X_n$ is a disk, $x \in X_n$ and $d(x, \partial X_n) = R$, then $X_n$ contains an embedded hyperbolic ball of radius $R$, so $X_n$ has area at least $2\pi R^2$. So $R \leq \sqrt{\frac{K'}{\pi}}$. It follows that if $x \in Y_n$, then $d(x, \partial Y_n) \leq \sqrt{\frac{K'}{\pi}} + 1$.

Now suppose that $X_n$ is an annulus. If $x \in X_n$, then again there is not an embedded hyperbolic ball of radius $\sqrt{\frac{K'}{\pi}}$ about $x$. If $x \in X_n$ and $inj_{N_n}(x) \geq \delta$, then $inj_{X_n}(x) \geq \delta$, so either $d(x, \partial X_n) \leq \sqrt{\frac{K'}{\pi}}$ or there exists a geodesic (in $X_n$) loop $\beta$ based at $x$ in $X$ of length at least $2\delta$ and at most $2\sqrt{\frac{K'}{\pi}}$. If $d(\beta, \partial X_n) = R$ and $d(x, \partial X_n) > \sqrt{\frac{K'}{\pi}}$, then by considering cylindrical coordinates about $\beta$, we see that $X_n$ must have area at least $\delta R$. It follows that $d(x, \partial X_n) \leq \sqrt{\frac{K'}{\pi}} + \frac{2K'}{\delta}$ in either case. Therefore, if $y \in Y_n$ and $inj_{N_n}(y) \geq \delta$, then $d(y, \partial Y_n) \leq \sqrt{\frac{K'}{\pi}} + \frac{2K'}{\delta} + 1$.

Let $k(\delta) = K' + \frac{2K'}{\delta} + \sqrt{\frac{K'}{\pi}} + 1$. In either case, if $y \in Y_n$ and $inj_{N_n}(y) \geq \delta$, then $d(y, \partial Y_n) \leq k(\delta)$.

Since the surface $Y_n$ is essential and the loops $\{\gamma_{i,n}\}$ represent generators of $\pi_1(N_n, y_{0,n})$ (where $y_{0,n}$ is the image of $x_0$ in $N_n$), at least one of these loops must meet $Y_n$. Since each of these loops has length at most $S$ and every point on each loop has injectivity radius at least $\delta$, the distance from $y_{0,n}$ to $\partial Y_n$ is at most $1 + S + k(\delta)$. This implies that the distance from $x_0$ to the boundary of $\partial CH(\Lambda(\rho_n))$ is uniformly bounded for all $n$.

Without loss of generality we may assume that we are working in the ball model and that $x_0$ is the origin. The inequalities above imply that there exists $\epsilon > 0$ depending only on $\delta$ and $S$ such that $\Omega(\rho_n)$ contains a disk of radius at least $\epsilon$ in the spherical metric on $S^2_\infty$. We may pass to a subsequence so that $(\Lambda(\rho_n)) = (\widehat{\mathbf{C}} - \Omega(\rho_n))$ converges, in the Hausdorff topology on closed subsets of $\widehat{\mathbf{C}}$, to $\widehat{\Lambda}$. It is immediate that $\widehat{\mathbf{C}} - \widehat{\Lambda}$ must contain a ball of radius at least $\epsilon$. Since $\widehat{\mathbf{C}} - \widehat{\Lambda} \subset \Omega(\rho)$, this contradicts our assumption that $\Omega(\rho) = \emptyset$.

$\boxed{3.1}$

# 4 Approximating by sequences with short pants decompositions

In this section we combine Lemma 2.1 and Lemma 3.1 to show that if $[\rho] \in \partial CC_0(M)$ and $\Omega(\rho) = \emptyset$, then $\rho$ may be approximated by convex cocompact uniformizations of $M$ whose conformal boundaries contain increasingly short pinchable pants decompositions.

**Proposition 4.1** *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold and let $[\rho] \in \partial CC_0(M)$, where $\Omega(\rho) = \emptyset$. There exists a sequence $(\rho_n)$ of convex cocompact uniformizations of $M$ converging to $\rho$ and a sequence $(C_n)$ of pinchable pants decompositions of $\partial_c N_{\rho_n}$ such that the length $l(C_n)$ of $C_n$ in $\partial_c N_{\rho_n}$ converges to 0.*

*Proof of 4.1:* Since $[\rho] \in \partial CC_0(M)$, we may choose a sequence $\bar{\rho}_n$ of convex cocompact uniformizations of $M$ which converge to $\rho$. Bers' inequality (Proposition 2.6) implies that, for all $n$, there exists a pants decomposition of $\partial_c N_{\bar{\rho}_n}$ of length at most $k_1$.

Let $B$ be the constant provided by Lemma 2.1 when $L_0 = k_1$ and let $K = \kappa B$ where $\kappa$ is the number of curves in a pants decomposition of $\partial M$. We may iteratively apply Lemma 2.1 to obtain a 1-Lipschitz path $\beta_n : [0, \infty) \to \mathcal{T}(\partial M)$ such that $q_M(\beta_n(0)) = [\rho_n]$ and the surface $\beta_n(Kj)$ has a pants decomposition of length at most $\frac{k_1}{2^j}$.

Let $[\rho_{n,j}] = q_M(\beta_n(Kj))$. Since the Teichmüller distance between $[\bar{\rho}_n]$ and $[\rho_{n,j}]$ is at most $Kj$, there exists a $e^{2Kj}$-quasiconformal map $f_{n,j} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $\bar{\rho}_{n,j} = f_{n,j}\bar{\rho}f_{n,j}^{-1}$.

For this paragraph let $j$ be any fixed positive integer. By post-composing each $f_{n,j}$ by a Mobius transformations, we may assume that a subsequence of $(f_{n,j})$ converges to a $e^{2Kj}$-quasiconformal map $f_j$ such that $f_j \rho f_j^{-1}$ is a discrete faithful representation. Since $\Lambda(\rho) = \widehat{\mathbf{C}}$, Sullivan's rigidity theorem ([Sull 1, Theorem VII]) implies that $f_j$ is a Mobius transformation. Therefore, a subsequence of $([\bar{\rho}_{n,j}])$ converges to $[\rho]$ in $CC_0(M)$.

A diagonalization argument then provides a sequence $(\rho_n)$ which converges to $\rho$, such that if $C_n$ is the shortest pants decomposition of $\partial_c N_{\rho_n}$, then $l(C_n)$ converges to 0. Proposition

3.1 implies that $C_n$ is pinchable for all large enough $n$, so by excising finitely many terms, we may assume that $C_n$ is pinchable for all $n$.

$\boxed{4.1}$

# 5 The main global estimate

In this section, we use Lemma 2.1 and the main local estimate to obtain an estimate on the distance between a representation with a "short" pants decomposition of its conformal boundary and its associated maximal cusp. This estimate is the key step in the proof of the main theorem.

**Proposition 5.1** *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold whose boundary is non-empty and contains no tori and let $\mathcal{A} = \{a_1, \dots, a_m\}$ be an allowable collection of test elements in $\pi_1(M)$. Given $D_0 > d_0 > 0$, there exists $L_1 > 0$ and $G > 0$ such that if $[\rho] \in CC_0(M)$ and*

1. *$C$ is a pinchable pants decomposition of $\partial_c N_\rho$ of length $L < L_1$,*
2. *no element of $\mathcal{A}$ represents a curve in $C$, and*
3. *$\frac{D_0}{2} \geq l_\rho(a_i) \geq 2d_0$ for all $i = 1, \dots, m$, where $l_\rho(a_i)$ denotes the real translation distance of $\rho(a_i)$,*

*then there exists a maximal cusp $[\widehat{\rho}] \in \partial CC_0(M)$ such that*

$$d_{\mathcal{A}}([\rho], [\widehat{\rho}]) \leq G\ L.$$

The proof of Proposition 5.1 follows rather quickly from the main local estimate:

**Theorem 14.1** *Given $d_0 > 0$, there exists $D_6 > 0$ and $K_0 > 0$ with the following properties. Suppose that $M$ is a compact, oriented 3-manifold, $a$ is a primitive element in $\pi_1(M)$, $[\rho] \in CC_0(M)$ and $l(\rho(a)) > d_0$. Suppose that $C$ is a pinchable collection of disjoint simple closed geodesics in $\partial_c N_\rho$, none of which represents $\rho(a)$, such that each element of $C$ has length at most $L$ where*

$$L \leq D_6 e^{-l(\rho(a))}.$$

*If $\mu$ is a unit-norm Beltrami differential on $\partial_c N_\rho$ which is supported on the union of the $2L$-thin parts associated to elements of $C$, then*

$$||D\Upsilon_a(D\bar{\Phi}(\mu))|| \leq K_0 L.$$

The proof of Theorem 14.1 will occupy Sections 8 - 14. An outline of the proof of Theorem 14.1 appears in section 7.

*Proof of Proposition 5.1:* (Assuming Theorem 14.1) We recall that $q_T : \mathcal{T}(T^2) \to CC_0(S^1 \times D^2)$ is the quotient map and we identify $\pi_1(S^1 \times D^2)$ with $\mathbf{Z}$. Let

$$E_s = \left\{\sigma \in \mathcal{T}(T^2)|\ 2d_0 \leq l(q_T(\sigma)(1)) \leq D_0/2\right\}$$

where $l(q_T(\sigma)(1))$ denote the real translation distance of $q_T(\sigma)(1)$. Similarly, we let

$$E_f = \left\{\rho \in \mathcal{T}(T^2)|\ d_0 \leq l(q_T(\sigma)(1)) \leq D_0\right\}.$$

By assumption, for each $i = 1, \cdots, m$, we have that $\Upsilon_{a_i}([\rho]) \in q_T(E_s)$. The sets $q_T(E_s)$ and $q_T(E_f)$ are compact subsets of $CC_0(S^1 \times D^2)$. Let $\delta$ be the distance, measured in the (quotient) Teichmüller metric on $CC_0(S^1 \times D^2)$, between $q_T(E_s)$ and the boundary of $q_T(E_f)$

Let $h : M \to \bar{N}_\rho$ be an orientation-preserving homeomorphism such that $[h_*] = [\rho]$. Let $B$ be the constant provided by Lemma 2.1 when $L_0 = D_6$ and let $B' = \kappa B$ where $\kappa$ is the number of curves in a pants decomposition of $\partial M$. Assuming that $L_1 \leq D_6$, we may apply Lemma 2.1 $\kappa$ times to obtain a path $\beta : [0, B'] \to \mathcal{T}(\partial M)$ with $\beta(t) = (X_t, g_t \circ h)$ such that (i) $\beta(0) = (\partial_c N_\rho, h)$, (ii) $l_{X_t}(g_t(C)) \leq L$ for all $t$, (iii) $l_{X_{B'}}(g_{B'}(C)) \leq \frac{L}{2}$, and (iv) $\beta'(t)$ is represented, for all $t$, by a unit norm Beltrami differential $\mu_t$ supported on the $2L$-thin part of $X_t$ associated to $g_t(C)$. If $[\rho_t] = q_M(\beta(t))$ and $\bar{N}_t = \bar{N}_{\rho_t}$, then $g_t$ extends to a homeomorphism $\bar{g}_t : \bar{N}_\rho \to \bar{N}_t$ such that $[\rho_t] = [(g_t \circ h)_*]$.

Let $D_6$ and $K_0$ be the constants associated to $d_0$ in the main local estimate, Theorem 14.1. Then assuming that $L_1 \leq D_6 e^{-D_0}$, Theorem 14.1 implies that

$$||D\Upsilon_{a_i}(D\bar{\Phi}(\mu_t))|| = ||D\widetilde{\Upsilon}_{a_i}(\beta'(t))|| \leq K_0 L$$

for all $t$ such that $\widetilde{\Upsilon}_{a_i}(\beta(t)) \in E_f$. Therefore, $\widetilde{\Upsilon}_{a_i}(\beta([0, B'])) \cap E_f$ has length at most $K_0 B' L$. If we have assumed that $L_1 < \frac{\delta}{K_0 B'}$, then this implies that the entire path lies in $E_f$.

We may iterate this process (reducing the length of $C$ by a factor of 2 at each stage) to produce an infinite path $\beta : [0, \infty) \to \mathcal{T}(\partial M)$ such that $l_{X_t}(g_t(C)) \leq \frac{L}{2^n}$ and $\beta'(t)$ is supported on the $\frac{L}{2^{n-1}}$-thin part of $X_t$ for all $t \in [nB', (n+1)B']$. Applying the argument above we see that $\widetilde{\Upsilon}_{a_i}(\beta([0, \infty)) \cap E_f$ has length at most

$$K_0 B' \left(L + \frac{L}{2} + \cdots + \frac{L}{2^k} + \cdots\right) = 2K_0 B' L.$$

In particular, if $L_1 \leq \frac{\delta}{2K_0 B'}$, then $\widetilde{\Upsilon}_{a_i}(\beta([0, \infty))$ lies entirely in $E_f$ for all $i$. Let

$$L_1 = \min\{D_6 e^{-D_0}, \frac{\delta}{2K_0 B'}\}.$$

If $[\nu] \in CC_0(S^1 \times D^2)$, then let $\bar{\tau}_0([\nu])$ denote the square of the trace of $\nu(1)$ where 1 denotes the generator of $\pi_1(S^1 \times D^2) \cong \mathbf{Z}$. Then $\bar{\tau}_0 : CC_0(S^1 \times D^2) \to \mathbf{C}$ is a smooth function and $\bar{\tau}_a([\rho]) = \bar{\tau}_0(\Upsilon_a([\rho]))$ for all $[\rho] \in CC_0(M)$ and all $a \in \pi_1(M)$. Since $q_T(E_f)$ is compact and $\bar{\tau}_0$ is smooth, there exists $K_3 > 0$ such that if $\nu_1, \nu_2 \in q_T(E_f)$, then

$$|\bar{\tau}_0(\nu_1) - \bar{\tau}_0(\nu_2)| \leq K_3 d(\nu_1, \nu_2)$$

(where the metric on the right is the quotient Teichmüller metric on $CC_0(S^1 \times D^2)$.) It follows that, for all $i$,

$$\bar{\tau}_0(q_T(\widetilde{\Upsilon}_{a_i}(\beta([0,\infty)))) = \tau_{a_i}(q_M(\beta([0,\infty)))$$

has length at most $2K_3K_0B'L$ in $\mathbf{C}$. By definition then $q_M(\beta([0,\infty)))$ has length at most $2mK_3K_0B'L$ in the $d_{\mathcal{A}}$-metric on $AH(\pi_1(M))$. Therefore, there is a conjugacy class $[\widehat{\rho}] \in AH(\pi_1(M))$ which is an accumulation point of $\{q_M(\beta(n))\}$.

Let $C' = h^{-1}(C)$. If $\eta$ is a curve in $C'$, then $l_{X_t}(g_t(h(\eta)))$ converges to 0. Let $b$ be an element of $\pi_1(M)$, such that $\eta$ is a representative of $\rho(b)$. A result of Sugawa [Su, Proposition 6.1], stated below as Proposition 11.2, implies that the complex translation length of $\rho_t(b)$ also converges to 0. Thus, $\widehat{\rho}(b)$ is parabolic. Since, $C'$ is a maximal, collection of disjoint simple closed curves in $\partial M$, Theorem III in [Ke-Mas-Se] guarantees that $\widehat{\rho}$ is a maximally cusped uniformization of $M$. Noticing that

$$d_{\mathcal{A}}([\widehat{\rho}],[\rho]) \leq 2mK_0K_3B'L,$$

completes the proof of the result if we take $G = 2mK_0K_3B'$.

$\boxed{5.1}$

# 6 Proof of the main theorem

We are now ready for the proof of our main theorem.

**Theorem 6.1** (Approximations by maximal cusps) *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold whose boundary is non-empty and contains no tori. If $[\rho] \in \partial CC_0(M)$ and $\Omega(\rho) = \emptyset$, then $[\rho]$ can be approximated by maximal cusps in $\partial CC_0(M)$.*

*Proof of 6.1:* Let $(\rho_n)$ and $(C_n)$ be the sequences of representations and pinchable pants decompositions given by Proposition 4.1. Let $F$ be the collection, necessarily finite, of conjugacy classes of primitive parabolic elements in $\rho(\pi_1(M))$. Let $\mathcal{A}$ be an allowable collection of test elements, provided by Proposition 2.2, which does not contain any elements of $F$.

We may choose positive constants $d_0$ and $D_0$ such that

$$4d_0 < l_\rho(a_i) < \frac{D_0}{4}$$

for all $i = 1, \ldots, m$. We use these values of $d_0$ and $D_0$ in all applications of Proposition 5.1. Let $L_1$ be the constant provided by Proposition 5.1 with our chosen values of $d_0$ and $D_0$.

Since $(\rho_n)$ converges to $\rho$ and $l(C_n)$ converges to 0, there exists $n_0$ such that if $n \geq n_0$, then

1. $2d_0 \le l_{\rho_n}(a_i) \le \frac{D_0}{2}$ for all $i = 1, \ldots, m$,
2. $45l(C_n)e^{l(C_n)/2} < d_0$, and
3. $l(C_n) < L_1$.

If $\rho_n(c_n)$ is an element of $\rho_n(\pi_1(M))$ representing a curve in $C_n$, then Theorem 3.2 implies that there is a representative of $\rho_n(c_n)$ in $\partial C(N_{\rho_n})$ of length at most $45l(C_n)e^{l(C_n)/2} < d_0$. It follows that the real translation length $l(\rho_n(c_n))$ of $\rho_n(c_n)$ is less than $d_0$. So, no curve in $C_n$ is represented by an element of $\mathcal{A}$.

Proposition 5.1 then implies that, for all $n > n_0$, there exists a maximal cusp $[\widehat{\rho}_n] \in \partial CC_0(M)$ such that

$$d_{\mathcal{A}}([\rho_n], [\widehat{\rho}_n]) \le Gl(C_n).$$

Since, $(d_{\mathcal{A}}([\rho_n], [\rho]))$ and $(l(C_n))$ both converge to 0, it is clear that $([\widehat{\rho}_n])$ is a sequence of maximal cusps in $\partial CC_0(M)$ converging to $[\rho]$.

$\boxed{6.1}$

# 7 An outline of the proof of the main local estimate

We now outline the proof of the main local estimate, Theorem 14.1. Recall that Theorem 14.1 asserts that if $\rho \in CC_0(M)$, $C$ is a pinchable collection of geodesics in $\partial_c N_\rho$ of length at most $L$, and $\mu$ is a Beltrami differential supported on the portion of the $2L$-thin part of $\partial_c N_\rho$ associated to $C$, then $D\Upsilon_a(D\Phi(\mu))$ has length $O(L)$, assuming that $L$ is sufficiently close to 0 and that $\rho(a)$ has moderate real translation length.

We previously observed that if $\widetilde{\mu}$ is the lift of $\mu$ to $\Omega(\rho)$, then $\widetilde{\mu}$ is also a lift of the representative of $D\Upsilon_a(D\Phi(\mu))$. The length of $D\Upsilon_a(D\Phi(\mu))$ is thus the supremum of the values of $\int_F \phi\widetilde{\mu}$ where $F$ is a fundamental domain for $\langle\rho(a)\rangle$ and $\phi$ is the pull-back of a unit-norm quadratic differential on $T(\rho(a)) = \Omega(\langle\rho(a)\rangle)/\langle\rho(a)\rangle$.

Since $\mu$ is supported on the $2L$-thin part of $\partial_c N_\rho$, $\widetilde{\mu}$ is supported on the union of the pre-images of these thin parts. If $\gamma$ is a hyperbolic element of $\rho(\pi_1(M))$ associated to a component of the $2L$-thin part, then we will define a seahorse $B_\gamma$ to be a pre-image of a specified annulus $A_\gamma$ on $T_\gamma = \Omega(\langle\gamma\rangle)/\langle\gamma\rangle$; see section 8 for a general discussion of seahorses. Each component of the pre-image of a thin part is contained in a seahorse $B_\gamma$; see Lemma 14.2. Although the components of the pre-images of the thin parts are disjoint, the associated seahorses need not be.

In section 13 we will explain how to modify a construction of McMullen, to choose a collection $\{E_\gamma\}$, indexed by a subset $\mathcal{G}'$ of $\mathcal{G}$, of *disjoint* sets each of which is contained in a seahorse $B_\gamma$, contains a slightly smaller seahorse, and is invariant under the "generator" $\gamma$ of the seahorse $B_\gamma$. Moreover, $\cup_{\gamma\in\mathcal{G}'} E_\gamma$ will be seen to contain the support of $\widetilde{\mu}$; see Lemma 14.2.

We thus reduce the proof of Theorem 14.1 to bounding $\int_{B_\gamma} \phi\nu_\gamma$ for each seahorse $B_\gamma$ where $\gamma \in \mathcal{G}'$ and $\nu_\gamma$ is the restriction of $\tilde{\mu}$ to $E_\gamma$. In particular, we need to show that

$$\left|\int_{B_\gamma} \phi\nu_\gamma\right| \leq D_3 L ||\phi||_{B_\gamma}. \tag{3}$$

for some constant $D_3 > 0$. After summing over all $\gamma \in \mathcal{G}'$ such that $B_\gamma$ intersects the fundamental domain $F$, this will show that

$$\left|\int_F \phi\tilde{\mu}\right| \leq D_3 L \sum ||\phi||_{B_\gamma} \leq K_0 L$$

for some positive constant $K_0$.

We use a duality principle (see equation 4 in section 9) to reduce equation 3 to an estimate on the image of $\phi$ by the Theta operator associated to the covering of $A_\gamma$ by $B_\gamma$. Our key tools are an estimate of McMullen's in a similar situation, stated here as Theorem 9.1, and an estimate coming from the Margulis lemma, see section 10.

## 8 Seahorses

In this section, we briefly review the theory of thickened spirals as developed by McMullen.

A hyperbolic Möbius transformation $\gamma$ will be said to have *complex translation length* $\lambda \in \mathbf{C}$ if $\gamma$ is conjugate in the group of Möbius transformations to $z \mapsto e^\lambda z$. We will, without loss of generality, always assume that the imaginary part of a complex translation length lies in the interval $(-\pi, \pi]$.

Let $\gamma$ be a Möbius transformation with complex translation length $\lambda$ and distinct fixed points $a$ and $b$. We set $\Omega_\gamma = \Omega(\langle\gamma\rangle) = \mathbf{C} - \{a, b\}$ and we observe that $T_\gamma = \Omega_\gamma/\langle\gamma\rangle$ is a torus. Consider the covering map $p_\gamma : \Omega_\gamma \to T_\gamma$. Given $q \in \Omega_\gamma$, we will construct an explicit covering $p_T : \mathbf{C} \to T_\gamma$ which factors through $p_\gamma$ and satisfies $p_T(0) = p_\gamma(q)$. Let $S_\gamma$ be the Möbius transformation which takes 0 and $\infty$ respectively to the fixed points $a$ and $b$ of $\gamma$ and satisfies $S_\gamma(1) = q$. The map $p_\Omega : \mathbf{C} \to \Omega(\langle\gamma\rangle)$ given by $p_\Omega(z) = S_\gamma(e^{\lambda z})$ is a covering map whose associated group of covering transformations is generated by $z \mapsto z + \frac{2\pi i}{\lambda}$. We set $p_T = p_\gamma \circ p_\Omega$. Note that the covering transformation $z \mapsto z + 1$ of $p_T : \mathbf{C} \to T_\gamma$ covers the deck transformation $\gamma$ of the intermediate cover $p_\gamma : \Omega_\gamma \to T_\gamma$.

The conformal structure on the torus $T_\gamma$ determines a flat metric on $T_\gamma$ which is unique up to scaling. The flat metric on $T_\gamma$ lifts to a flat metric on $\Omega_\gamma$. If $s \in \Omega_\gamma$ and $w \in p_\Omega^{-1}(s)$, then $g_s = p_\Omega(\{z \in \mathbf{C} | \mathrm{Im}(z) = \mathrm{Im}(w)\})$ is a geodesic in the flat metric on $\Omega_\gamma$ and projects to a closed geodesic on $T_\gamma$ that passes through $p_\gamma(s)$. One may easily check that $T_\gamma - p_\gamma(g_s)$ is an annulus of conformal modulus $M(\gamma) = 4\pi^2 \mathrm{Re}(\frac{1}{\lambda})$.

If $m \in (0, M(\gamma))$, then we can define $C(m) = \{z \in \mathbf{C} | \frac{m}{4\pi} < \mathrm{Im}(z) < \frac{M(\gamma)}{2\pi} - \frac{m}{4\pi}\}$. We let $B(\gamma, m, q) = p_\Omega(C(m)) \subset \Omega_\gamma$ and $A(\gamma, m, q) = p_T(C(m)) \subset T_\gamma$. One may also define $A(\gamma, m, q)$ to be the annulus obtained by removing from $T_\gamma$ a right cylinder of modulus $m$ with central circle $p_\gamma(g_q)$. In particular, $q$ projects to a point in $T_\gamma - A(\gamma, m, q)$ at maximal distance from $A(\gamma, m, q)$.

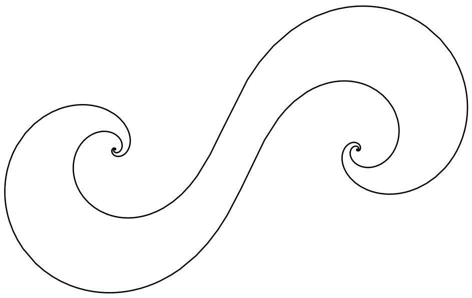

Figure 1: The seahorse $B(\gamma, m, q)$ where $\gamma$ has fixed points $\{0, 1\}$ and complex length $\lambda = 0.5 + (75\pi/243)i$, $m = 4.004\pi$ and $q = \infty$.

We will refer to $B(\gamma, m, q)$ as a *seahorse*. Our definition of a seahorse coincides with McMullen's "thickened spiral." If $\mathrm{Im}(\lambda)$ is non-zero, the seahorse is bounded by two exponential spirals connecting the fixed points of $\gamma$.

The shapes of seahorses are studied in Section 3.1 of McMullen [Mc 1].

**Proposition 8.1** ([Mc 1, Proposition 3.2]) *There exist positive constants $c_1$, $C_1$, $c_2$ and $C_2$ such that if $\gamma$ is a hyperbolic Möbius transformation with fixed points $\{0, 1\}$ and $q = \infty$, then*

1. $$\frac{c_1}{|m\lambda|} < \mathrm{diam}(B(\gamma, m, q)) < \frac{C_1}{|m\lambda|}$$
   *where* $\mathrm{diam}(B(\gamma, m, q))$ *denotes the Euclidean diameter of* $B(\gamma, m, q)$*, and*

2. *if $m < M(\gamma)/2$, then*
   $$c_2 \left(\mathrm{diam}(B(\gamma, m, q))\right)^2 \leq \mathrm{area}(B(\gamma, m, q)) \leq C_2 \left(\mathrm{diam}(B(\gamma, m, q))\right)^2.$$

# 9 The Theta operator

Given a holomorphic covering $\pi : Y \to X$ of Riemann surfaces and $\phi \in Q(Y)$, one may define a quadratic differential $\pi_*(\phi)$ as follows. If $x \in X$ and $U$ is an evenly covered neighborhood of $x$, then we define $\pi_*(\phi)(x)$ by summing $(\pi^{-1})^*(\phi)$ over the components of $\pi^{-1}(U)$. In other words,

$$\pi_*(\phi)(x) = \sum_{w \in \pi^{-1}(x)} (\pi_w^{-1})^* \phi.$$

The map $\phi \to \pi_*(\phi)$ defines an operator, called the Poincaré Theta operator, which is denoted

$$\Theta_{Y/X} : Q(Y) \to Q(X).$$

Since,

$$||\Theta_{Y/X}(\phi)|| \leq \int_X \pi_* |\phi| = \int_Y |\phi| = ||\phi||,$$

$\Theta_{Y/X}$ has operator norm at most 1.

If $\mu \in B(X)$ is a Beltrami differential and $\phi \in Q(Y)$, then

$$\int_Y \phi \pi^* \mu = \int_X \Theta(\phi)\mu.$$

In particular, if $||\mu|| \leq 1$, then

$$\left|\int_Y \phi \pi^* \mu\right| = \left|\int_X \Theta(\phi)\mu\right| \leq \int_X |\Theta(\phi)\mu| \leq ||\Theta(\phi)|| \,. \tag{4}$$

Inequality 4 plays a key role in the proof of Theorem 14.1.

The following estimate is one of the key tools in McMullen's work.

**Theorem 9.1** (Theorem 3.1 in McMullen [Mc 1]) *Let $\gamma$ be a hyperbolic Möbius transformation with fixed points 0 and 1. Let $m$ be a real number such that $M(\gamma) > m > 4\pi$ and set $B = B(\gamma, m, \infty)$, $A = A(\gamma, m, \infty)$, and $\Theta = \Theta_{B/A}$. There exists a constant $C_3 > 0$ such that for any $\gamma$ and $m$ as above we have*

$$\frac{\|\Theta(dz^2)\|_A}{\|dz^2\|_B} \leq C_3 \left(\frac{m^2}{M(\gamma)^2} + \frac{m^2}{e^{\frac{m}{2}}}\right).$$

We will need a similar bound on the image under $\Theta$ of the quadratic differential $\frac{dz^2}{(z-t)^2}$. This differential will arise as the pull-back of a quadratic differential on the torus associated to one of our test elements. McMullen's original estimate will be the key tool used in our proof.

**Proposition 9.2** *Let $\gamma$ be a hyperbolic Möbius transformation with fixed points 0 and 1 and complex translation length $\lambda$. Let $m$ be a real number such that $M(\gamma) > m > 4\pi$ and set $A = A(\gamma, m, \infty)$, $B = B(\gamma, m, \infty)$, and $\Theta = \Theta_{B/A}$. There exist positive constants $c_4$ and $C_4$ such that if $\gamma$ and $m$ are as above, $t$ is a complex number and $|t| > c_4/|m\lambda|$, then*

$$\frac{\|\Theta\left(\frac{dz^2}{(z-t)^2}\right)\|_A}{\|\frac{dz^2}{(z-t)^2}\|_B} \leq C_4 \left(\frac{m^2}{M(\gamma)^2} + \frac{m^2}{e^{\frac{m}{2}}}\right).$$

*Proof of 9.2:* Let $B = B(\gamma, m, q)$ and let $\beta_0 = \sup_{z \in B} |z|$, so $\beta_0 \leq diam(B) \leq 2\beta_0$. Proposition 8.1 gives that $\mathrm{diam}(B) < C_1/|m\lambda|$. Choose $c_4 = 2C_1$. Since $|t| > c_4/|m\lambda|$, then $|t| > 2\beta_0$. Thus, for every $z \in B$ we have

$$2|t| \geq |z - t| \geq \frac{|t|}{2}.$$

We also observe that

$$\frac{dz^2}{(z-t)^2} - \frac{dz^2}{t^2} = \frac{2zt - z^2}{t^2(z-t)^2} dz^2.$$

and that
$$\left|\frac{z(2t-z)}{t^2(z-t)^2}\right| dz^2 < \frac{10\beta_0}{|t|^3}dz^2$$
for all $z \in B$. Since $\Theta$ is a linear operator we have
$$\|\Theta\left(\frac{dz^2}{(z-t)^2}\right)\|_A \leq \|\Theta\left(\frac{dz^2}{t^2}\right)\|_A + \|\Theta\left(\frac{10\beta_0}{|t|^3}dz^2\right)\|_A.$$

Since $\frac{1}{(z-t)^2} \geq \frac{1}{4t^2}$ for all $z \in B$, $\|\frac{dz^2}{(z-t)^2}\|_B \geq \frac{1}{4}\|\frac{dz^2}{t^2}\|_B$. Combining the last two observations with the fact that $\Theta$ is a linear operator, we obtain
$$\frac{\|\Theta(\frac{dz^2}{(z-t)^2})\|_A}{\|\frac{dz^2}{(z-t)^2}\|_B} \leq 4\left(\frac{\frac{1}{|t|^2}\|\Theta(dz^2)\|_A}{\frac{1}{|t|^2}\|dz^2\|_B} + \frac{\frac{10\beta_0}{|t|^3}\|\Theta(dz^2)\|_A}{\frac{1}{|t|^2}\|dz^2\|_B}\right) = 4\left(1+\frac{10\beta_0}{|t|}\right)\frac{\|\Theta(dz^2)\|_A}{\|dz^2\|_B}.$$

Since $|t| > 2\beta_0$, we see that $4\left(1+\frac{10\beta_0}{|t|}\right) < 24$. Combining this with McMullen's estimate, we obtain
$$\frac{\|\Theta(\frac{dz^2}{(z-t)^2})\|_A}{\|\frac{dz^2}{(z-t)^2}\|_B} \leq 24C_3\left(\frac{m^2}{M(\gamma)^2} + \frac{m^2}{e^{\frac{m}{2}}}\right)$$
and the result follows if we take $C_4 = 24C_3$.

$\boxed{9.2}$

## 10 An application of the Margulis lemma

The Margulis lemma gives a lower bound on the distance between the axes of hyperbolic elements of a Kleinian group when one of the hyperbolic elements has a short translation distance. In this section we will derive an explicit form of this observation which applies to our situation.

There is a universal constant $\epsilon_3 > 0$, called the *Margulis constant*, such that any two infinite order elements of a Kleinian group $\Gamma$ which both translate some point $x \in \mathbf{H}^3$ by a distance less than $\epsilon_3$ lie in an abelian subgroup of $\Gamma$ (see [BP, Chapter D] for more details.) Let $\gamma$ be a primitive hyperbolic element of a Kleinian group $\Gamma$ and set
$$Z^n_{\epsilon_3}(\gamma) = \{x \in \mathbf{H}^3 | d(x, \gamma^n(x)) < \epsilon_3\}.$$
The *Margulis tube* for $\gamma$ is then defined to be
$$Z_{\epsilon_3}(\gamma) = \cup_{n\in\mathbf{Z}_+} Z^n_{\epsilon_3}(\gamma).$$

If $\beta \in \Gamma - \langle\gamma\rangle$, then no non-trivial power of $\beta$ commutes with a non-trivial power of $\gamma$, so $Z_{\epsilon_3}(\beta) \cap Z_{\epsilon_3}(\gamma) = \emptyset$. Moreover, since $\beta(Z_{\epsilon_3}(\gamma)) = Z_{\epsilon_3}(\beta\gamma\beta^{-1})$, it follows that $\beta(Z_{\epsilon_3}(\gamma)) \cap Z_{\epsilon_3}(\gamma) = \emptyset$.

**Proposition 10.1** *There exist positive constants $D_1$ and $d_1$ with the following property. Suppose that $\Gamma$ is a Kleinian group, $\gamma$ is a primitive hyperbolic element in $\Gamma$ with fixed points $0$ and $1$ and complex translation length $\lambda$ and $\alpha$ is a primitive hyperbolic element in $\Gamma$ with real translation length $l(\alpha)$ and fixed points $t$ and $\infty$. If*

$$|\lambda| \le D_1 e^{\frac{-l(\alpha)}{2}},$$

*then*

$$|t| \ge \frac{d_1}{|\lambda|} e^{\frac{-l(\alpha)}{2}}.$$

*Proof of 10.1:* Let $\lambda = l + i\theta$. Then $Z^1_{\epsilon_3}(\gamma)$ is empty if $l \ge \epsilon_3$ and otherwise is a solid cylinder about $A_\gamma$ with radius $R_\gamma$ satisfying ([CS5, 1.3, page 1283])

$$\sinh^2(R_\gamma) = \frac{\cosh \epsilon_3 - \cosh l}{\cosh l - \cos\theta} \tag{5}$$

If $D_1 \le \frac{\epsilon_3}{2}$, then $l \le \frac{\epsilon_3}{2}$, so

$$\sinh^2(R_\gamma) \ge \frac{J_0}{\cosh l - \cos\theta},$$

where $J_0 = \cosh \epsilon_3 - \cosh \frac{\epsilon_3}{2}$. Moreover, if $D_1 \le 1$, then $|\lambda| \le 1$, a direct computation shows that $\cosh l \le 1 + l^2$ (when $l < 1$) and that $\cos\theta \ge 1 - \theta^2/2$, hence

$$\cosh l - \cos\theta \le (1 + l^2) - (1 - \theta^2/2) \le |\lambda|^2.$$

We therefore have

$$\frac{e^{2R_\gamma}}{4} \ge \sinh^2(R_\gamma) \ge \frac{J_0}{|\lambda|^2}.$$

Therefore,

$$R_\gamma \ge \log\left(\frac{J_1}{|\lambda|}\right) \tag{6}$$

where $J_1 = 2\sqrt{J_0}$.

Let $L$ be the unique common perpendicular to the axes $A_\gamma$ and $A_\alpha$, with intersection points $Q_\gamma$ and $Q_\alpha$ respectively. Since $\alpha(Z_{\epsilon_3}(\gamma)) \cap Z_{\epsilon_3}(\gamma) = \emptyset$, $A_\alpha \cap Z^1_{\epsilon_3}(\gamma)$ has length less than $l(\alpha)$, so we may conclude that

$$d(Q_\gamma, Q_\alpha) \ge R_\gamma - \frac{l(\alpha)}{2}.$$

If $|t| < 2$, then $d(Q_\gamma, Q_\alpha) < 5$. If we choose

$$D_1 = \min\left\{1, \frac{\epsilon_3}{2}, \frac{J_1}{e^5}\right\},$$

then equation (6) implies that

$$R_\gamma \geq 5 + \frac{l(\alpha)}{2},$$

which implies that $d(Q_\gamma, Q_\alpha) \geq 5$. So we may assume that $|t| \geq 2$.

Let $b$ denote the distance from $A_\alpha$ to the point $x_0$ at height $\frac{1}{2}$ above the point $\frac{1}{2} \in \mathbf{C}$. Basic hyperbolic geometry (see, for example, section 7.20 in Beardon [Be]) gives

$$\sinh b = |2t - 1|.$$

Therefore, since $b \geq d(Q_\gamma, Q_\alpha)$, $|t| \geq 2$ and $d(Q_\gamma, Q_\alpha) \geq 5$, we get that

$$3|t| \geq |2t-1| \geq \sinh(d(Q_\gamma, Q_\alpha)) \geq \frac{e^{d(Q_\gamma, Q_\alpha)}}{4}.$$

So,

$$|t| \geq \frac{e^{d(Q_\gamma, Q_\alpha)}}{12} \geq \frac{e^{R_\gamma} e^{\frac{-l(\alpha)}{2}}}{12} \geq \frac{J_1}{12|\lambda|} e^{\frac{-l(\alpha)}{2}}.$$

We then take $d_1 = \frac{J_1}{12}$ to complete the proof.

$\boxed{10.1}$

## 11 Bounds on the Theta operator

In this section, we will consider the pull-back $\phi$ of a quadratic differential on the quotient torus associated to a test element. We restrict $\phi$ to a seahorse associated to a pinching element and bound its image under the corresponding Theta operator. We will later use this estimate to bound the pairing of a Beltrami differential with $\phi$ on the seahorse.

**Proposition 11.1** *There exist positive constants $D_2$ and $D_3$ with the following property. Suppose that $\Gamma$ is a Kleinian group, $\alpha$ and $\gamma$ are non-commuting primitive hyperbolic elements of $\Gamma$, $\phi$ is the pull-back to $\Omega(\Gamma)$ of a quadratic differential on $T(\alpha)$ and $q$ is a fixed point of $\alpha$. Let $A = A(\gamma, \frac{1}{\sqrt{L}}, q)$, $B = B(\gamma, \frac{1}{\sqrt{L}}, q)$ and $\Theta = \Theta_{B/A}$. If $\gamma$ has a representative in the conformal boundary of $\mathbf{H}^3/\Gamma$ with length at most $L$ and*

$$L \leq D_2 e^{\frac{-l(\alpha)}{2}},$$

*then*

$$\frac{||\Theta(\phi)||_A}{||\phi||_B} \leq D_3 L$$

*Proof of 11.1:* We notice that we may first normalize the situation so that the fixed points of $\gamma$ are 0 and 1 and $q = \infty$. Let $t$ denote the other fixed point of $\alpha$ and let $\lambda$ denote the complex length of $\gamma$. We notice that $\phi$ must be a complex multiple of $\frac{dz^2}{(z-t)^2}$. Since $\Theta$ is a linear operator we may simply assume that $\phi = \frac{dz^2}{(z-t)^2}$.

In order to apply Proposition 9.2, in the case where $m = \frac{1}{\sqrt{L}}$, we need to produce bounds on $|\lambda|$ and on $t$. The following result of Sugawa allows us to translate our bounds on $L$ into bounds on $|\lambda|$.

**Proposition 11.2** (Proposition 6.1 of Sugawa [Su]) *If $\gamma$ is a hyperbolic element of a Kleinian group $\Gamma$ with complex translation length $\lambda = l + i\theta$, which has a representative of length $L$ in the conformal boundary $\partial_c N$, then*

$$Le^{\frac{L}{2}} \geq \frac{|\lambda|^2}{2l} \geq \frac{|\lambda|}{2}.$$

Since

$$M(\gamma) = 4\pi^2 \mathrm{Re}\left(\frac{1}{\lambda}\right) = \frac{4\pi^2 l}{|\lambda|^2},$$

Sugawa's result implies that

$$\frac{1}{M(\gamma)} = \frac{|\lambda|^2}{4\pi^2 l} \leq \frac{Le^{\frac{L}{2}}}{2\pi^2}.$$

In particular, if $L < 1$, then, since $\frac{2\pi^2}{e^{\frac{L}{2}}} > 1$,

$$M(\gamma) > \frac{1}{L} > \frac{1}{\sqrt{L}}.$$

Moreover, if $L < \frac{1}{16\pi^2}$, then $\frac{1}{\sqrt{L}} > 4\pi$.

Sugawa's result also implies that if $L < 1$ and $L \leq \frac{D_1}{2e} e^{\frac{-l(\alpha)}{2}}$, then $|\lambda| \leq D_1 e^{\frac{-l(\alpha)}{2}}$. Proposition 10.1 then implies that $|t| \geq \frac{d_1}{|\lambda|} e^{\frac{-l(\alpha)}{2}}$. In order to apply Proposition 9.2 we need to check that $|t| \geq \frac{c_4}{|m\lambda|}$. Since we have chosen $m = \frac{1}{\sqrt{L}}$ this will hold if $c_4\sqrt{L} \leq d_1 e^{\frac{-l(\alpha)}{2}}$. So, if $L \leq D_2 e^{-l(\alpha)}$, where

$$D_2 = \min\left\{\frac{1}{16\pi^2}, \frac{D_1}{2e}, \left(\frac{d_1}{c_4}\right)^2\right\},$$

then $|t| \geq \frac{c_4}{|m\lambda|}$ and $M(\gamma) > m > 4\pi$. Proposition 9.2 then gives that

$$\frac{||\Theta\left(\frac{dz^2}{(z-t)^2}\right)||_A}{||\frac{dz^2}{(z-t)^2}||_B} \leq C_4\left(\frac{m^2}{M(\gamma)^2} + \frac{m^2}{e^{m/2}}\right).$$

Substituting $m = \frac{1}{\sqrt{L}}$ and using the inequality $\frac{1}{M(\gamma)} \leq \frac{\sqrt{e}}{2\pi^2}L$ we conclude that

$$\frac{\|\Theta\left(\frac{dz^2}{(z-t)^2}\right)\|_A}{\|\frac{dz^2}{(z-t)^2}\|_B} \leq C_4\left(\frac{e}{4\pi^4}L + \frac{1}{Le^{\frac{1}{2\sqrt{L}}}}\right)$$

But, there exists a constant $J_2 > 0$ such that $\frac{1}{Le^{\frac{1}{2\sqrt{L}}}} \leq J_2L$ for all $L > 0$. Thus, if $D_3 = C_4(\frac{e}{4\pi^4} + J_2)$, then

$$\frac{\|\Theta\left(\frac{dz^2}{(z-t)^2}\right)\|_A}{\|\frac{dz^2}{(z-t)^2}\|_B} \leq D_3L$$

and we have completed our proof.

$\boxed{11.1}$

## 12 Coralling the seahorses

We now begin the process of organizing the seahorses which will contain the pre-image of the $2L$-thin part. We first observe that, as a consequence of the Margulis Lemma and McMullen's bounds on the diameter of a seahorse, the seahorses we are considering are not too large when viewed from the point of view of our test element.

We introduce the notion of a standard fundamental domain in order to make this claim precise. If $\alpha$ is a hyperbolic Möbius transformation, we will say that an annulus $F \subset \widehat{\mathbf{C}}$ is a *standard fundamental domain* for $\langle\alpha\rangle$ if it is bounded by 2 circles $S_1$ and $S_2 = \alpha(S_1)$ and the hyperbolic planes $H_1$ and $H_2$ bounded by $S_1$ and $S_2$ are each perpendicular to the axis of $\alpha$. We will show that if $\gamma$ is represented by a short curve in the conformal boundary and if the seahorse associated to $\gamma$ intersects a standard fundamental domain $F$ for $\langle\alpha\rangle$, then the seahorse is contained in the union of $F$ and the two adjacent standard fundamental domains $\alpha(F)$ and $\alpha^{-1}(F)$.

**Lemma 12.1** *Given $d_0 > 0$ there exists $D_4 > 0$ with the following property. Suppose that $\Gamma$ is a Kleinian group, $\alpha$ and $\gamma$ are primitive hyperbolic elements of $\Gamma$ and $q$ is a fixed point of $\alpha$ which is not a fixed point of $\gamma$. Suppose that $\gamma$ is represented by a curve of length at most $L$ in $\Omega(\Gamma)/\Gamma$, where*

1. *$L \leq D_4e^{-l(\alpha)}$, and*

2. *$l(\alpha) > d_0$.*

*If $B(\gamma, \frac{1}{\sqrt{L}}, q)$ intersects a standard fundamental domain $F$ for $\langle\alpha\rangle$, then*

$$B(\gamma, \frac{1}{\sqrt{L}}, q) \subset \alpha^{-1}(F) \cup F \cup \alpha(F).$$

*Proof of 12.1:* We first normalize the situation so that $\gamma$ has fixed points 0 and 1 and $q = \infty$. Let $t$ be the other fixed point of $\alpha$ and let $\lambda$ be the complex translation length of $\gamma$.

Let $F_0$ be the standard fundamental domain for $\langle\alpha\rangle$ bounded by circles about $t$ of radius $e^{l(\alpha)/2}|t|$ and $e^{-l(\alpha)/2}|t|$. It is an easy calculation to show that the disk $D$ centered at 0 of radius $|t|(1 - e^{-l(\alpha)/2})$ is contained in $F_0$. Since $l(\alpha) > d_0$, the radius of $D$ is at least $J_3|t|$ where $J_3 = 1 - e^{-d_0/2}$. We will show that we can choose $D_4$ so as to guarantee that $B \subset D \subset F_0$.

If we assume that $D_4 \leq \frac{D_1}{2e}$ and that $D_4 \leq 1$, then then we may apply Proposition 11.2 and the second assumption in the lemma to see that

$$|\lambda| \leq 2Le^{\frac{L}{2}} < 2eL \leq D_1 e^{-l(\alpha)} < D_1 e^{\frac{-l(\alpha)}{2}}.$$

Proposition 10.1 implies that

$$|t| \geq \frac{d_1}{|\lambda|} e^{\frac{-l(\alpha)}{2}}.$$

Hence,

$$\text{radius}(D) \geq \frac{J_3 d_1}{|\lambda|} e^{\frac{-l(\alpha)}{2}}.$$

On the other hand, Proposition 8.1 gives that

$$\text{diam}(B) \leq \frac{C_1\sqrt{L}}{|\lambda|}.$$

If, in addition, $D_4 \leq \left(\frac{J_3 d_1}{C_1}\right)^2$, then

$$L \leq \left(\frac{J_3 d_1}{C_1}\right)^2 e^{-l(\alpha)},$$

which implies that

$$C_1\sqrt{L} \leq J_3 d_1 e^{\frac{-l(\alpha)}{2}}.$$

So the diameter of $B$ is less than the radius of $D$, which implies that $B \subset D \subset F_0$ But since $F$ and $F_0$ are intersecting standard fundamental domains, $F_0 \subset \alpha^{-1}(F) \cup F \cup \alpha(F)$ and $B \subset \alpha^{-1}(F) \cup F \cup \alpha(F)$ as claimed. Hence we have established the theorem if we take

$$D_4 = \min\left\{\frac{D_1}{2e}, 1, \left(\frac{J_3 d_1}{C_1}\right)^2\right\}.$$

$\boxed{12.1}$

Proposition 8.1 assures us that the area of $B(\gamma, \frac{1}{\sqrt{L}}, q)$ is comparable to the area of $B(\gamma, \frac{2}{\sqrt{L}}, q)$. We will also need to know that $\|\phi\|_{B(\gamma,\frac{1}{\sqrt{L}},q)}$ is comparable to $\|\phi\|_{B(\gamma,\frac{2}{\sqrt{L}},q)}$, where $\phi$ and $\gamma$ are as in proposition 11.1.

**Lemma 12.2** *There exists a positive constant $C_5$ with the following property. Suppose that $\Gamma$ is a Kleinian group, $\alpha$ and $\gamma$ are non-commuting primitive hyperbolic elements of $\Gamma$, $\phi$ is the pull-back of a quadratic differential on $T(\alpha)$ and $q$ is a fixed point of $\alpha$. Suppose that $\gamma$ is represented by a curve of length at most $L$ in $\Omega(\Gamma)/\Gamma$, with*

$$L \leq D_2 e^{\frac{-l(\alpha)}{2}},$$

*where $D_2$ is the constant in Proposition 11.1. Then*

$$||\phi||_{B(\gamma,\frac{1}{\sqrt{L}},q)} \leq C_5 ||\phi||_{B(\gamma,\frac{2}{\sqrt{L}},q)}.$$

*Proof of 12.2:* As in the proof of Proposition 11.1, we may normalize so that that the fixed points of $\gamma$ are 0 and 1, $q = \infty$ and $\phi = \frac{dz^2}{(z-t)^2}$ where $t$ is a fixed point of $\alpha$. In the proof of Proposition 11.1, we saw that if $L \leq D_2 e^{\frac{-l(\alpha)}{2}}$ and we choose $m = \frac{1}{\sqrt{L}}$, then $|t| \geq \frac{c_4}{|m\lambda|}$ and $M(\gamma) > m > 4\pi$.

In the proof of Proposition 9.2, we saw that this guaranteed that $2|t| \geq |z-t| \geq |t|/2$ for all $z \in B(\gamma, \frac{1}{\sqrt{L}}, q)$. In particular,

$$||\phi||_{B(\gamma,\frac{1}{\sqrt{L}},q)} \leq \frac{4}{|t|^2}\text{area}(B(\gamma, \frac{1}{\sqrt{L}}, q))$$

and

$$||\phi||_{B(\gamma,\frac{2}{\sqrt{L}},q)} \geq \frac{1}{4|t|^2}\text{area}(B(\gamma, \frac{2}{\sqrt{L}}, q)).$$

Since $L \leq D_2 e^{\frac{-l(\alpha)}{2}}$ guarantees that $L < 1$, Proposition 11.2 gives that

$$M(\gamma) \geq \frac{2\pi^2}{Le^{\frac{L}{2}}} \geq \frac{4}{\sqrt{L}}.$$

Therefore, we may apply Proposition 8.1 to see that

$$C_2 \left(\frac{C_1\sqrt{L}}{|\lambda|}\right)^2 > \text{area}(B(\gamma, \frac{1}{\sqrt{L}}, q))$$

and that

$$c_2 \left(\frac{c_1\sqrt{L}}{2|\lambda|}\right)^2 < \text{area}(B(\gamma, \frac{2}{\sqrt{L}}, q)).$$

It follows that

$$||\phi||_{B(\gamma,\frac{1}{\sqrt{L}},q)} \leq 64 \left(\frac{C_2C_1^2}{c_2c_1^2}\right) ||\phi||_{B(\gamma,\frac{2}{\sqrt{L}},q)}.$$

Hence, the lemma holds with $C_5 = 64\left(\frac{C_2C_1^2}{c_2c_1^2}\right)$.

$\boxed{12.2}$

## 13 Organizing the seahorses

Let $\Gamma$ be a discrete torsion-free Kleinian group and let $C$ be a disjoint pinchable collection of simple closed geodesics in the conformal boundary of $\mathbf{H}^3/\Gamma$. Let $\mathcal{G}$ be the $\Gamma$-invariant collection of primitive hyperbolic elements of $\Gamma$ which are represented by some geodesic in $C$. Recall that since $C$ is pinchable, the curves in $C$ are associated to distinct conjugacy class of primitive hyperbolic elements of $\Gamma$.

In the proof of our main local estimate we will consider the collection $\cup_{\gamma\in\mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q)$. This collection of seahorses will contain the support of the lift $\widetilde{\mu}$ of the Beltrami differential $\mu$ in that estimate. We will be pairing $\widetilde{\mu}$ with a quadratic differential $\phi$ and we would like to estimate the contribution to the pairing on each seahorse and then sum to obtain our estimate. However, these seahorses need not be disjoint, so we will encounter difficulties when attempting to sum our estimates on the individual seahorses.

In Theorem 4.5 of [Mc 1], McMullen shows how to find a subset $\mathcal{G}'$ of $\mathcal{G}$ and a collection of disjoint sets $\{E_\gamma\}_{\gamma\in\mathcal{G}'}$ covering $\cup_{\gamma\in\mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q)$, such that, for all $\gamma \in \mathcal{G}'$, $E_\gamma$ is $\gamma$-invariant, contains $B(\gamma, \frac{2}{\sqrt{L}}, q)$ and is contained in $B(\gamma, \frac{1}{\sqrt{L}}, q)$. We will then be able to look at the restriction of the Beltrami differential to $E_\gamma$ for all $\gamma \in \mathcal{G}'$, perform estimates in $B(\gamma, \frac{1}{\sqrt{L}}, q)$ and sum these estimates to obtain the desired bounds.

The following theorem is essentially a version of Theorem 4.5 in [Mc 1].

**Theorem 13.1** (McMullen) *Given $d_0 > 0$ there exists $D_5 > 0$ with the following property. Suppose that $\Gamma$ is a Kleinian group and $\mathcal{G}$ is a $\Gamma$-invariant collection of primitive hyperbolic elements of $\Gamma$ constructed from a pinchable collection $C$ of geodesics as above. Suppose that $q \in \widehat{\mathbf{C}}$ is a fixed point of a primitive hyperbolic element $\alpha \in \Gamma$ which is not a fixed point of any element in $\mathcal{G}$. If each element of $C$ has length at most $L$,*

1. *$L \leq D_5 e^{-l(\alpha)}$, and*
2. *$l(\alpha) \geq d_0$,*

*then there exists a subset $\mathcal{G}'$ of $\mathcal{G}$ and $\gamma$-invariant sets $\{E_\gamma\}_{\gamma\in\mathcal{G}'}$ such that*

1. *$E_\gamma \cap E_{\gamma'} = \emptyset$ for distinct elements $\gamma, \gamma' \in \mathcal{G}'$,*
2. $$B(\gamma, \frac{2}{\sqrt{L}}, q) \subset E_\gamma \subset B(\gamma, \frac{1}{\sqrt{L}}, q)$$ *for all $\gamma \in \mathcal{G}'$, and*
3. $$\cup_{\gamma\in\mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q) \subset \cup_{\gamma\in\mathcal{G}'} E_\gamma.$$

**Remark.** In this remark, we simply indicate the mild changes needed to establish our version of McMullen's result. McMullen's proof goes through exactly as written to show that there exists a uniform constant $J_4$ such that if $\frac{2}{\sqrt{L}} > J_4$, then there exists a collection of $\gamma$-invariant sets $\{E_\gamma\}_{\gamma\in\mathcal{G}}$ such that

$$B(\gamma, \frac{2}{\sqrt{L}}, q) \subset E_\gamma \subset B(\gamma, \frac{1}{\sqrt{L}}, q)$$

for all $\gamma \in \mathcal{G}$ and such that if two of the sets $E_{\gamma_1}$ and $E_{\gamma_2}$ intersect, then one is a subset of the other. Moreover, he shows that there exists some constant $C_6 > 1$ such that if $E_{\gamma_1} \subset E_{\gamma_2}$, then $\mathrm{diam}(E_{\gamma_2}) \geq C_6 \mathrm{diam}(E_{\gamma_1})$. In his setting, McMullen uses the fact that the union of the seahorses is bounded to see that one can choose a collection of maximal elements of $\{E_\gamma\}_{\gamma\in\mathcal{G}}$ (with respect to inclusion). This collection of maximal elements forms the desired $\{E_\gamma\}_{\gamma\in\mathcal{G}'}$.

In our situation, $\cup_{\gamma\in\mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q)$ is not bounded. However, given $d_0 > 0$, Lemma 12.1 provides $D_4 > 0$ such that if $L \leq D_4 e^{-l(\alpha)}$, $\gamma_0 \in \mathcal{G}$ and $B(\gamma_0, \frac{2}{\sqrt{L}}, q)$ intersects a standard fundamental domain $F$ for $\langle\alpha\rangle$, then $B(\gamma_0, \frac{1}{\sqrt{L}}, q)$, and hence $E_{\gamma_0}$, is contained in $\alpha(F) \cup F \cup \alpha^{-1}(F)$. It follows that any nested ascending sequence in $\{E_\gamma\}_{\gamma\in\mathcal{G}}$, which contains $B(\gamma_0, \frac{2}{\sqrt{L}}, q)$ is contained in $\alpha(F) \cup F \cup \alpha^{-1}(F)$ and is thus bounded. So we can again choose a collection of maximal elements of $\{E_\gamma\}_{\gamma\in\mathcal{G}}$. Hence, our version of McMullen's theorem holds if we choose

$$D_5 = \min\left\{D_4, \left(\frac{2}{J_4}\right)^2\right\}.$$

## 14 The main local estimate

In this section, we give the proof of the main local estimate, Theorem 14.1. We have already indicated, in sections 5 and 6, how this estimate is used to establish our main result.

We recall that if $M$ is a compact oriented 3-manifold and $a$ is a primitive element of $\pi_1(M)$, then we define $\Upsilon_a : CC_0(M) \to CC_0(S^1 \times D^2)$ by letting $\Upsilon_a([\rho])$ be the conjugacy class of the restriction $\rho_a$ of $\rho$ to $\langle a\rangle$. The map $\Upsilon_a$ essentially records the complex length of $\rho(a)$. In the following estimate we think of the unit norm Beltrami differential $\mu$ as a representative of a tangent vector to $CC_0(M)$ at a point $[\rho]$. Our estimate asserts that if $\mu$ is supported on the $2L$-thin part of $\partial_c N_\rho$, $l(\rho(a))$ is not small and $L$ is small enough, then $D\Upsilon_a(D\bar\Phi(\mu))$ has length at most $O(L)$. We interpret this as saying that if one deforms $[\rho]$ in the direction determined by $\mu$, then the complex length of the image of $a$ changes very little.

**Theorem 14.1** *Given $d_0 > 0$, there exists $D_6 > 0$ and $K_0 > 0$ with the following properties. Suppose that $M$ is a compact, oriented 3-manifold, $a$ is a primitive element in $\pi_1(M)$, $[\rho] \in CC_0(M)$ and $l(\rho(a)) > d_0$. Suppose that $C$ is a pinchable collection of disjoint simple closed geodesics in $\partial_c N_\rho$, none of which represents $\rho(a)$, such that each element of $C$ has length at most $L$ where*

$$L \leq D_6 e^{-l(\rho(a))}.$$

*If $\mu$ is a unit-norm Beltrami differential on $\partial_c N_\rho$ which is supported on the union of the $2L$-thin parts associated to elements of $C$, then*

$$||D\Upsilon_a(D\bar{\Phi}(\mu))|| \leq K_0 L.$$

*Proof of 14.1:* Let $\mathcal{G}$ denote the collection of primitive hyperbolic elements of $\Gamma = \rho(\pi_1(M))$ which are represented by elements of $C$. Let $q$ be a fixed point of $\alpha = \rho(a)$. If we assume that $D_6 \leq D_5$ , then Theorem 13.1 provides a subset $\mathcal{G}'$ of $\mathcal{G}$ and a disjoint collection of $\gamma$-invariant subsets $\{E_\gamma\}_{\gamma \in \mathcal{G}'}$ such that

$$B(\gamma, \frac{2}{\sqrt{L}}, q) \subset E_\gamma \subset B(\gamma, \frac{1}{\sqrt{L}}, q)$$

for all $\gamma \in \mathcal{G}'$, and

$$\cup_{\gamma \in \mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q) \subset \cup_{\gamma \in \mathcal{G}'} E_\gamma.$$

Let $\tilde{\mu}$ be the lift of $\mu$ to $\Omega(\Gamma)$ and let $B_\gamma = B(\gamma, \frac{1}{\sqrt{L}}, q)$ and $A_\gamma = A(\gamma, \frac{1}{\sqrt{L}}, q)$. We first prove that $\tilde{\mu}$ is supported on $\cup_{\gamma \in \mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q)$.

**Lemma 14.2** *There exists a constant $J_6$ such that if $L < J_6$, then the Beltrami differential $\tilde{\mu}$ is supported on $\cup_{\gamma \in \mathcal{G}} B(\gamma, \frac{2}{\sqrt{L}}, q)$.*

*Proof of 14.2:* Let $\epsilon_2$ denote the 2-dimensional Margulis constant. We will assume that $2L < \epsilon_2$.

Suppose that $z$ is contained in the support of $\tilde{\mu}$. Since $\mu$ is supported on the $2L$-thin part associated to the pinchable collection of geodesics $C$, there exists an annular component $Q'$ of the $2L$-thin part of $\partial_c N_\rho$ containing $p(z)$, where $p : \mathbf{H}^3 \cup \Omega(\Gamma) \to \bar{N}_\rho$ is the obvious covering map. Let $\eta$ be the geodesic contained in $Q'$ and let $S$ denote the component of the pre-image of $Q'$ which contains $z$. There exists an element $\gamma \in \mathcal{G}$, which is represented by $\eta$, such that $S$ is $\gamma$-invariant. We will show that $z \in B(\gamma, \frac{2}{\sqrt{L}}, q)$.

Let $Q$ be the component of the $\epsilon_2$-thin part of $\partial_c N_\rho$ containing $\eta$. One may compute, as in Maskit [Mas 2], that the modulus of each component of $Q - Q'$ is $\frac{2\pi(\theta_1 - \theta_2)}{l(\eta)}$ where $\cos\theta_1 = \frac{\sinh(l(\eta)/2)}{\sinh(\epsilon_2)}$ and $\cos\theta_2 = \frac{\sinh(l(\eta)/2)}{\sinh(2L)}$. One may use basic trigonometric formulas to show that there exists positive constants $J_7$ and $C_7$ such that if $L \leq J_7$ then $\theta_1 - \theta_2 \geq \frac{C_7 l(\eta)}{L}$. Therefore, in this case, the modulus of each component of $Q - Q'$ is at least $\frac{2\pi C_7}{L}$.

The annuli $Q$ and $Q'$ lift to annuli $\widetilde{Q}$ and $\widetilde{Q}'$ in $T_\gamma$ and $p_\gamma^{-1}(\widetilde{Q}') = S$ where $p_\gamma : \Omega_\gamma \to T_\gamma$ is the obvious covering map. Since $q$ does not lie in $\Omega(\Gamma)$, $p_\gamma(q)$ does not lie in $\widetilde{Q}$. Lemma 5.1 of [Mc 1] guarantees that there exists a constant $J_8$ such that, if $\frac{2\pi C_7}{L} > J_8$, then each component of $\widetilde{Q} - \widetilde{Q}'$ contains a right cylinder of modulus at least $\frac{2\pi C_7}{L} - J_8$. If $L < \frac{\pi C_7}{J_8}$, then each component of $\widetilde{Q} - \widetilde{Q}'$ contains a right cylinder of modulus $\frac{\pi C_7}{L}$. Let $R_1$ and $R_2$ denote these right cylinders.

The annulus $A^c(\gamma, \frac{2}{\sqrt{L}}, q) = T_\gamma - A(\gamma, \frac{2}{\sqrt{L}}, q)$ is a right cylinder of modulus $\frac{2}{\sqrt{L}}$. It is easy to check that no Euclidean torus contains two right cylinders of modulus more than $4\pi$ which are not homotopic, so if $L < \min\{\frac{C_7}{4}, \frac{1}{4\pi^2}\}$, then $A^c(\gamma, \frac{2}{\sqrt{L}}, q)$, $R_1$ and $R_2$ must all be homotopic right cylinders. If $\widetilde{Q}'$ intersects $A^c(\gamma, \frac{2}{\sqrt{L}}, q)$, then, since $p_\gamma(q)$ does not lie in $\widetilde{Q}$, either $R_1$ or $R_2$ is contained entirely in $A^c(\gamma, \frac{2}{\sqrt{L}}, q)$, which implies that $\frac{2}{\sqrt{L}} \geq \frac{\pi C_7}{L}$. However, this is impossible if we assume that $L < \frac{\pi^2 C_7^2}{4}$. Thus if we take

$$J_6 = \min\{\frac{\epsilon_2}{2}, J_7, \frac{\pi C_7}{J_8}, \frac{C_7}{4}, \frac{1}{4\pi^2}, \frac{\pi^2 C_7}{5}\}$$

and assume $L < J_6$, then $\widetilde{Q}' \subset A(\gamma, \frac{2}{\sqrt{L}}, q)$ which in turn implies that

$$z \in S = p_\gamma^{-1}(\widetilde{Q}') \subset p_\gamma^{-1}(A(\gamma, \frac{2}{\sqrt{L}}, q)) = B(\gamma, \frac{2}{\sqrt{L}}, q)$$

as desired.

$\boxed{14.2}$

It follows that if $D_6 \leq J_6$, then the support of $\tilde{\mu}$ is contained in $\cup_{\gamma \in \mathcal{G}'} E_\gamma$.

We recall that

$$||D\Upsilon_a(D\bar{\Phi}(\mu))|| = \sup\left\{\left\langle \bar{\phi}, d\widehat{\Upsilon}_a(\mu)\right\rangle \mid \bar{\phi} \in Q(T(\rho(a))),\ ||\bar{\phi}|| = 1\right\}$$

So let $\bar{\phi}$ be a unit norm quadratic differential on $T(\rho(a))$ and let $\phi$ denote its pull-back to $\Omega(\langle\rho(a)\rangle)$. If $F$ is a standard fundamental domain for $\langle\rho(a)\rangle$, then since $\tilde{\mu}$ is a lift of $D\widehat{\Upsilon}_a(\mu)$ (see the end of section 2.3),

$$\left\langle \bar{\phi}, D\widehat{\Upsilon}_a(\mu)\right\rangle = Re\left(\int_F \phi\tilde{\mu}\right) \leq \left|\int_F \phi\tilde{\mu}\right|.$$

Let $\nu_\gamma$ be the restriction of $\tilde{\mu}$ to $E_\gamma$. Then, $\nu_\gamma$ is a $\gamma$-invariant Beltrami differential of norm at most 1. We will estimate the integral above by estimating the integral over each $E_\gamma$ and summing. For each $\gamma \in \mathcal{G}'$, let $\Theta_\gamma = \Theta_{B_\gamma/A_\gamma}$. The duality expressed in Equation (4), see section 9, gives that

$$\left|\int_{E_\gamma} \phi\tilde{\mu}\right| = \left|\int_{B_\gamma} \nu_\gamma\phi\right| \leq ||\Theta_\gamma(\phi)||_{A_\gamma}$$

If we assume that $D_6 \leq D_2$, then Proposition 11.1 implies that

$$||\Theta_\gamma(\phi)||_{A_\gamma} \leq D_3 L \,||\phi||_{B_\gamma}$$

Let $\mathcal{G}''$ be the set of elements $\gamma \in \mathcal{G}'$ with the property that $E_\gamma \cap F$ is non-empty. Then

$$\left|\int_F \phi\tilde{\mu}\right| \leq \sum_{\gamma \in \mathcal{G}''} \left|\int_{E_\gamma} \phi\tilde{\mu}\right| \leq D_3 L \sum_{\gamma \in \mathcal{G}''} ||\phi||_{B_\gamma}$$

Lemma 12.2 implies that if $D_6 \le D_2$, then

$$||\phi||_{B_\gamma} \le C_5 \, ||\phi||_{B(\gamma, \frac{2}{\sqrt{L}}, q)} .$$

Since $B(\gamma, \frac{2}{\sqrt{L}}, q) \subset E_\gamma \subset B_\gamma$, this implies that

$$||\phi||_{B_\gamma} \le C_5 \, ||\phi||_{E_\gamma} .$$

Lemma 12.1 implies that if $D_6 \le D_4$ and $\gamma \in \mathcal{G}''$, then

$$E_\gamma \subset B(\gamma, \frac{1}{\sqrt{L}}, q) \subset \alpha^{-1}(F) \cup F \cup \alpha(F).$$

Since the $\{E_\gamma\}$ are pairwise disjoint and $\cup_{\gamma \in \mathcal{G}''} E_\gamma \subset \alpha^{-1}(F) \cup F \cup \alpha(F)$,

$$\sum_{\gamma \in \mathcal{G}''} ||\phi||_{E_\gamma} \le 3.$$

Therefore,

$$\left\langle \bar{\phi}, D\widehat{\Upsilon}_a(\mu) \right\rangle \le \left| \int_F \phi \tilde{\mu} \right| \le 3 D_3 C_5 L$$

Since this holds for an arbitrary unit norm quadratic differential on $T(\alpha)$, the theorem holds if we choose $D_6 = \min\{D_2, D_4, D_5, J_6\}$ and $K_0 = 3D_3C_5$.

$\boxed{14.1}$

## 15 Corollaries

If $H_g$ is a handlebody of genus $g \ge 2$, then Marden (see section 7.4 of [Mar 2]) observed that there is a dense set of conjugacy classes in $\partial CC(H_g)$ whose associated representations have empty domain of discontinuity. The following corollary is then immediate from our main result.

**Corollary 15.1** *If $H_g$ is a handlebody of genus $g \ge 2$, then maximal cusps are dense in the boundary of Schottky space $CC(H_g)$.*

If $\partial M$ is connected, then we may similarly observe that there is a dense set $\mathcal{D}$ of conjugacy classes in $\partial CC(M)$ such that if $[\rho] \in \mathcal{D}$, then $\Omega(\rho) = \emptyset$.

**Lemma 15.2** *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold whose (non-empty) boundary is a connected surface which is not a torus. If $\mathcal{D}$ is the set of conjugacy classes of purely hyperbolic representations in $\partial CC_0(M)$, then $\mathcal{D}$ is dense in $\partial CC_0(M)$. Furthermore, if $[\rho] \in \mathcal{D}$, then $\Omega(\rho) = \emptyset$.*

*Proof of 15.2:* It will be convenient to work in the pre-image $\widehat{CC}_0(M)$ of $CC_0(M)$ in the full representation variety $\widehat{R} = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$. Since $\widehat{CC}_0(M)$ is a $\mathrm{PSL}_2(\mathbf{C})$-bundle over $CC_0(M)$ it suffices to prove that the set $\widehat{\mathcal{D}}$ of purely hyperbolic representations in $\partial\widehat{CC}_0(M)$ is dense and that if $\rho \in \widehat{\mathcal{D}}$, then $\Omega(\rho) = \emptyset$.

The proof relies on two foundational results about the topology of $\widehat{CC}_0(M)$. Sullivan [Sull 2] proved that $\widehat{CC}_0(M)$ is the interior of its closure in the representation variety. Kapovich [Ka] proved that every point in the closure of $\widehat{CC}_0(M)$ is a smooth point of $\widehat{R}$.

We first show that $\widehat{\mathcal{D}}$ is dense in $\partial\widehat{CC}_0(M)$. If $\rho_0 \in \partial\widehat{CC}_0(M)$ does not lie in the closure of $\widehat{\mathcal{D}}$, then $\rho_0$ has a smooth, connected open neighborhood $U$ in $\widehat{R}$ such that $U \cap \partial\widehat{CC}_0(M) \subset X$, where $X \subset \widehat{R}$ is the set of all representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(\gamma)$ is parabolic or the identity for some nontrivial $\gamma \in \pi_1(M)$. Note that $X$ is a countable union of complex algebraic subsets of $\widehat{R}$, and that $X \cap \widehat{CC}_0(M) = \emptyset$, so that $U \not\subset X$. Hence if $\widehat{R}_0$ denotes the irreducible component of $\widehat{R}$ containing $U$, the set $X_0 = X \cap \widehat{R}_0$ is a countable union of proper complex algebraic subvarieties of $\widehat{R}_0$. Since $U \subset \widehat{R}_0$ is smooth and connected, $W = U - (U \cap X)$ is a connected, dense subset of $U$. As $W$ meets $\widehat{CC}_0(M)$ but is disjoint from $\partial\widehat{CC}_0(M)$, connectedness guarantees that $W \subset \widehat{CC}_0(M)$. Hence $U$ is contained in the closure of $\widehat{CC}_0(M)$ in $\widehat{R}$. But, since $\widehat{CC}_0(M)$ is the interior of its closure, this implies that $\rho$ lies in the interior of $\widehat{CC}_0(M)$, which is a contradiction.

Now suppose that $\rho \in \widehat{\mathcal{D}}$ and $\Omega(\rho)$ is non-empty. There exists a sequence $\{\rho_i\}$ in $CC(M)$ converging to $\rho$. Since $\rho$ is purely hyperbolic and $\Omega(\rho)$ is non-empty, Theorem E of Anderson-Canary [AC 1] implies that $\{\rho_i\}$ converges strongly to $\rho$ (i.e. that $\{\rho_i(\pi_1(M))\}$ converges geometrically to $\rho(\pi_1(M)).$) The main theorem of Canary-Minsky [CaMi] then implies that there is a homeomorphism from the interior of $M$ to $N_\rho$. In particular, $N_\rho$ has only one end. Since $\Omega(\rho)$ is non-empty, the one end of $N_\rho$ must be geometrically finite, so $N_\rho$ is convex cocompact. Thus, $\rho$ is a convex cocompact uniformization of $M$ and hence lies in $\widehat{CC}_0(M)$ which is again a contradiction. Therefore, if $\rho \in \widehat{\mathcal{D}}$, then $\Omega(\rho) = \emptyset$ as desired.

$\boxed{15.2}$

Combining this observation with our main result we obtain the desired generalization of Corollary 15.1.

**Corollary 15.3** *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold whose (non-empty) boundary is a connected surface which is not a torus. Then maximal cusps are dense in the boundary of $CC(M)$.*

As another corollary of our results we see that hyperbolic 3-manifolds with arbitrarily short geodesics are dense in $\partial CC_0(M)$.

**Corollary 15.4** *Let $M$ be a compact, oriented, irreducible, atoroidal 3-manifold whose (non-empty) boundary is a connected surface which is not a torus. The set $\mathcal{S}$ in $\partial CC_0(M)$ consisting of all $[\rho] \in \mathcal{S}$ such that $N_\rho$ contains arbitrarily short geodesics, is a dense $G_\delta$ in $\partial CC_0(M)$.*

*Proof of 15.4:* If $g$ is an element of $\pi_1(M)$ which represents a simple closed curve in $\partial M$, then $l_{[\rho]}(g)$, the real translation length of $\rho(g)$, is a non-constant real analytic function on $AH(\pi_1(M))$. Therefore, the set $\mathcal{U}_n$ of conjugacy classes in $\partial CC_0(M)$ whose associated manifolds contained a closed geodesic of length at most $\frac{1}{n}$ is open in $\partial CC_0(M)$. Since, by corollary 15.3, maximal cusps are dense in $\partial CC_0(M)$, $\mathcal{U}_n$ is also dense in $\partial CC_0(M)$. The Baire category theorem then applies to show that $\mathcal{S}$ is a dense $G_\delta$ in $\partial CC_0(M)$.

$\boxed{15.4}$

## 16 Pared manifolds

In this section, we will explain how to extend our main result to the setting of deformation spaces of general geometrically finite Kleinian groups. In particular, we extend our results to the setting where our manifold $M$ is allowed to have a toroidal boundary component. In order to do so, we introduce the formalism of pared manifolds.

**Definition 16.1** ([Mo]) *A pared manifold is a pair $(M, P)$, where*

- *$M$ is a compact, irreducible 3-manifold and*
- *$P \subset \partial M$ is a union of incompressible annuli and tori,*

*such that*

1. *If $A$ is an abelian subgroup of $\pi_1(M)$ which is not cyclic, then $A$ is conjugate into the fundamental group of a component of $P$, and*
2. *every map $\phi : (S^1 \times I, S^1 \times \partial I) \to (M, P)$ that is injective on the fundamental groups, is homotopic, as a map of pairs, into $P$.*

Let $(M, P)$ be a pared manifold. A representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ is a *geometrically finite uniformization* of $(M, P)$ if there exists a homeomorphism $h : M - P \to \bar{N}_\rho$ such that $[h_*] = [\rho]$. Let $GF_0(M, P)$ denote the space of (conjugacy classes of) geometrically finite uniformizations of $(M, P)$. The space $GF_0(M, P)$ may be identified with an open subset of the space of conjugacy classes of representations of $\pi_1(M)$ in which each element of $\pi_1(P)$ is taken to a parabolic element or the identity. If we let $\mathrm{Mod}_0(M, P)$ denote the group of isotopy classes of pared homeomorphisms of $(M, P)$ which are homotopic to the identity, then $GF_0(M, P)$ may be identified with the quotient of $\mathcal{T}(\partial M - P)$ by $\mathrm{Mod}_0(M, P)$.

A conjugacy class $[\rho] \in AH(\pi_1(M))$ is said to be a *maximal cusp* if there exists a homeomorphism $h : M' - P' \to \bar{N}_\rho$ such that $(M', P')$ is a pared 3-manifold and each component of $\partial M' - P'$ is an open pair of pants. Our main results generalizes to the setting of pared 3-manifolds as follows:

**Theorem 16.2** *Let $(M,P)$ be any pared 3-manifold. If $[\rho]\in\partial GF_0(M,P)$ and $\Omega(\rho)=\emptyset$, then $[\rho]$ may be approximated by maximal cusps in $\partial GF_0(M,P)$.*

The proof of Theorem 16.2 is largely an immediate generalization of the proof of our main theorem. The main technical difference comes in the definition of a pinchable pants decomposition and in the proof of Proposition 3.1.

Suppose that $[\rho]\in GF_0(M,P)$ and $h: M-P\to \bar N_\rho$ is an orientation-preserving homeomorphism. If $A$ is a collection of disjoint simple closed geodesics in $\partial_c N_\rho$, then let $\mathcal{N}(h^{-1}(A))$ be a closed regular neighborhood of $h^{-1}(A)$. We say that $A$ is *pinchable* if $(M, P\cup\mathcal{N}(h^{-1}(A)))$ is a pared 3-manifold. One may readily check that this is equivalent to our original definition when $P=\emptyset$.

In the generalization of Proposition 3.1 one assumes that $(\rho_n)$ is a sequence of geometrically finite uniformizations of a pared 3-manifold $(M,P)$, such that $(\rho_n)$ converges to $\rho$ and $\Omega(\rho)=\emptyset$. One proves that if $C_n$ is a collection of disjoint simple closed geodesics in $\partial_c N_{\rho_n}$ and $l(C_n)<K$ for all $n$, then $C_n$ is pinchable for all sufficiently large $n$. Let $\delta$ and $S$ be chosen as in the original proof and, for each $n$, let $h_n: M-P\to\bar N_{\rho_n}$ be an orientation-preserving homeomorphism. The original proof of Proposition 3.1 must only be modified to rule out essential annuli in $M$ with one boundary component in $h_n^{-1}(C_n)$ and the other boundary component in $P$. To do so, one considers the $\epsilon_n$-thick part $C(N_{\rho_n})_{thick(\epsilon_n)}$ of the convex core, where $\epsilon_n$ is chosen to be much less than $\delta$ and less than the length of the shortest closed geodesic in $N_{\rho_n}$. Any essential annulus with one boundary component in $h_n^{-1}(C_n)$ and the other boundary component in $P$ gives rise to an essential annulus $B_n'$ in $C(N_{\rho_n})_{thick(\epsilon_n)}$ with one boundary component which is a curve in $\partial C(N_n)$, of length at most $K'=45Ke^{\frac{K}{2}}$, which is homotopic to a component of $r_n(C_n)$ and the other boundary component lies in the portion of $\partial C(N_{\rho_n})_{thick(\epsilon_n)}$ which abuts the $\epsilon_n$-thin part of $N_{\rho_n}$. One again constructs an annulus $Y_n$ which is homotopic rel boundary to $B_n'$ and shows that the distance from the basepoint to $\partial Y_n\cap\partial C(N_{\rho_n})$ is uniformly bounded by a constant which depends only on $\delta$ and $S$. More details on the generalization of Proposition 3.1 and our main theorem can be found in [CaHe]

The following lemma is the natural generalization of Lemma 15.2 to this setting. We will say that representation $\rho\in\partial GF_0(M,P)$ is *minimally parabolic* if, for all $g\in\pi_1(M)$, $\rho(g)$ is parabolic if and only if $g$ is conjugate to an element of $\pi_1(P)$

**Lemma 16.3** *Let $(M,P)$ be a pared 3-manifold such that $\partial M-P$ is connected. If $\mathcal{D}$ is the set of minimally parabolic representations in $\partial GF_0(M,P)$, then $\mathcal{D}$ is dense in $\partial GF_0(M,P)$. Furthermore, if $\rho\in\mathcal{D}$, then $\Omega(\rho)=\emptyset$.*

*Proof of 16.3:* In this situation one lets $\widehat R$ be the subset of $\mathrm{Hom}(\pi_1(M),\mathrm{PSL}_2(\mathbf{C}))$ consisting of homomorphisms such that the image of any element of $\pi_1(P)$ is either parabolic or the identity. Then, $\widehat{GF}_0(M,P)$ is the pre-image of $GF_0(M,P)$ in $\widehat R$ and $\widehat{\mathcal{D}}$ is the pre-image of $\mathcal{D}$ in $\widehat R$. The proof that the set $\widehat{\mathcal{D}}$ of minimally parabolic representations in $\partial\widehat{GF}_0(M,P)$ is dense in $\partial\widehat{GF}_0(M,P)$ is virtually the same as the proof that $\widehat{\mathcal{D}}$ is dense in $\partial\widehat{CC}_0(M)$ given in Lemma 15.2.

In order to prove that $\Omega(\rho) = \emptyset$ if $\rho \in \widehat{\mathcal{D}}$ we follow the same outline as in the proof of Lemma 15.2. Suppose that $\rho \in \widehat{\mathcal{D}}$ and $\Omega(\rho) \neq \emptyset$. The main theorem of Anderson-Canary [AC 2] implies that any sequence $\{\rho_i\}$ in $\widehat{GF}_0(M, P)$ which converges algebraically to $\rho$ also converges strongly. Let $\epsilon$ be a positive constant which is less than the Margulis constant and let $N_\rho^0$ be obtained from $N_\rho$ be removing all the non-compact(cuspidal) components of its $\epsilon$-thin part. A recent result of Evans [Ev] implies that there exists a relative compact core $(M', P')$ for $N_\rho^0$ and an orientation-preserving homeomorphism $h' : (M, P) \to (M', P')$ such that $(h')_* = \rho$. (A relative compact core $(M', P')$ for $N_\rho^0$ is a compact core $M'$ for $N_\rho^0$ whose intersection $P'$ with $\partial N_0^\rho$ is a collection of compact cores for the components of $\partial N_\rho^0$, one for each component.) In particular, $N_\rho^0$ has only one end. Since, $\Omega(\rho)$ is non-empty, that end must be geometrically finite. It follows that $\rho$ is geometrically finite.

Marden's Stability theorem [Mar 1] implies that, since $\rho$ is geometrically finite and minimally parabolic, there is a neighborhood of $\rho$ in $\widehat{R}$ consisting of representations quasiconformally conjugate to $\rho$. Since $\rho$ is a limit of representations in $\widehat{GF}_0(M, P)$, it follows that $\rho \in \widehat{GF}_0(M, P)$, which is a contradiction.

$\boxed{16.3}$

We thus obtain the following natural generalization of Corollary 15.3:

**Corollary 16.4** *Let $(M, P)$ be a pared 3-manifold such that $\partial M - P$ is connected. Then maximal cusps are dense in the boundary of $GF_0(M, P)$.*

We recall that any torsion-free geometrically finite Kleinian group may be thought of as the image of a geometrically finite uniformization of some pared 3-manifold $(M, P)$ (see Corollary 10.6 in Morgan [Mo].) Its quasiconformal deformation space may then be identified with $GF_0(M, P)$. So, one may summarize Corollary 16.4 by saying that if the conformal boundary of any torsion-free geometrically finite Kleinian group is connected, then maximal cusps are dense in the boundary of its quasiconformal deformation space.

We also get an analogue of Corollary 15.4:

**Corollary 16.5** *Let $(M, P)$ be a pared 3-manifold such that $\partial M - P$ is connected. Then the set of conjugacy classes $[\rho] \in \partial GF_0(M, P)$ such that $N_\rho$ contains arbitrarily short geodesics is a dense $G_\delta$ subset of $\partial GF_0(M, P)$.*

## References

[AC 1] J. W. Anderson and R. D. Canary, "Cores of hyperbolic 3-manifolds and limits of Kleinian groups," *Amer. J. Math.* **118**(1996), 745–779.

[AC 2] J. W. Anderson and R. D. Canary, "Cores of hyperbolic 3-manifolds and limits of Kleinian groups II," *Journal of the L.M.S.*, **61**(2000), 489–505.

[ACCS] J. W. Anderson, R. D. Canary, M. Culler, and P.B. Shalen, "Free Kleinian groups and volumes of hyperbolic 3-manifolds," *J. Diff. Geom.* **44**(1996), 738–782.

[Be] A.F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag Graduate Texts in Mathematics, vol. 91, 1983.

[BP] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag Universitext, 1992.

[Bers 1] L. Bers, "On boundaries of Teichmüller spaces and on Kleinian groups: I," *Ann. of Math.* **91**(1970), 570–600.

[Bers 2] L. Bers, "On moduli of Kleinian groups," *Russian Math Surveys* **29**(1974), 88–102.

[Bers 3] L. Bers, "An inequality for Riemann surfaces," in *Differential geometry and complex analysis*, Springer-Verlag, 1985.

[BZ] S. Boyer and X. Zhang, "On Culler-Shalen seminorms and Dehn filling," *Ann. of Math.* **148**(1998), 737–801.

[Ca] R.D. Canary, "The conformal boundary and the boundary of the convex core," *Duke J. Math.*, **106**(2001), 193–207.

[CaHe] R.D. Canary and S. Hersonsky, in preparation.

[CaMc] R.D. Canary and D. McCullough, "Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups," preprint, available at:
http://www.math.ou.edu/~dmccullo/research/cm.html

[CaMi] R.D. Canary and Y.N. Minsky, "On limits of tame hyperbolic 3-manifolds," *J. Diff. Geom.* **43**(1996), 1–41.

[CHS] M. Culler, S. Hersonsky, and P.B. Shalen, "The first Betti number of the smallest closed hyperbolic 3-manifold," *Topology* **37**(1998), 805–849.

[CS1] M. Culler and P.B. Shalen, "Varieties of group representations and splittings of 3-manifolds," *Ann. of Math.* **117**(1983), 109–146.

[CS2] M. Culler and P.B. Shalen, "Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds," *J. Amer. Math. Soc.* **5**(1992), 231–288.

[CS3] M. Culler and P.B. Shalen, "Hyperbolic volume and mod p homology," *Comm. Math. Helv.* **68**(1993), 494–509.

[CS4] M. Culler and P.B. Shalen, "Volumes of hyperbolic Haken manifolds, I," *Invent. Math.* **118**(1994), 284–329.

[CS5] M. Culler and P.B. Shalen, "The volume of a hyperbolic 3-manifold with Betti number 2," *Proc. Amer. Math. Soc.* **120**(1994), 1281–1288.

[CS6] M. Culler and P.B. Shalen, "Volumes of Hyperbolic Haken Manifolds, II," *Proc. Amer. Math. Soc.* **125**(1997), 3059–3067.

[Ep-Mar] D.B.A. Epstein and A. Marden, "Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces," in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 113–253.

[Ev] R. Evans, "Deformation spaces of hyperbolic 3-manifolds: strong convergence and tameness." Ph.D. thesis, University of Michigan, Ann Arbor, 2000.

[Ga] F.P. Gardiner, *Teichmüller theory and quadratic differentials*, Wiley, 1987.

[ImTa] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces,* Springer-Verlag, 1992.

[Jo] T. Jørgensen, "On discrete groups of Möbius transformations," *Amer. J. Math.* **98** (1976), 739–749.

[Ka] M. Kapovich, *Hyperbolic manifolds and discrete groups: lectures on Thurston's hyperbolization*, Birkhäuser, 2000.

[Ke-Mas-Se] L. Keen, B. Maskit & C. Series, "Geometric finiteness and uniqueness of groups with circle packing limit sets," *J. Reine Angew. Math.* **436**(1993), 209–219.

[Mar 1] A. Marden, "The geometry of finitely generated Kleinian groups," *Ann. of Math.* **99**(1974), 383–462.

[Mar 2] A. Marden "Geometrically finite Kleinian groups," in *Discrete groups and automorphic forms*, ed. by W. Harvey, Academic Press, 1977, 259–293.

[Mas 1] B. Maskit, "Self-maps of Kleinian groups," *Amer. J. Math.* **93**(1971), 840–856.

[Mas 2] B. Maskit, "Comparison of hyperbolic and extremal lengths," *Ann. Acad. Sci. Fenn.* **10**(1985), 381–396.

[Mc 1] C.T. McMullen, "Cusps are dense," *Ann. of Math.* **133**(1991), 217–247.

[Mc 2] C.T. McMullen, "Complex earthquakes and Teichmüller theory," *J. Amer. Math. Soc.* **11**(1998), 283–320.

[Mo] J.W. Morgan, "On Thurston's uniformization theorem for three-dimensional manifolds," in *The Smith Conjecture*, ed. by J. Morgan and H. Bass, Academic Press, 1984, 37–125.

[Oh] K. Ohshika, "Geometrically finite Kleinian groups and parabolic elements," *Proc. Edinburgh Math. Soc.* **41**(1998), 141–159.

[Sh] P. B. Shalen, "Representations of 3-manifold groups," in *Handbook of Geometric Topology*, R. Daverman and R. Sher, eds., Elsevier, 2001.

[Su] T. Sugawa, "Uniform perfectness of the limit sets of Kleinian groups," *Trans. Amer. Math. Soc.*, **353**(2001), 3603–3615.

[Sull 1] D.P. Sullivan, "On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions," in *Riemann surfaces and related topics*, Ann. of Math. Stud. **97**(1980), 465–496.

[Sull 2] D. P. Sullivan, "Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity of Kleinian groups," *Acta Math.* **155**(1985), 243–260.

[Tan] H. Tanigawa, "Grafting, harmonic maps and projective structures on surfaces," *J. Diff. Geom.* **47**(1997), 399–419.