Cores of hyperbolic 3-manifolds and limits of Kleinian groups II

James W. Anderson and Richard D. Canary*

Faculty of Mathematical Studies, University of Southampton
Highfield, Southampton, SO17 1BJ, England
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

1 Introduction

Troels Jörgensen conjectured that the algebraic and geometric limits of an algebraically convergent sequence of isomorphic Kleinian groups agree if there are *no new parabolics* in the algebraic limit. We prove that this conjecture holds in "most" cases. In particular, we show that it holds when the domain of discontinuity of the algebraic limit of such a sequence is non-empty (see Theorem 3.1). We further show, with the same assumptions, that the limit sets of the groups in the sequence converge to the limit set of the algebraic limit. As a corollary, we verify the conjecture for finitely generated Kleinian groups which are not (non-trivial) free products of surface groups and infinite cyclic groups (see Corollary 3.3). These results are extensions of similar results for purely loxodromic groups which can be found in [4]. Thurston [32] previously established these results in the case that the Kleinian groups are freely indecomposable (see also Ohshika [24], [25], [27]). Using different techniques than ours, Ohshika [26] has proven versions of these results for purely loxodromic function groups.

Both authors would like to thank the Institut Henri Poincaré for its hospitality during the writing of this paper, as well as the referee for useful comments.

2 Preliminaries

The purpose of this section is to describe the background material used in this paper.

*Research supported in part by the National Science Foundation and a fellowship from the Sloan Foundation

2.1 Convergence of Kleinian groups

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, which we view as acting either on hyperbolic 3-space $\mathbf{H}^3$ via isometries or on the Riemann sphere $\overline{\mathbf{C}}$ via Möbius transformations. The action of Γ partitions $\overline{\mathbf{C}}$ into the *domain of discontinuity* $\Omega(\Gamma)$, which is the largest open subset of $\overline{\mathbf{C}}$ on which Γ acts properly discontinuously, and the *limit set* $\Lambda(\Gamma)$. A Kleinian group is *non-elementary* if its limit set contains at least three points, and is *elementary* otherwise. A Kleinian group is elementary if and only if it is virtually abelian; recall that a group is *virtually abelian* if it contains a finite index abelian subgroup. We refer the reader to Maskit [22] for a more detailed discussion of the theory of Kleinian groups.

It is often convenient to view a Kleinian group as the image of a discrete, faithful representation of a group into $\mathrm{PSL}_2(\mathbf{C})$. Given a finitely generated group G, let $\mathcal{D}(G)$ denote the space of all discrete, faithful representations of G into $\mathrm{PSL}_2(\mathbf{C})$. A sequence $\{\rho_j\}$ in $\mathcal{D}(G)$ converges *algebraically* to ρ if $\{\rho_j(g)\}$ converges to $\rho(g)$ for each $g \in G$. It is a fundamental result of Jørgensen [14] that $\mathcal{D}(G)$ is a closed subset of $Hom(G, \mathrm{PSL}_2(\mathbf{C}))$ when G is finitely generated and not virtually abelian.

An isomorphism $\alpha : \Phi \to \Theta$ between Kleinian groups Φ and Θ is *type-preserving* if φ is parabolic if and only if $\alpha(\varphi)$ is parabolic for all $\varphi \in \Phi$. More generally, an algebraically convergent sequence $\{\rho_j\} \subset \mathcal{D}(G)$ with limit $\rho \in \mathcal{D}(G)$ is *type-preserving* if $\rho_j \circ \rho^{-1} : \rho(G) \to \rho_j(G)$ is type-preserving for all j.

A sequence $\{\Gamma_j\}$ of Kleinian groups converges *geometrically* to a Kleinian group $\widehat{\Gamma}$ if every element of $\widehat{\Gamma}$ is the limit of a sequence $\{\gamma_j \in \Gamma_j\}$ and if every accumulation point of every sequence $\{\gamma_j \in \Gamma_j\}$ lies in $\widehat{\Gamma}$. We make use of the following proposition, which assures that algebraically convergent sequences in $\mathcal{D}(G)$ have geometrically convergent subsequences.

Proposition 2.1 (Proposition 3.8 in Jørgensen and Marden [15]) *If G is not virtually abelian and $\{\rho_j\} \subset \mathcal{D}(G)$ is an algebraically convergent sequence with limit ρ, then there exists a subsequence $\{\rho_{j_k}\}$ of $\{\rho_j\}$ so that $\{\rho_{j_k}(G)\}$ converges geometrically to a Kleinian group $\widehat{\Gamma}$ with $\rho(G) \subset \widehat{\Gamma}$.*

A sequence of closed sets $\{X_j\}$ in $\overline{\mathbf{C}}$ *converges in the Hausdorff topology* to a closed set X in $\overline{\mathbf{C}}$ if every point of X is the limit of a sequence of points $\{x_j \in X_j\}$ and if every accumulation point of a sequence $\{x_j \in X_j\}$ is contained in X. With this topology, the collection of non-empty closed subsets of $\overline{\mathbf{C}}$ is compact. Convergence of closed sets is always assumed to be in the Hausdorff topology.

If $\{\rho_j\} \subset \mathcal{D}$ converges algebraically to ρ and $\{\rho_j(G)\}$ converges geometrically to $\rho(G)$, we say that $\{\rho_j\}$ converges *strongly* to ρ. It is conjectured that if G is not virtually abelian and if $\{\rho_j\} \subset \mathcal{D}(G)$ converges algebraically to ρ, then $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$ if and only if

$\{\rho_j\}$ converges strongly to ρ. We make use of the following partial result in the direction of this conjecture.

Proposition 2.2 (Proposition 4.2 in Jørgensen and Marden [15]) *Let G be a finitely generated group which is not virtually abelian. If $\{\rho_j\} \subset \mathcal{D}(G)$ converges algebraically to $\rho \in \mathcal{D}(G)$, $\Omega(\rho(G))$ is non-empty, and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$, then $\{\rho_j\}$ converges strongly to ρ.*

The following lemma indicates the geometric significance, on the level of the quotient manifolds, of the geometric convergence of a sequence of Kleinian groups. (For a proof, see Theorem 3.2.9 of Canary, Epstein, and Green [10], and Theorem E.1.13 and Remark E.1.19 of Benedetti and Petronio [6].) Let 0 denote a fixed choice of basepoint for $\mathbf{H}^3$, and let $B_R(0)$ denote the ball of radius R centered at 0.

Lemma 2.3 *A sequence of torsion-free Kleinian groups $\{\Gamma_j\}$ converges geometrically to a torsion-free Kleinian group $\widehat{\Gamma}$ if and only if there exists a sequence $\{(R_j, K_j)\}$ and a sequence of maps $\widetilde{f}_j : B_{R_j}(0) \to \mathbf{H}^3$ such that*

1. *$R_j \to \infty$ and $K_j \to 1$ as $j \to \infty$;*
2. *the map $\widetilde{f}_j$ is a K_j-bilipschitz diffeomorphism onto its image, $\widetilde{f}_j(0) = 0$, and for any compact subset A of $\mathbf{H}^3$, $\{\widetilde{f}_j|_A\}$ converges to the identity;*
3. *if $V_j = B_{R_j}(0)/\Gamma_j$, then V_j is a submanifold of $N_j = \mathbf{H}^3/\Gamma_j$ and $\widetilde{f}_j$ descends to a map $f_j : V_j \to \widehat{N}$, where $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$; moreover, f_j is also a K_j-bilipschitz diffeomorphism onto its image.*

2.2 Types of Kleinian groups

Given a set $X \subset \mathbf{H}^3 \cup \overline{\mathbf{C}}$ and a Kleinian group Γ, define the *stabilizer* of X in Γ to be

$$\mathrm{st}_\Gamma(X) = \{\gamma \in \Gamma \,:\, \gamma(X) = X\}.$$

A *component subgroup* of Γ is the stabilizer in Γ of a component Δ of $\Omega(\Gamma)$. A set $X \subset \mathbf{H}^3 \cup \overline{\mathbf{C}}$ is *precisely invariant* under a subgroup Φ of Γ if $\mathrm{st}_\Gamma(X) = \Phi$ and if $X \cap \gamma(X)$ is empty for all $\gamma \in \Gamma - \Phi$.

There are several classes of Kleinian groups of particular interest in our paper. A *quasifuchsian* group is a finitely generated Kleinian group whose limit set is a Jordan curve and which contains no element interchanging the components of its domain of discontinuity, while an *extended quasi-fuchsian* group is a finitely generated Kleinian group whose limit set is a Jordan curve and which does contain an element interchanging the components of its domain of discontinuity. Note that an extended quasifuchsian group contains a canonical quasifuchsian subgroup of index two, namely the component subgroup associated to either of the components of its domain of discontinuity.

A *degenerate* group is a finitely generated Kleinian group whose domain of discontinuity and limit set are both non-empty and connected. A *web* group is a finitely generated Kleinian group whose domain of discontinuity contains infinitely many components, and each component subgroup is quasifuchsian. A finitely generated Kleinian group is a *generalized web* group if it is either quasifuchsian, extended quasifuchsian or a web group.

The convex core $C(N)$ of a hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ is the quotient of the convex hull $CH(\Lambda(\Gamma))$ of the limit set of Γ by Γ. (The convex core can also be defined to be the smallest convex subset of N whose inclusion is a homotopy equivalence.) A finitely generated Kleinian group is *geometrically finite* if its convex core has finite volume. A torsion-free Kleinian group is *topologically tame* if its quotient 3-manifold $N = \mathbf{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold. We note that geometrically finite Kleinian groups are topologically tame (see Marden [18]). A theorem of Scott [29] guarantees that any hyperbolic 3-manifold with finitely generated fundamental group contains a compact submanifold, called a *compact core*, whose inclusion is a homotopy equivalence. It is conjectured (see Marden [18]) that every hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of its compact core, and hence topologically tame.

2.3 Decompositions of Kleinian groups

In Section 3, we make use of two related decompositions, due to Abikoff and Maskit [1], of a finitely generated, torsion-free Kleinian group. We first discuss how a non-elementary, finitely generated, torsion-free Kleinian group with connected limit set and non-empty domain of discontinuity can be built from generalized web groups and degenerate groups without accidental parabolic elements. We then discuss the decomposition of a finitely generated Kleinian group into groups with connected limit sets and elementary groups.

We begin with a few definitions. Let Γ be a torsion-free Kleinian group, and let $\gamma \in \Gamma$ be a parabolic element. A *cusp region* for γ is a closed Jordan domain D which is precisely invariant under $\langle\gamma\rangle$ in Γ and intersects $\Lambda(\Gamma)$ only at the fixed point of γ. Note that this immediately implies that γ is primitive in Γ and that γ cannot lie in a rank two abelian subgroup of Γ. The interior of a cusp region descends to a punctured disc neighborhood of a cusp on the Riemann surface $\Omega(\Gamma)/\Gamma$.

As the choice of a cusp region is by no means canonical, we introduce a notion of equivalence. Say that cusp regions D and D' for parabolic elements γ and γ' of Γ are *equivalent* if their images on $\Omega(\Gamma)/\Gamma$ are neighborhoods of the same cusp on $\Omega(\Gamma)/\Gamma$. This implies in particular that γ and γ'

are conjugate in Γ; we note that the converse need not hold, as a single primitive parabolic element may stabilize two inequivalent cusp regions.

Let Γ be a non-elementary, finitely generated, torsion-free Kleinian group with connected limit set; in particular, every component of $\Omega(\Gamma)$ is simply connected. Let Δ be a component of $\Omega(\Gamma)$, and let $\Phi = \mathrm{st}_\Gamma(\Delta)$ be its component subgroup. By the classical Uniformization Theorem for Riemann surfaces, there exists a conformal homeomorphism $f : \Delta \to \mathbf{H}^2$. The elements of $f\Phi f^{-1}$ are conformal homeomorphisms of $\mathbf{H}^2$, and so are necessarily Möbius transformations. An *accidental parabolic element* θ of Φ is a parabolic element whose conjugate $f\theta f^{-1}$ by f is a primitive hyperbolic element of $f\Phi f^{-1}$. Denoting by l the line in $\mathbf{H}^2$ joining the fixed points of $f\theta f^{-1}$, we define the *axis* of θ to be $c_\theta = f^{-1}(l)$ and the *full axis* of θ to be the Jordan curve $C_\theta = c_\theta \cup \mathrm{fix}(\theta)$; for a detailed discussion of accidental parabolics, we refer the reader to Maskit [22], particularly Chapter IX.D.10. By construction, c_θ is precisely invariant under $\Theta = \langle\theta\rangle$ in Γ, and C_θ separates $\Lambda(\Phi)$, and hence separates $\Lambda(\Gamma)$.

Before describing the decomposition of a Kleinian group along an accidental parabolic element, we make a couple of remarks. First, a given parabolic element of Γ may lie in several component subgroups, being accidental in some and not accidental in others; hence, we make the convention that whenever we refer to an accidental parabolic element of Γ, we actually refer to the parabolic element along with the implicit choice for the component of $\Omega(\Gamma)$ containing its axis. Second, a primitive parabolic element of a Kleinian group which keeps invariant a component of the domain of discontinuity must either be accidental in that component or must have a cusp region in that component. Third, a finitely generated, non-elementary Kleinian group with connected limit set and non-empty domain of discontinuity contains no accidental parabolic elements if and only if it is either a degenerate group without accidental parabolics or a generalized web group (this follows immediately from Theorem 4 of Maskit [20]).

Given an accidental parabolic element θ of Γ with full axis C_θ, let P_1 and P_2 be the two components of $\overline{\mathbf{C}} - \Gamma(C_\theta)$ which contain C_θ in their boundaries, and let Γ_m be the stabilizer of P_m in Γ. Let E_1 and E_2 be the closed discs in $\overline{\mathbf{C}}$ determined by C_θ, labeled so that C_θ separates the interior of E_m from P_m.

Suppose that P_1 and P_2 are not equivalent under Γ. In this case, we say that C_θ is a *separating* full axis for θ, and we refer to Γ_1 and Γ_2 as the *factor subgroups* of this decomposition. It is easy to show that Γ is generated by its factor subgroups Γ_1 and Γ_2, that both factor subgroups are non-elementary and finitely generated, that $\Gamma_1 \cap \Gamma_2 = \langle\theta\rangle$, and that C_θ is precisely invariant under $\langle\theta\rangle$ in Γ_m for both m. The statement of the Klein-Maskit combination theorem of type I given below, adapted from the statement in Maskit [22], essentially states that this operation of decomposition can be reversed.

Recall that a *fundamental domain* for a finitely generated Kleinian group Γ is an open subset D of $\Omega(\Gamma)$ so that D is precisely invariant under the identity, every point of $\Omega(\Gamma)$ is equivalent under the action of Γ to a point of $\overline{D}$ and the boundary of D is a finite collection of analytic arcs.

Theorem 2.4 *(Klein-Maskit combination I) Let Γ_1 and Γ_2 be non-elementary, finitely generated Kleinian groups whose intersection $\Gamma_1 \cap \Gamma_2$ is the parabolic cyclic group $\langle\theta\rangle$. Let C be a Jordan curve in $\overline{\mathbf{C}}$ determining closed discs E_1 and E_2, so that E_m is a cusp region for θ in Γ_m.*

Then, the following hold. First, the group $\Gamma = \langle\Gamma_1, \Gamma_2\rangle$ is a Kleinian group isomorphic to the amalgamated free product of Γ_1 and Γ_2 along $\langle\theta\rangle$. Second, if D_m is a fundamental domain for Γ_m so that $D_m \cap E_m$ is a fundamental domain for the action of $\langle\theta\rangle$ on E_m, $D_m \cap E_{3-m}$ has non-empty interior, and $D_1 \cap C_\theta = D_2 \cap C_\theta$, then $D = (D_1 \cap E_2) \cup (D_2 \cap E_1)$ is a fundamental domain for the action of Γ. Third, every cusp region for Γ_m which does not intersect any Γ_m-translate of E_m is a cusp region for Γ, and every cusp region for Γ is a cusp region for either Γ_1 or Γ_2.

Suppose, on the other hand, that P_1 and P_2 are equivalent by $\xi \in \Gamma$, so that $\xi(P_1) = P_2$. In this case we say that C_θ is a *non-separating* full axis for θ, and we refer to Γ_2 and $\langle\xi\rangle$ as the *factor subgroups* of this decomposition. It is easy to show that Γ is generated by its factor subgroups Γ_2 and $\langle\xi\rangle$, that Γ_2 is finitely generated and non-elementary, that $\Gamma_2 \cap \xi^{-1}\Gamma_2\xi = \langle\theta\rangle$, and that C_θ is precisely invariant under $\langle\theta\rangle$ in Γ_2 (even though we may have that $\xi(C_\theta) \cap C_\theta$ is non-empty if ξ and θ commute). The statement of the Klein-Maskit combination theorem of type II, adapted from the statement in Maskit[22], essentially states that this decomposition can be reversed.

Theorem 2.5 *(Klein-Maskit combination II) Let Γ^0 be a non-elementary, finitely generated Kleinian group, and let $\langle\theta_1\rangle$ and $\langle\theta_2\rangle$ be parabolic cyclic subgroups of Γ^0. Let C_1 and C_2 be Jordan curves in $\overline{\mathbf{C}}$ so that, for both m, C_m determines a closed disc E_m which is a cusp region for $\langle\theta_m\rangle$ in Γ^0. We also require that E_1 and E_2 are inequivalent cusp regions for Γ^0. Let ξ be a Möbius transformation so that $\xi\theta_1\xi^{-1} = \theta_2$, $\xi(C_1) = C_2$, and the image of the interior of E_1 under ξ is disjoint from the interior of E_2.*

Then, the following hold. First, the group $\langle\Gamma^0, \xi\rangle$ is a Kleinian group isomorphic to the HNN extension of Γ^0 by ξ. Second, if D^0 is a fundamental domain for Γ^0 so that $D^0 \cap E_m$ is a fundamental domain for the action of $\langle\theta_m\rangle$ on E_m and $\xi(D_1 \cap C_1) = D_2 \cap C_2$, then $D = D^0 \cap A$ is a fundamental domain for Γ, where $A = \overline{\mathbf{C}} - (E_1 \cup E_2)$. Third, every cusp region for Γ^0 which does not intersect any Γ^0-translate of E_1 or E_2 is a cusp region for Γ, and every cusp region for Γ is a cusp region for Γ^0.

Abikoff and Maskit [1] showed that given a finitely generated, non-elementary Kleinian group Γ with connected limit set and non-empty domain of discontinuity, one can produce a finite collection

$\{\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}, \ldots, \Gamma_l\}$ of subgroups of Γ, where Γ_i is either a generalized web group or a degenerate group without accidental parabolic elements for $i \leq k$, and Γ_i is a cyclic group for $i > k$, so that Γ can be built from $\{\Gamma_1, \ldots, \Gamma_l\}$ by repeatedly performing Klein-Maskit combinations of types I and II. Specifically, set $\Gamma^1 = \Gamma_1$; for $j \leq k$, form Γ^j from Γ^{j-1} and Γ_j by using a Klein-Maskit combination of type I along a common parabolic cyclic subgroup of Γ^{j-1} and Γ_j, and for $j > k$, form Γ^j from Γ^{j-1} and Γ_j by a Klein-Maskit combination of type II, where the generator of Γ_j pairs inequivalent cusp regions of Γ^{j-1}. The final result of this process Γ^l is the original group Γ.

Abikoff and Maskit [1] also showed that any torsion-free, finitely generated Kleinian group with non-empty domain of discontinuity can be constructed from a finite collection of elementary groups and groups with connected limit set by a finite number of applications of the Klein combination theorem. We recall the statement of the Klein combination theorem below for reference, and refer the reader to Maskit [22] for a complete discussion.

Theorem 2.6 (Klein combination theorem) *Let Γ_1 and Γ_2 be Kleinian groups with non-empty domains of discontinuity, and suppose there exist fundamental domains D_1 and D_2 for Γ_1 and Γ_2 so that each contains the exterior of the other. Then, $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group isomorphic to the free product $\Gamma_1 * \Gamma_2$, and $D = D_1 \cap D_2$ is a fundamental domain for Γ.*

2.4 Relative compact cores and ends of Kleinian groups

A *horoball* in $\mathbf{H}^3$ is an open Euclidean ball (or half-space) B in $\mathbf{H}^3$ whose (Euclidean) closure in $\mathbf{H}^3 \cup \overline{\mathbf{C}}$ intersects $\overline{\mathbf{C}}$ in a single point, which is the *center* of the horoball. A *precisely invariant system of horoballs* $\mathcal{H}$ for a Kleinian group Γ is a Γ-invariant collection of disjoint horoballs centered at parabolic fixed points of Γ, such that there is one based at every parabolic fixed point. It is a consequence of the Margulis Lemma (see Benedetti and Petronio [6] or Maskit [22]) that every Kleinian group has a precisely invariant system of horoballs. Given a precisely invariant system of horoballs $\mathcal{H}$ for Γ, set $N_{\mathcal{H}} = (\mathbf{H}^3 - \mathcal{H})/\Gamma$.

A *relative compact core* of $N_{\mathcal{H}}$ is a compact submanifold M of $N_{\mathcal{H}}$ so that the inclusion of M into $N_{\mathcal{H}}$ is a homotopy equivalence and so that ∂M intersects every non-compact component of $\partial N_{\mathcal{H}}$ in an incompressible annulus and contains every toroidal component of $\partial N_{\mathcal{H}}$. (Results of McCullough [23] or Kulkarni and Shalen [17] guarantee that $N_{\mathcal{H}}$ has a relative compact core whenever $\pi_1(N)$ is finitely generated.) Set $P = \partial M \cap \partial N_{\mathcal{H}}$. The *ends* of $N_{\mathcal{H}}$ are in one-to-one correspondence with the components of $\partial M - P$ (see Bonahon [7]). An end E of $N_{\mathcal{H}}$ is *geometrically finite* if it has a neighborhood U such that $U \cap C(N) = \emptyset$, and is *geometrically infinite* otherwise. We also refer to the corresponding components of $\partial M - P$ as geometrically finite or geometrically infinite.

The following generalization of Thurston's covering theorem [32] appears in [9].

Theorem 2.7 *Let $N = \mathbf{H}^3/\Gamma$ be a topologically tame hyperbolic 3-manifold which covers another hyperbolic 3-manifold $\widehat{N}$ by a local isometry $\pi : N \to \widehat{N}$, and suppose that $\Omega(\Gamma)$ is non-empty. If E is a geometrically infinite end of $N_{\mathcal{H}}$, then E has a neighborhood U such that π is finite-to-one on U.*

The relative compact core encodes much of the structure of the Kleinian group. For example, a finitely generated, torsion-free Kleinian group Γ with associated relative compact core (M, P) has connected limit set if and only if every geometrically finite component of $\partial M - P$ is incompressible. In this language, an accidental parabolic gives rise to an essential annulus A in M which has one boundary component in P and the other in a geometrically finite component of $\partial M - P$. If Γ has connected limit set, then the non-cyclic subgroups in the Abikoff-Maskit decomposition of Γ are simply the fundamental groups of the components of $M - \mathcal{A}$, where $\mathcal{A}$ is a maximal collection of disjoint, non-parallel essential annuli associated to accidental parabolics. If Γ has disconnected limit set, the non-cyclic subgroups in an Abikoff-Maskit decomposition of Γ into groups with connected limit set and elementary groups arise as fundamental groups of components of $M - \mathcal{C}$, where $\mathcal{C}$ is a maximal collection of disjoint, non-parallel compressing disks for (M, P) whose boundary curves lie in geometrically finite components of $\partial M - P$.

3 The main results

In this section we prove our main result, namely that a type-preserving sequence converges strongly, under the assumption that the domain of discontinuity of the algebraic limit is non-empty.

Theorem 3.1 *Let G be a finitely generated, torsion-free group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to $\rho \in \mathcal{D}(G)$. If $\Omega(\rho(G))$ is non-empty, then $\{\rho_j\}$ converges strongly to ρ and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$.*

Before beginning the proof, we recall the statement of Theorem F from [4].

Theorem 3.2 *(Theorem F from [4]) Let G be a finitely generated, torsion-free, non-abelian group and let $\{\rho_j\}$ be a sequence in $\mathcal{D}(G)$ converging to ρ. If $\Lambda(\rho(G)) = \overline{\mathbf{C}}$ and G is not a non-trivial free product of (orientable) surface groups and cyclic groups, then $\{\rho_j\}$ converges strongly to ρ. Moreover, $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G)) = \overline{\mathbf{C}}$.*

Combining Theorem 3.1 with the above theorem yields the following immediate corollary.

Corollary 3.3 *Let G be a finitely generated, torsion-free group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to ρ. If G is not a (non-trivial) free product of (orientable) surface groups and infinite cyclic groups, then $\{\rho_j\}$ converges strongly to ρ and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$.*

We further note that we obtain the same conclusion if we replace the group-theoretic assumption in the statement of the corollary with the weaker topological assumption that the relative compact core of the algebraic limit is not a relative compression body. For a discussion of the relation between the group-theoretic and topological conditions, we refer the reader to Section 11 of [4].

Proof of Theorem 3.1: The proof divides naturally into three steps. In Step 1, we show that Theorem 3.1 holds in the case that $\rho(G)$ is either a generalized web group, a degenerate group without accidental parabolics or an elementary group. In Step 2, we show that Theorem 3.1 holds in the case that $\rho(G)$ has connected limit set, using the result of Step 1 and the Abikoff-Maskit decomposition of $\rho(G)$ into generalized web groups, degenerate groups without accidental parabolic elements and elementary groups. In Step 3, we pass from the case of connected limit set to the general case, making use of the Abikoff-Maskit decomposition of a torsion-free finitely generated Kleinian group into groups with connected limit sets and elementary groups.

Step 1: We recall that Thurston [32] and Kerckhoff and Thurston [16] established Theorem 3.1 in the case that the algebraic limit is a degenerate group without accidental parabolic elements. One can obtain this by combining Theorem 9.6.1 in Thurston [32] with the proof of Corollary 2.2 in Kerckhoff and Thurston [16]; see also Corollary 6.1 in Ohshika [27].

We next establish Theorem 3.1 for generalized web groups.

Proposition 3.4 *Let G be a finitely generated, torsion-free, non-abelian group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to ρ. If $\rho(G)$ is a generalized web group, then $\{\rho_j\}$ converges strongly to ρ and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$.*

Proof of Proposition 3.4: Let Γ^0 be a geometrically finite subgroup of $\rho(G)$. Since each $\rho_j \circ \rho^{-1}$ is type-preserving, Marden's quasiconformal stability theorem (Proposition 9.1 in Marden [18]) implies that $\Gamma^0_j = \rho_j \circ \rho^{-1}(\Gamma^0)$ is geometrically finite and quasiconformally conjugate to $\rho(G)$ for sufficiently large j. In particular, there exists a sequence $\{\phi_j\}$ of k_j-quasiconformal maps of $\overline{\mathbf{C}}$ converging to the identity map, with $\{k_j\}$ converging to 1, so that $\rho_j \circ \rho^{-1}(\gamma) = \phi_j \circ \gamma \circ \phi_j^{-1}$ for all $\gamma \in \Gamma^0$. One sees immediately that $\{\Gamma^0_j\}$ converges geometrically to Γ^0 and that $\{\Lambda(\Gamma^0_j)\}$ converges to $\Lambda(\Gamma^0)$. Hence, Theorem 3.1 holds whenever $\rho(G)$ is geometrically finite, and in particular for $\rho(G)$ quasifuchsian or extended quasifuchsian.

Suppose now that $\rho(G)$ is a web group. Proposition 2.2 implies that it suffices to prove that $\{\Lambda(\rho_j(G)\}$ converges to $\Lambda(\rho(G))$. If $\{\Lambda(\rho_j(G))\}$ does not converge to $\Lambda(\rho(G))$, then there exists a subsequence of $\{\rho_j\}$, again called $\{\rho_j\}$, so that $\{\Lambda(\rho_j(G))\}$ converges to a set $\widehat{\Lambda}$ which contains $\Lambda(\rho(G))$ as a proper subset. In particular, there must exist a quasifuchsian component subgroup of $\rho(G)$ whose limit set separates $\widehat{\Lambda}$. Proposition 2.1 implies that we may pass to a further subsequence of $\{\rho_j\}$, still called $\{\rho_j\}$, such that $\{\rho_j(G)\}$ converges geometrically to a Kleinian group $\widehat{\Gamma}$. The definitions of geometric and Hausdorff convergence imply that $\Lambda(\rho(G)) \subset \Lambda(\widehat{\Gamma}) \subset \widehat{\Lambda}$.

Let $x \in \widehat{\Lambda} - \Lambda(\rho(G))$, let Δ be the component of $\Omega(\rho(G))$ which contains x, and set $\Phi = \mathrm{st}_{\rho(G)}(\Delta)$, so that $\Lambda(\Phi)$ separates $\widehat{\Lambda}$. Set $\Phi_j = \rho_j \circ \rho^{-1}(\Phi)$. Since $\rho(G)$ is a web group, Φ is quasifuchsian, and hence geometrically finite. Let $\{\phi_j\}$ be the sequence of quasiconformal maps conjugating Φ_j to Φ produced in the first paragraph of this proof. Since $\Lambda(\Phi)$ separates $\widehat{\Lambda}$, $\{\Lambda(\rho_j(G))\}$ converges to $\widehat{\Lambda}$, and $\{\phi_j\}$ converges to the identity map, we see that $\Lambda(\Phi_j) = \phi_j(\Lambda(\Phi))$ separates $\Lambda(\rho_j(G))$ for all sufficiently large j.

Recall that a *spanning disc* for the quasifuchsian subgroup Φ of the Kleinian group $\rho(G)$ is a properly embedded open disc D in $\mathbf{H}^3$ which is precisely invariant under Φ in $\rho(G)$ and which extends to a closed disc $\overline{D}$ in $\mathbf{H}^3 \cup \overline{\mathbf{C}}$ with boundary $\Lambda(\Phi)$. Let $N = \mathbf{H}^3/\rho(G)$ and $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$, and $\pi : N \to \widehat{N}$ and $\widehat{p} : \mathbf{H}^3 \to \widehat{N}$ be the associated covering maps. It is shown in the proof of Proposition 6.1 of [4] that, for any algebraically convergent sequence in $\mathcal{D}(G)$ whose algebraic limit is a web group, such as $\rho(G)$, and for any component subgroup of $\rho(G)$, such as Φ, there exists a spanning disk D for Φ and a compact core M for N such that π is an embedding restricted to M, $\widehat{p}(D)$ is a properly embedded surface in $\widehat{N}$, and $\pi(M)$ is disjoint from $\widehat{p}(D)$.

Set $N_j = \mathbf{H}^3/\rho_j(G)$. Let $\{V_j \subset N_j\}$ and $\{f_j : V_j \to \widehat{N}\}$ be the sequence of submanifolds and K_j-bilipschitz diffeomorphisms produced by Lemma 2.3. Since $\pi(M)$ lies in $f_j(V_j)$ for all sufficiently large j, the submanifold $M_j = f_j^{-1}(\pi(M))$ of N_j is defined. In Corollary C of [4], we observe that M_j is a compact core for N_j for all sufficiently large j.

One may extend each ϕ_j to a L_j-bilipschitz diffeomorphism $\psi_j : \mathbf{H}^3 \to \mathbf{H}^3$ conjugating the action of Φ to the action of Φ_j (see Douady and Earle [11], Reimann [28], or Tukia [33]) in such a way that $\{L_j\}$ converges to 1 and $\{\psi_j\}$ converges to the identity map. Let $D_j = \psi_j(D)$, and note that D_j is a properly embedded open disc in $\mathbf{H}^3$ which is invariant under Φ_j and which extends to a closed disc in $\mathbf{H}^3 \cup \overline{\mathbf{C}}$ with boundary $\Lambda(\Phi_j)$. However, D_j need not be a spanning disk for Φ_j, as it need not be precisely invariant under Φ_j in $\rho_j(G)$.

We next show that $p_j(D_j)$ is disjoint from M_j for sufficiently large j, where $p_j : \mathbf{H}^3 \to N_j$ is the covering map. Choose $\delta > 0$ so that $\min_{x\in\pi(M)} \mathrm{inj}_{\widehat{N}}(x) > 3\delta$. Since $f_j : V_j \to \widehat{N}$ is K_j-bilipschitz and $\{K_j\}$ converges to 1, we see that $\min_{x\in M_j} \mathrm{inj}_{N_j}(x) > 2\delta$ for sufficiently large j.

Let $X \subset D$ denote the set of points for which there exists a non-trivial $\gamma \in \Phi$ such that $\mathrm{d}(x, \gamma(x)) < \delta$, and let $Y = D - X$. Consider the subsets $Y_j = \psi_j(Y)$ and $X_j = \psi_j(X)$ of D_j. For each $x \in X_j$, there exists a non-trivial $\gamma_j \in \Phi_j$ so that $\mathrm{d}(x, \gamma_j(x)) < L_j\delta$. Since $L_j\delta < \min_{x \in M_j} \mathrm{inj}_{N_j}(x)$ for sufficiently large j, we have that $p_j(X_j)$ is disjoint from M_j. Moreover, since $\widehat{p}(Y)$ is compact and is disjoint from $\pi(M)$, and since both $\{f_j\}$ and $\{\psi_j\}$ converge to the identity map, $p_j(Y_j)$ is disjoint from M_j for sufficiently large j. This completes the proof that $p_j(D_j)$ is disjoint from M_j for sufficiently large j.

We now assume that we have chosen j large enough so that $\Lambda(\Phi_j)$ separates $\Lambda(\rho_j(G))$, M_j is a compact core for N_j, and $p_j(D_j)$ is disjoint from M_j. Since $\Lambda(\Phi_j)$ separates $\Lambda(\rho_j(G))$ and since pairs of fixed points of loxodromic elements of $\rho_j(G)$ are dense in $\Lambda(\rho_j(G)) \times \Lambda(\rho_j(G))$ (see Proposition V.E.5 in [22]), there exists a hyperbolic element $\gamma_j \in \rho_j(G)$ whose fixed points are separated by $\Lambda(\Phi_j)$. If C_j is the closed geodesic in N_j which is the projection of the axis of γ_j, then every curve homotopic to C_j must intersect $p_j(D_j)$. However, since M_j is a compact core for N_j which is disjoint from $p_j(D_j)$, there exists a curve in M_j which is homotopic to C_j and disjoint from $p_j(D_j)$. This contradiction establishes the result.

$\boxed{\text{Proposition 3.4}}$

We note the following corollary of the proof of Proposition 3.4.

Corollary 3.5 *Let G be a finitely generated, torsion-free, non-abelian group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to ρ. Suppose that $\rho(G)$ is a generalized web group, and let Φ be a component subgroup of $\rho(G)$. Then, $\Phi_j = \rho_j \circ \rho^{-1}(\Phi)$ is quasifuchsian and is a component subgroup of $\rho_j(G)$ for all sufficiently large j.*

We complete Step 1 by establishing Theorem 3.1 for elementary groups, under the additional assumption that the limit representation ρ is discrete and faithful.

Lemma 3.6 *Let G be a free abelian group of rank at most two, and let $\{\rho_j\}$ be a type-preserving sequence in $\mathcal{D}(G)$ converging algebraically to $\rho \in \mathcal{D}(G)$. Then, $\{\rho_j\}$ converges strongly to ρ.*

Proof of Lemma 3.6: Suppose there exists a sequence of non-trivial elements $\{g_j\}$ of G so that $\{\rho_j(g_j)\}$ converges to a Möbius transformation γ. We wish to show that $\gamma \in \rho(G)$. We begin by noting that $\mathrm{fix}(\gamma) = \Lambda(\rho(G))$, for otherwise, the fixed point set $\mathrm{fix}(\rho_j(g_j))$ of $\rho_j(g_j)$ would be distinct from $\Lambda(\rho_j(G))$ for all sufficiently large j, which cannot occur.

Suppose that $G = \langle g \rangle$ is infinite cyclic and that $\rho(g)$ is loxodromic. As G is cyclic, we may write $g_j = g^{n_j}$ for $n_j \in \mathbf{Z}$. Since $\{\rho_j(g)\}$ converges to $\rho(g)$, and since $\rho_j(g)$ is loxodromic for all j and $\rho(g)$

is loxodromic, there exists a sequence of Möbius transformations $\{\beta_j\}$ converging to the identity so that the axis of $\beta_j\rho_j(g)\beta_j^{-1}$ in $\mathbf{H}^3$ is equal to the axis l of $\rho(g)$ in $\mathbf{H}^3$ for all j. Since $\{\beta_j\}$ converges to the identity, $\{\beta_j\rho_j(g_j)\beta_j^{-1}\}$ converges to γ, and so the translation length of $\beta_j\rho_j(g_j)\beta_j^{-1}$ along l converges to the translation length of γ.

If $\{n_j\}$ is not bounded, there exists a subsequence of $\{n_j\}$, again called $\{n_j\}$, converging to either ∞ or $-\infty$. Since the translation length of $\rho_j(g_j)$ is n_j times the translation length of $\beta_j\rho_j(g)\beta_j^{-1}$, we see that the translation length of $\beta_j\rho_j(g)\beta_j^{-1}$ along l converges to zero, and so $\rho(g)$ fixes l pointwise. In particular, $\rho(g)$ is either the identity or elliptic. However, $\rho(g)$ cannot be the identity, as this would contradict the faithfulness of ρ. Also, $\rho(g)$ cannot be elliptic, as this would contradict either the faithfulness of ρ if $\rho(g)$ has finite order or the discreteness of $\rho(G)$ if $\rho(g)$ has infinite order. This contradiction gives that $\{n_j\}$ is bounded. If $\{n_j\}$ is not eventually constant, there exist two subsequences with limits $m \neq n$, and so there are two subsequences of $\{\rho_j(g_j)\}$ whose limits are $\rho(g)^m$ and $\rho(g)^n$, which cannot occur. Hence, $\{n_j\}$ is eventually constant, and so $\gamma \in \rho(G)$.

The proof in the case that $\rho(G)$ is a parabolic subgroup is similar. Suppose that G is infinite cyclic and $\rho(g)$ is parabolic. Since $\{\rho_j(g)\}$ converges to $\rho(g)$, and since $\rho_j(g)$ is parabolic for all j and $\rho(g)$ is parabolic, there exists a sequence of Möbius transformations $\{\beta_j\}$ converging to the identity so that $\beta_j\rho_j(g)\beta_j^{-1} = \rho(g)$ for all j. Normalizing so that $\rho(g)(z) = z+1$ and writing $g_j = g^{n_j}$ for $n_j \in \mathbf{Z}$, we see that $\beta_j\rho_j(g_j)\beta_j^{-1}(z) = z + n_j$. Since $\{\beta_j\rho_j(g_j)\beta_j^{-1}\}$ converges to a Möbius transformation γ, it is easy to see that $\{n_j\}$ must be eventually constant, and so $\gamma \in \rho(G)$.

In the case that G has rank two, choose generators a and b. Since $\rho_j(a)$ and $\rho_j(b)$ are parabolic for all j, and since $\rho(a)$ and $\rho(b)$ are parabolic, there exists a sequence of Möbius transformations $\{\beta_j\}$ converging to the identity so that $\beta_j\rho_j(a)\beta_j^{-1} = \rho(a)$ for all j. Normalizing so that $\rho(a)(z) = z+1$, we see that $\beta_j\rho_j(b)\beta_j^{-1}(z) = z + \tau_j$ and $\rho(b)(z) = z + \tau$, where $\{\tau_j\}$ converges to τ and τ has non-zero imaginary part. Write $g_j = a^{n_j}b^{m_j}$ for $n_j, m_j \in \mathbf{Z}$, so that $\beta_j\rho_j(g_j)\beta_j^{-1}(z) = z + n_j + m_j\tau_j$. Since $\{\beta_j\rho_j(g_j)\beta_j^{-1}\}$ converges to a Möbius transformation γ, it is easy to see that both $\{n_j\}$ and $\{m_j\}$ must be eventually constant, and so $\gamma \in \rho(G)$.

$\boxed{\text{Lemma 3.6}}$

Step 2: In this step, we establish Theorem 3.1 in the case in which the algebraic limit $\rho(G)$ has connected limit set. As we have already shown that Theorem 3.1 holds for elementary groups, generalized web groups, and degenerate groups without accidental parabolics, it will suffice to show that it is preserved by Klein-Maskit combination along a parabolic cyclic subgroup.

In order to cleanly handle comparisons of cusp regions of the limit of a convergent sequence and of the approximates, we use the fact that we are working with type-preserving sequences to make a convenient normalization. Let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to ρ, and let $\rho(h)$ be a primitive parabolic element of $\rho(G)$. Since $\{\rho_j(h)\}$ is a sequence of parabolic Möbius transformations converging to $\rho(h)$, there exists a sequence $\{\beta_j\}$ of Möbius transformations converging to the identity so that $\beta_j\rho_j(h)\beta_j^{-1}(z) = \rho(h)$ for all j. Since $\{\beta_j\}$ converges to the identity, the modified sequence $\{\beta_j\rho_j\beta_j^{-1}\}$ has the same algebraic limit as the original sequence $\{\rho_j\}$, namely ρ, and the geometric limit of $\{\beta_j\rho_j(G)\beta_j^{-1}\}$ is equal to the geometric limit of the original sequence $\{\rho_j(G)\}$, if the geometric limit exists. Hence, we may replace the original sequence with the modified sequence without effect. We refer to this process as *normalizing the sequence about* $\rho(h)$.

We now wish to describe what we mean by cusp regions persisting in the approximates. Let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging to ρ and let $D \subset \Omega(\rho(G))$ be a cusp region for $\rho(h)$. Say that D *persists in the approximates* if, whenever $\{\rho_j\}$ is normalized about $\rho(h)$ to obtain a new sequence $\{\rho'_j\}$, D is a cusp region for $\rho'_j(h)$ for all sufficiently large j.

In order to show that the conclusions of Theorem 3.1 are preserved under the application of the Klein-Maskit combination theorems, we need to carry along the information about the persistence of cusp regions. We begin by showing in Proposition 3.7 that cusp regions persist in the approximates for degenerate groups without accidental parabolic elements and for generalized web groups. In Proposition 3.9, we show that if $\rho(G)$ is obtained from $\rho(G_1)$ and $\rho(G_2)$ by Klein-Maskit combination along an accidental parabolic element, and if $\{\Lambda(\rho_j(G_m)\}$ converges to $\Lambda(\rho(G_m)$ and all cusp regions of $\rho(G_m)$ persist in the approximates for both m, then $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$, $\{\rho_j\}$ converges strongly to ρ, and all cusp regions of $\rho(G)$ persist in the approximates. Combining these two propositions, we construct an inductive proof of Theorem 3.1 in the case that $\rho(G)$ has connected limit set.

Proposition 3.7 *Let G be a finitely generated, torsion-free group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to $\rho \in \mathcal{D}(G)$. Suppose that $\rho(G)$ is either degenerate without accidental parabolics, generalized web, or elementary. Then, cusp regions persist in the approximates.*

Proof of Proposition 3.7: We begin by noting that we have already shown that $\{\rho_j\}$ converges strongly to ρ and that $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$. Let D be a cusp region for the primitive parabolic $\rho(h)$. We assume that the sequence $\{\rho_j\}$ is normalized about $\rho(h)$. In the case that $\rho(G)$ is elementary, there is nothing to prove. Hence, we may assume that $\rho(G)$ is either degenerate without accidental parabolics or generalized web.

Let $c = \partial D \cap \Omega(\rho(G))$. The first key observation is that c lies in $\Omega(\rho_j(G_m))$ and is precisely invariant under $\langle\rho_j(h)\rangle$ in $\rho_j(G)$ for sufficiently large j. As this argument is used later, we state it as a lemma.

Lemma 3.8 *Let G be a finitely generated, torsion-free, non-abelian group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging strongly to ρ such that $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$. Suppose that D is a cusp region for $\rho(G)$ associated to the primitive parabolic element $\rho(h)$, that $\{\rho_j\}$ is normalized about $\rho(h)$, and set $c = \partial D \cap \Omega(\rho(G))$. Then, for sufficiently large j, c lies in $\Omega(\rho_j(G))$ and is precisely invariant under $\langle\rho_j(h)\rangle$ in $\rho_j(G)$.*

Proof of 3.8: We may assume that $\rho(h)(z) = z + 1$. Choose a point z on c, and let c_f be the closed arc in c joining z to $z + 1$, so that the interior of c_f is a fundamental domain for the action of $\langle\rho(h)\rangle$ on c. Since c_f is compact and since $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$, we see that c_f lies in $\Omega(\rho_j(G))$ for sufficiently large j, and hence that c lies in $\Omega(\rho_j(G))$ for sufficiently large j.

In order to show that c is precisely invariant under $\langle\rho_j(h)\rangle$ in $\rho_j(G)$, we need to make use of a standard *convergence property* for Möbius transformations, which we describe here. Let K be a compact set in $\overline{\mathbf{C}}$ and let $\{M_j\}$ be a sequence of distinct Möbius transformations so that $K \cap M_j(K)$ is non-empty for all j. Then, there exists a subsequence of $\{M_j\}$, again called $\{M_j\}$, so that either there exists a sequence of points $\{x_j \in \mathrm{fix}(M_j)\}$ converging to a point in K or $\{M_j\}$ converges to a Möbius transformation (see Marden [19] or Gehring and Martin [13].)

If c is not precisely invariant under $\langle\rho_j(h)\rangle$ in $\rho_j(G)$ for all sufficiently large j, there exists a sequence $\{g_j\}$ of elements of $G - \langle h\rangle$ so that $c \cap \rho_j(g_j)(c)$ is non-empty for infinitely many j. We may pre- and post-multiply each g_j by appropriate powers of h to produce a sequence $\{g_j'\}$ of elements of $G - \langle h\rangle$ so that $c_f \cap \rho_j(g_j')(c_f)$ is non-empty for infinitely many j. The convergence property for Möbius transformations implies that there exists a subsequence of $\{\rho_j(g_j')\}$, again called $\{\rho_j(g_j')\}$, for which either there exist points $\{x_j \in \Lambda(\rho_j(G))\}$ converging to a point in c_f, or $\{\rho_j(g_j')\}$ converges to a Möbius transformation γ. In the latter case, note that $c_f \cap \gamma(c_f)$ is necessarily non-empty.

The former case cannot occur, as $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$, which is disjoint from c_f. The latter case cannot occur, as the precise invariance of c under $\langle\rho(h)\rangle$ in $\rho(G)$ and the strong convergence of $\{\rho_j\}$ to ρ together imply that $\gamma = \rho(h^n)$ for some $n \neq 0$; Lemma 3.6 of Jørgensen and Marden [15] then implies that $g_j' = h^n$ for all sufficiently large j, contradicting our assumption that g_j' does not lie in $\langle h\rangle$.

$\boxed{3.8}$

We now show that D persists in the approximates. Suppose first that $\rho(G)$ is a degenerate group without accidental parabolics. As each isomorphism $\rho_j \circ \rho^{-1} : \rho(G) \to \rho_j(G)$ is type-preserving and $\Omega(\rho_j(G))$ is non-empty for all large enough j, Theorem 6 of Maskit [21] gives that, for all large enough j, $\rho_j(G)$ does not contain accidental parabolic elements. However, since c is precisely invariant under $\langle\rho_j(h)\rangle$ in $\rho_j(G)$ for all sufficiently large j, by Lemma 3.8, we see that either D is a cusp region for $\rho_j(h)$ in $\rho_j(G)$, or $C = c \cup \text{fix}(\rho(h))$ separates $\Lambda(\rho_j(G))$, in which case $\rho_j(h)$ is an accidental parabolic element of $\rho_j(G)$. As the latter case cannot occur, it must be that D is a cusp region for $\rho_j(h)$ in $\rho_j(G)$ for all sufficiently large j.

Suppose now that $\rho(G)$ is a generalized web group, let Δ be the component of $\Omega(\rho(G))$ containing D, and let $\Phi = \text{st}_{\rho(G)}(\Delta)$. By Lemma 3.8, we know that c lies in $\Omega(\rho_j(G))$ and is precisely invariant under $\langle\rho_j(h)\rangle$ for all sufficiently large j. By Corollary 3.5, $\Phi_j = \rho_j \circ \rho^{-1}(\Phi)$ is a quasifuchsian component subgroup of $\rho_j(G)$, stabilizing the component Δ_j of $\rho_j(G)$. Since $\{\Lambda(\Phi_j) = \partial\Delta_j\}$ converges to $\Lambda(\Phi) = \partial\Delta$, we see that c is contained in Δ_j for j sufficiently large. Since quasifuchsian component subgroups cannot contain accidental parabolic elements (see Theorem 4 of Maskit [20]), the Jordan curve $C = \partial D$ cannot separate $\Lambda(\rho_j(G))$, and so D is a cusp region for $\rho_j(h)$ in $\rho_j(G)$ for all sufficiently large j.

$\boxed{\text{Proposition 3.7}}$

The following proposition shows that the conclusions of Theorem 3.1 are preserved by Klein-Maskit combination along a cyclic parabolic subgroup.

Proposition 3.9 *Let G be a finitely generated, torsion-free, non-abelian group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to ρ with $\Omega(\rho(G))$ non-empty. Suppose that $\Theta = \langle\rho(h)\rangle$ is an accidental parabolic subgroup of $\rho(G)$, and that $\rho(G_1)$ and $\rho(G_2)$ are the factor subgroups of the Klein-Maskit decomposition of $\rho(G)$ along Θ. If $\{\Lambda(\rho_j(G_m))\}$ converges to $\Lambda(\rho(G_m))$ for both m and if all cusp regions for $\rho(G_m)$ persist in the approximates, then $\{\rho_j\}$ converges strongly to ρ, $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$, and all cusp regions for $\rho(G)$ persist in the approximates.*

Proof of Proposition 3.9: We begin by showing that $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$. Proposition 2.2 then implies the strong convergence of $\{\rho_j\}$ to ρ. In order to show that the limit sets converge, we first show that the approximates are also Klein-Maskit combinations along (the same) cyclic parabolic subgroup. We use this to construct explicit fundamental domains for the approximates which can be used to show that the limit sets converge. We then show that the cusp regions for $\rho(G)$ persist in the approximates. The details for the two Klein-Maskit combinations are similar; we give full details for the case of a Klein-Maskit combination of type I, and briefly sketch the argument for a Klein-Maskit combination of type II at the end of the proof.

If $\{\Lambda(\rho_j)\}$ does not converge to $\Lambda(\rho(G))$, we may pass to a subsequence so that $\{\Lambda(\rho_j(G))\}$ converges to a closed set $\widehat{\Lambda}$ containing $\Lambda(\rho(G))$ as a proper subset. Without loss of generality, we may assume that $\{\rho_j\}$ is normalized about $\rho(h)$ and that $\rho(h)(z) = z + 1$. Let c be the axis of

$\rho(h)$ and let $C = c \cup \{\infty\}$ be the full axis of $\rho(h)$. Let E_1 and E_2 be the two closed discs in $\overline{\mathbf{C}}$ determined by C, labeled so that E_m is a cusp region for $\rho(h)$ in $\rho(G_m)$.

Since cusp regions persist in the approximates for both $\rho(G_m)$, we have that E_m is a cusp region for $\rho_j(h)$ in $\rho_j(G_m)$ for both m and for all sufficiently large j. Therefore, $\rho_j(G)$ is formed from $\rho_j(G_1)$ and $\rho_j(G_2)$ by a Klein-Maskit combination of type I along C for all sufficiently large j.

Choose a point z on c and consider the open vertical band V in $\mathbf{C}$ whose two bounding lines pass through z and $z+1$. For both m, we can choose a fundamental domain F^m for $\rho(G_m)$ which lies in V and contains $V \cap E_m$. Since $\rho(G)$ is formed from $\rho(G_1)$ and $\rho(G_2)$ by Klein-Maskit combination of type I along C, Theorem 2.4 assures us that $F = F^1 \cap F^2$ is a fundamental domain for $\rho(G)$. Choose a point $x \in \widehat{\Lambda} - \Lambda(\rho(G))$ which lies in the interior of F (noting that one may need to alter the choice of F to guarantee that x lies in F rather than on its boundary). Let K be a compact set which contains an open neighborhood of x and which is contained entirely in the interior of F. Since K lies in F, it is precisely invariant under the identity in $\rho(G)$.

By construction, K lies in the strip V. Combining the convergence property for Möbius transformations (see the proof of Lemma 3.8) with the assumptions that $\{\rho_j(G_m)\}$ converges to $\rho(G_m)$ and $\{\Lambda(\rho_j(G_m))\}$ converges to $\Lambda(\rho(G_m))$, we see that K lies in $\Omega(\rho_j(G_m))$ and is precisely invariant under the identity in $\rho_j(G_m)$ for sufficiently large j. Moreover, since E_m is precisely invariant under $\langle\rho_j(h)\rangle$ in $\rho_j(G_m)$ for sufficiently large j, we may construct a fundamental domain F_j^m for $\rho_j(G_m)$ containing both $V \cap E_m$ and K for sufficiently large j; in particular, $F_j^1 \cap c = F_j^2 \cap c$. Since $\rho_j(G)$ is formed from $\rho_j(G_1)$ and $\rho_j(G_2)$ by Klein-Maskit combination of type I along C, Theorem 2.4 implies that $F_j = F_j^1 \cap F_j^2$ is then a fundamental domain for $\rho_j(G)$ for sufficiently large j.

Since $\{\Lambda(\rho_j(G))\}$ converges to $\widehat{\Lambda}$, there exists a sequence of points $\{x_j \in \Lambda(\rho_j(G))\}$ converging to x. This implies that $x_j \in K \subset F_j \subset \Omega(\rho_j(G))$ for sufficiently large j, a contradiction. Hence, $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$, and so $\{\rho_j\}$ converges strongly to ρ.

In order to complete the proof in the case of a Klein-Maskit combination of type I, it remains only to argue that cusp regions for $\rho(G)$ persist in the approximates. Let D' be a cusp region for a primitive parabolic element $\rho(g)$ of $\rho(G)$. After suitably normalizing, we first use Klein-Maskit Combination Theorem 2.4 and our assumption that cusps persist in the approximates of the factors to show that a cusp region D'' contained in D' (and missing all translates of one of the cusp regions we are combining along) does persist in the approximates. We then use the convergence property and the fact that limit sets converge to show that D' itself must persist in the approximates.

Let $\{\beta_j\}$ be a sequence of Möbius transformations converging to the identity so that $\beta_j\rho_j(g)\beta_j^{-1} = \rho(g)$ for all j. Set $\rho_j' = \beta_j \circ \rho_j \circ \beta_j^{-1}$, and let $c_j = \beta_j(c)$ and $E_m^j = \beta_j(E_m)$ for all j and for both m. Notice that, for all large enough j, E_m^j is a cusp region for $\rho_j'(h)$ in $\rho_j'(G_m)$ and $\rho_j'(G)$ is a Klein-Maskit combination of type I of $\rho_j'(G_1)$ and $\rho_j'(G_2)$ along $\langle\rho_j'(h)\rangle$ with associated axis c_j.

Moreover, $\{\rho'_j\}$ converges strongly to ρ and $\{\Lambda(\rho'_j(G))\}$ converges to $\Lambda(\rho(G))$.

Let ξ be the fixed point of $\rho(g)$, and let $d' = \partial D' - \{\xi\}$. Lemma 3.8 implies that d' is precisely invariant under $\langle\rho_j(g)\rangle$ in $\rho_j(G)$ for sufficiently large j. By Theorem 2.4, every cusp region for $\rho(G)$ is a cusp region for either $\rho(G_1)$ or $\rho(G_2)$, and so we may assume that D' is a cusp region for $\rho(G_1)$. Since we have assumed that cusp regions for $\rho_j(G_1)$ persist in the approximates, we see that D' is a cusp region for $\rho'_j(G_1)$ for all sufficiently large j. Moreover, since E_1 is not a cusp region for $\rho(G)$, we see that D' and E_1 are inequivalent cusp regions for $\rho(G_1)$, and hence that D' and E^1_j are inequivalent cusp regions for $\rho_j(G_1)$ for all sufficiently large j.

Let $D'' \subset D'$ be a cusp region for $\rho(g)$ in $\rho(G_1)$ which does not intersect any $\rho(G_1)$-translate of E_1, and suppose that there exists a $\rho'_j(G_1)$-translate of E^j_1 which intersects D'' for infinitely many j. Since D'' and E^j_1 are inequivalent for all sufficiently large j, D'' cannot contain or be contained in any translate of E^j_1. Hence, $d'' = \partial D'' \cap \Omega(\rho'_j(G_1))$ must intersect some $\rho'_j(G_1)$-translate of $e'' = \partial E^j_1 \cap \Omega(\rho'_j(G_1))$ for infinitely many j. This, however, can be ruled out by the argument in the proof of Lemma 3.8. Hence, for all sufficiently large j, D'' does not intersect any $\rho'_j(G_1)$-translate of E^j_1, and so Theorem 2.4 guarantees that D'' is a cusp region for $\rho'_j(G)$ for sufficiently large j. That is, D'' persists in the approximates of $\rho(G)$.

Suppose that D' itself does not persist in the approximates of $\rho(G)$. Then, for infinitely many j, there exists $g_j \in G - \langle g\rangle$ such that $\rho'_j(g_j)(D')$ intersects D'. Since d' is precisely invariant under $\langle\rho_j(g)\rangle$ in $\rho_j(G)$ for sufficiently large j, this implies that $\rho_j(g_j)(\xi)$ is contained in the interior of D' for infinitely many j. However, the argument in the proof of Lemma 3.8 shows that $D' - D'' \subset \Omega(\rho_j(G))$ for all sufficiently large j. (The key point is that there is a compact fundamental domain for the action of $\langle\rho(g)\rangle$ on $D' - int(D'')$, where $int(D'')$ is the interior of D''.) Since D'' is a cusp region for $\rho_j(G)$ for all sufficiently large j, we see that the interior of D' must be contained in $\Omega(\rho_j(G))$ for all sufficiently large j, a contradiction. Hence, D' is a cusp region for $\rho'_j(G)$ for all sufficiently large j, and we have completed the proof of Proposition 3.9 in the case that $\rho(G)$ is formed from $\rho(G_1)$ and $\rho(G_2)$ by a Klein-Maskit combination of type I.

The proof in the case of Klein-Maskit combination of type II is quite similar. Suppose that the factor subgroups of the decomposition are $\rho(H)$ and $\langle\rho(g)\rangle$. Let C_1, C_2, E_1, E_2, $\theta_1 = \rho(h_1)$ and $\theta_2 = \rho(h_2)$ be as in the statement of the Klein-Maskit combination theorem of type II. We may assume that our sequence is normalized about $\rho(h_1)$. Let $c_m = C_m \cap \Omega(H)$. Let $c^j_1 = c_1$, $E^j_1 = E_1$, $c^j_2 = \rho_j(g)(c_1)$ and $E^j_2 = \rho_j(g)(E_2)$. By assumption, for sufficiently large j, E^j_m is a cusp region for $\rho_j(h_m)$ in $\rho_j(H)$. As before, we see that $\rho_j(G)$ is a Klein-Maskit combination of type II with factor subgroups $\rho_j(H)$ and $\langle\rho_j(g)\rangle$, for sufficiently large j, and that $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$. This again implies that $\{\rho_j\}$ converges strongly to ρ. The proof that cusp regions persist in the approximates is also much as above.

$\boxed{\text{Proposition 3.9}}$

We are now prepared to establish Theorem 3.1 in the case when $\rho(G)$ has connected limit set. We suppose that $\{\rho_j\} \subset \mathcal{D}(G)$ is a type-preserving sequence converging algebraically to ρ, that $\Omega(\rho(G))$ is non-empty, and that $\Lambda(\rho(G))$ is connected. Let $\{\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}, \ldots, \Gamma_l\}$ be the Abikoff-Maskit decomposition of $\rho(G)$ as described in Section 2.3, so that each Γ_i is either degenerate without accidental parabolics, generalized web or cyclic. Let $H_i = \rho^{-1}(\Gamma_i)$. By the results of Step 1, each of the restricted sequences $\{\rho_j|_{H_i}\}$ converges strongly to $\rho|_{H_i}$ and $\{\Lambda(\rho_j(H_i))\}$ converges to $\Lambda(\rho(H_i))$. Proposition 3.7 implies that all cusp regions of $\rho(H_i)$ persist in the approximates. We set $\Gamma^1 = \Gamma_1$, and recall that Γ^i is formed from Γ^{i-1} and Γ_i by Klein-Maskit combination along a parabolic cyclic subgroup, of type I for $1 \leq i \leq k$ and of type II for $k < i$, and $\rho(G) = \Gamma^l$. Let $H^i = \rho^{-1}(\Gamma^i)$. Applying Proposition 3.9 inductively shows, for all i, that $\{\rho_j|_{H^i}\}$ converges strongly to $\rho|_{H^i}$, that $\{\Lambda(\rho_j(H^i))\}$ converges to $\Lambda(\rho(H^i))$, and that all cusp regions of $\rho(H^i)$ persist in the approximates. This completes the proof in the case that $\rho(G)$ has connected limit set.

Step 3: We conclude the proof of the theorem by considering the case in which $\rho(G)$ has non-connected limit set. In this case we may decompose $\rho(G)$ into a finite collection of subgroups, each of which either has connected limit set or is elementary, so that $\rho(G)$ is built up from the subgroups by repeatedly applying the operation of Klein combination. One may then complete the proof of the theorem by applying the following result from [4] finitely many times.

Proposition 3.10 *(Proposition 10.2 in [4]) Let G be a finitely generated, torsion-free group, and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to $\rho \in \mathcal{D}(G)$. Suppose that $\Omega(\rho(G))$ is non-empty and $\rho(G)$ is obtained from $\rho(G_1)$ and $\rho(G_2)$ by Klein combination. If $\{\Lambda(\rho_j(G_m))\}$ converges to $\Lambda(\rho(G_m))$ for both m, then $\{\rho_j\}$ converges strongly to ρ and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$.*

$\boxed{\text{Theorem 3.1}}$

4 Generalizations

One may use Selberg's lemma (see Selberg [30]) and results of Jørgensen and Marden to remove the assumption that G is torsion-free in the statement of Theorem 3.1.

Theorem 4.1 *Let G be a finitely generated group and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to $\rho \in \mathcal{D}(G)$. If $\Omega(\rho(G))$ is non-empty, then $\{\rho_j\}$ converges strongly to ρ and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$.*

Proof of 4.1: Selberg's lemma guarantees that there exists a finite index, torsion-free subgroup H of G. Theorem 3.1 implies that $\{\rho_j|_H\}$ converges strongly to ρ_H and that $\{\Lambda(\rho_j(H))\}$ converges to $\Lambda(\rho(H))$. Since $\Lambda(\rho_j(G)) = \Lambda(\rho_j(H))$ for all j and $\Lambda(\rho(G)) = \Lambda(\rho(H))$, we see that $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$. If G is not virtually abelian, then Proposition 2.2 implies that $\{\rho_j\}$ converges strongly to ρ. If G is virtually abelian, we may prove that $\{\rho_j\}$ converges strongly to ρ by extending the arguments given in the proof of Lemma 3.6.

$\boxed{4.1}$

We can similarly generalize Corollary 3.3 to obtain:

Corollary 4.2 *Let G be a finitely generated group and let $\{\rho_j\} \subset \mathcal{D}(G)$ be a type-preserving sequence converging algebraically to ρ. If G contains a finite index, torsion-free subgroup which is not a (non-trivial) free product of (orientable) surface groups and infinite cyclic groups, then $\{\rho_j\}$ converges strongly to ρ and $\{\Lambda(\rho_j(G))\}$ converges to $\Lambda(\rho(G))$.*

References

[1] W. Abikoff and B. Maskit, 'Geometric decompositions of Kleinian group', *Amer. J. Math.* **99** (1977), 687–697.

[2] L. V. Ahlfors, 'Finitely generated Kleinian groups', *Amer. J. Math.* **86** (1964), 413–29.

[3] J. W. Anderson and R. D. Canary, 'Algebraic limits of Kleinian groups which rearrange the pages of a book', *Invent. Math.*, **126** (1996), 205–214.

[4] J. W. Anderson and R. D. Canary, 'Cores of hyperbolic 3-manifolds and limits of Kleinian groups', *Amer. J. Math.* **118** (1996), 745–779.

[5] J. W. Anderson, R. D. Canary, M. Culler, and P. Shalen, 'Free Kleinian groups and volumes of hyperbolic 3-manifolds', *J. Differential Geom.* **44** (1996), 738–782.

[6] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag Universitext, 1992.

[7] F. Bonahon, 'Bouts des variétés hyperboliques de dimension 3', *Annals Math.* **124** (1986), 71–158.

[8] R. D. Canary, 'Ends of hyperbolic 3-manifolds', *J.A.M.S.* **6** (1993), 1–35.

[9] R. D. Canary, 'A covering theorem for hyperbolic 3-manifolds and its applications', *Topology*, **35** (1996), 751–778.

[10] R. D. Canary, D. B. A. Epstein, and P. Green, 'Notes on notes of Thurston', in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 3–92.

[11] A. Douady and C.J. Earle, 'Conformally natural extensions of homeomorphisms of the circle', *Acta Math.* **157** (1986) 23–48.

[12] D. B. A. Epstein and A. Marden, 'Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces', in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, 113–253.

[13] F.W. Gehring and G.J. Martin, 'Discrete quasiconformal groups I', *Proceedings L. M. S.* **55** (1987), 331–358.

[14] T. Jørgensen, 'On discrete groups of Möbius transformations', *Amer. J. Math.* **98** (1976), 739–749.

[15] T. Jørgensen and A. Marden, 'Algebraic and geometric convergence of Kleinian groups', *Math. Scand.* **66** (1990), 47–72.

[16] S. Kerckhoff and W. P. Thurston, 'Non-continuity of the action of the mapping class group at Bers' boundary of Teichmuller space', *Invent. Math.* **100** (1990), 25–47.

[17] R. Kulkarni and P. Shalen, 'On Ahlfors' finiteness theorem', *Adv. in Math.* **76** (1989), 155–169.

[18] A. Marden, 'The geometry of finitely generated Kleinian groups', *Annals Math.* **99** (1974), 383–462.

[19] A. Marden, 'Geometrically finite Kleinian groups and their deformation spaces', in *Discrete Groups and Automorphic Functions,* ed. by W. J. Harvey, Academic Press, New York, 1977, 259–293.

[20] B. Maskit, 'On boundaries of Teichmüller spaces and on Kleinian groups II', *Annals Math.* **91** (1970), 607–639.

[21] B. Maskit, 'On the classification of Kleinian groups: I - Koebe groups', *Acta Math.* **135** (1975), 249–270.

[22] B. Maskit, *Kleinian groups*, Springer-Verlag, 1988.

[23] D. McCullough, 'Compact submanifolds of 3-manifolds with boundary', *Quart. J. Math. Oxford* **37** (1986), 299–307.

[24] K. Ohshika, 'Strong convergence of Kleinian groups and Caratheodory convergence of domains of discontinuity', *Math. Proc. Camb. Phil. Soc.* **112** (1992), 297–307.

[25] K. Ohshika, 'Geometric behaviour of Kleinian groups on boundaries for deformation spaces', *Quart. J. Math. Oxford* **43** (1992), 97–111.

[26] K. Ohshika, 'Kleinian groups which are limits of geometrically finite groups', preprint.

[27] K. Ohshika, 'Divergent sequences of Kleinian groups', preprint.

[28] H. M. Reimann, 'Invariant extensions of quasiconformal deformations', *Ann. Acad. Sci. Fenn.* **10** (1985), 477–492.

[29] G. P. Scott, 'Compact submanifolds of 3-manifolds', *Journal L. M. S.* **7** (1973), 246–250.

[30] A. Selberg, 'On Discontinuous groups in higher dimensional symmetric spaces', in *Colloquim Function Theory,* Tata Institute, 1960.

[31] T. Soma, 'Function groups in Kleinian groups', *Math. Annalen* **292** (1992), 181–190.

[32] W. P. Thurston, *The geometry and topology of 3-manifolds*, lecture notes, Princeton University.

[33] P. Tukia, 'Quasiconformal extensions of quasisymmetric mappings compatible with a Möbius group', *Acta Math.* **154** (1985), 153–193.