Bending, entropy and proper affine actions of surface groups

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Overview

We study bending deformations of quasifuchsian representations using old-fashioned hyperbolic geometry and obtain two more modern sounding applications.

1) An explicit criterion guaranteeing that a quasifuchsian representation is not a critical point of the entropy function on quasifuchsian space.

2) An explicit criterion guaranteeing that a quasifuchsian representation ρ has the property that $\mathrm{Ad}\rho$ is the linear part of a proper affine action of a surface group on $\mathfrak{sl}(2,\mathbb{C})$.

In each case, we obtain an open neighborhood of the Fuchsian locus in quasifuchsian space, so that any nonfuchsian representation in the neighborhood satisfies the criterion.

Quasifuchsian representations

- Let S be a closed surface of genus $g \geq 2$ and recall that $\mathsf{PSL}(2,\mathbb{C}) = \mathrm{Isom}_+(\mathbb{H}^3)$.
- A **quasifuchsian representation** is a discrete, faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$ so that $\rho(\pi_1(S))$ leaves invariant a Jordan curve in $\mathbb{CP}^1$ which we call $\Lambda(\rho)$. Here, $\Lambda(\rho)$ is called the **limit set**.
- $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ is a hyperbolic 3-manifold diffeomorphic to $S \times R$.
- $\rho(\pi_1(S))$ acts properly discontinuously on the **domain of discontinuity** $\Omega(\rho) = \mathbb{CP}^1 \setminus \Lambda(\rho)$ and its quotient

$$\partial_c N_\rho = \Omega(\rho)/\rho(\pi_1(S))$$

is a Riemann surface diffeomorphic to $S \sqcup \bar{S}$.

Quasifuchsian space

- There exists an orientation-preserving diffeomorphism

$$h_\rho : S \times [0,1] \to N \cup \partial_c N_\rho = \left(\mathbb{H}^3 \cup \Omega(\rho)\right)/\rho(\pi_1(S))$$

so that $(h_\rho)_*$ is conjugate to ρ.

- **Quasifuchsian space** $QF(S)$ is the space of conjugacy classes of quasifuchsian representations which we may regard a complex submanifold of the character variety

$$\mathbf{X}(S) = \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))//\mathsf{PSL}(2,\mathbb{C})$$

- (Bers) The map $B : QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ given by $B(\rho) = (\partial_c N_\rho, h_\rho)$ is a well-defined real analytic diffeomorphism.

The bending lamination

- The **convex hull** $CH(\rho)$ is the convex hull of the limit set $\Lambda(\rho)$.
- The *convex core* $C(N) = CH(\rho)/\rho(\pi_1(S))/\rho(\pi_1(S))$ has two boundary components $\partial C^{\pm}(\rho)$. Each boundary component is totally geodesic in the complement of a geodesic lamination $\beta^{\pm}$, called the **bending lamination**.
- Each bending lamination $\beta^{\pm}$ inherits a transverse measure which records how much $\partial C^{\pm}(\rho)$ bends along ρ.
- If β^{ν} is a multicurve, this transverse measure is just the the atomic measure given by the angle of bending of $\partial C^{\pm}(\rho)$ along β^{ν}.
- Bonahon and Otal characterized which pairs of lamination can be the bending laminations of a quasifuchsian represenation
- Dular and Schlenker showed that the bending laminations determine the representation.

Moderately bent Jordan domains

- Let U be a Jordan domain with boundary the Jordan curve C.
- We say that a pair (x, y) of distinct points in C is a **bending pair** if there is an open round disk D in U so that $x, y \in \partial D$. Geometrically, D bounds a support plane to the convex hull of C.
- Leaves of the bending laminations of $\partial C^\nu(\rho)$ give rise to bending pairs of $\Omega^\nu(\rho)$, but the converse need not hold.
- We say that U is **moderately bent** if for every bending pair (x, y) in C there exists a circle L which is transverse to C and intersects C only at x and y.
- A component of the domain of discontinuity $\Omega(\rho)$ of a quasifuchsian representation need not be moderately bent, but if ρ is close enough to being quasifuchsian, then both components of the domain of discontinuity are moderately bent. We will later describe a numerical invariant which makes this explicit.

Bending deformations

- If $\rho \in \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$ is quasifuchsian and α is a simple closed curve on S one may define the **bending deformation** $\{\rho_t\}_{t\in\mathbb{R}} \subset \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$ of ρ along α.
- If α separates S, then $\pi_1(S) = \pi_1(S_1) *_{\langle\alpha\rangle} \pi_1(S_2)$. If A_t is the a rotation of angle t in the axis of $\rho(\alpha)$, we let $\rho_t = \rho$ on $\pi_1(S_1)$ and $\rho_t(\gamma) = A_t\rho(g)A_t^{-1}$ if $\gamma \in \pi_1(S_2)$. If α does not separate, then we use a HNN decomposition of $\pi_1(S)$.
- Moreover, for any measured lamination λ we can define a bending deformation of ρ along λ and ρ_t is quasifuchsian for all t close enough to 0.
- We can also define an **infinitesmal bending deformation** of ρ along λ as

$$\left.\frac{d}{dt}\right|_{t=0}\rho_t \in T_\rho\mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C})).$$

The entropy function

- If $\rho \in QF(S)$, then the **topological entropy** $h(\rho)$ is simply the exponential growth rate of the number of closed geodesics in N_ρ of length at most T, i.e.

$$h(\rho) = \lim_{T\to\infty} \frac{\log \#\{[\gamma] \in [\pi_1(S)] : \ell_\gamma(\rho) \le T\}}{T}$$

  where $\ell_\gamma(\rho)$ is the real translation length of $\rho(\gamma)$.
- (Sullivan) $h(\rho)$ is the Hausdorff dimension of the limit set $\Lambda(\rho)$.
- (Bowen) $h(\rho) \ge 1$ with equality if and only if ρ is Fuchsian.
- (Ruelle) h is analytic on $QF(S)$.
- (Bridgeman) At any critical point, the Hessian of the entropy function is positive definite on at least a half-dimensional subspace of $QF(S)$. In particular, h has no local maxima.
- One hopes that the only critical points of h lie on the Fuchsian locus.

Our first application

- (Sambarino) If ρ is quasifuchsian and there exists $\vec{v} \in T_\rho \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))$ so that $D\ell_\gamma(\vec{v}) \leq 0$ for all $\gamma \in \pi_1(S)$, then $[\rho]$ is not a critical point of h.
- We generalize Kourounitis to obtain a formula for the derivative of length with respect to the bending deformation and show:
- **Theorem (BCS):** If a component, $\Omega^\nu(\rho)$ is moderately bent and $\vec{v}$ is the infinitesmal bending deformation for β^ν at ρ, then $D\ell_\gamma(\vec{v}) \leq 0$ for all $\gamma \in \pi_1(S)$. Moreover, $D\ell_\gamma(\vec{v}) < 0$ if γ intersects β^ν transversely.
- **Corollary (BCS):** If $[\rho] \in QF(S)$ is not Fuchsian and either component of $\Omega(\rho)$ is moderately bent, then $[\rho]$ is not a critical point of the entropy function h.

Affine actions

- The group $\mathrm{Aff}(V)$ of affine actions on a finite-dimensional vector space V is identified with $\mathrm{GL}(V) \rtimes V$ where $(A, \vec{w})$ acts on V by

$$(A, \vec{w})(\vec{v}) = A(\vec{v}) + \vec{w} \quad \text{for all} \quad \vec{v} \in V.$$

- Auslander conjectured that if $\Gamma \subset \mathrm{Aff}(V)$ acts properly and cocompactly on V, then Γ is virtually solvable. Milnor asked if something similar might be true for general proper actions.
- Margulis produced proper affine action of non-abelian free groups on $\mathbb{R}^3$.
- Danciger-Guéritaud-Kassel showed that right-angled Coxeter groups with k generators admit proper affine actions on $\mathbb{R}^{k(k-1)/2}$. Moreover, surface groups admit proper affine actions on $\mathbb{R}^6$.
- Mess showed that there are no proper affine actions of surface groups on $\mathbb{R}^3$.

Constructing affine actions

- Suppose that $\rho : \Gamma \to \mathsf{PSL}(2,\mathbb{K})$ is convex cocompact (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\vec{v} \in T_\rho \mathrm{Hom}(\Gamma, \mathsf{PSL}(2,\mathbb{K}))$.
- If $\gamma \in \Gamma$, let $E_\gamma : \mathrm{Hom}(\Gamma, \mathsf{PSL}(2,\mathbb{K})) \to \mathsf{PSL}(2,\mathbb{K})$ be given by $E_\gamma(\rho) = \rho(\gamma)$. We then define a cocycle $\mathbf{u}_{\vec{v}} : \Gamma \to \mathfrak{sl}(2,\mathbb{K})$ by setting

$$\mathbf{u}_{\vec{v}}(\gamma) = DE_\gamma(\vec{v})E_\gamma(\rho)^{-1}.$$

- One may then check that $F_{(\rho,\vec{v})} : \Gamma \to \mathrm{Aff}(\mathfrak{sl}(2,\mathbb{K}))$ given by

$$F_{(\rho,v)}(\gamma) = \Big(\mathrm{Ad}\big(\rho(\gamma)\big), \mathbf{u}_{\vec{v}}(\gamma)\Big)$$

is a discrete, faithful representation.
- However, since $\mathrm{Aff}(V)$ does not act by isometries of a metric on V, this does not imply that $\rho(\Gamma)$ acts properly discontinuously.

A properness criterion

- (Margulis, Goldman-Labourie-Margulis, Smilga, Ghosh, Kassel-Smilga) If there exists $C > 0$ so that

$$|D\ell_\gamma(\vec{v})| \geq C\ell_\gamma(\rho) \quad \text{for all} \quad \gamma \in \Gamma$$

  then $F_{(\rho,v)}(\Gamma)$ acts properly discontinuously on $\mathfrak{sl}(2,\mathbb{K})$.
- This criterion is usually expressed in terms of the Margulis invariant.
- In this ahistorical presentation, Margulis produced a deformation $\{\rho_t : F_2 \to \mathrm{PSL}(2,\mathbb{R})\}$ so that, for some $C > 0$,

$$D\ell_\gamma(\vec{v}) \geq C\ell_\gamma(\rho) \quad \text{for all} \quad \gamma \in F_2.$$

- One may construct such a deformation by taking a pair of pants and inserting a strip along each common perpendicular between the cuffs. This viewpoint was developed by Danciger-Guéritaud-Kassel and exploited to classify proper affine actions of free groups on $\mathbb{R}^3$.

Our second application

- **Theorem (BCS):** Suppose that $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is quasifuchsian but not Fuchsian and both components of $\Omega(\rho)$ are moderately bent. If $\vec{v}^{\pm}$ is the infinitesmal bending deformation arising from bending ρ along $\beta^{\pm}$ and $\vec{w} = \vec{v}^{+} + \vec{v}^{-}$, then there exists $C > 0$ so that

$$D\ell_\gamma(\vec{w}) \leq -C\ell_\gamma(\rho) \quad \text{for all} \quad \gamma \in \pi_1(S).$$

- **Corollary (BCS):** if $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is quasifuchsian but not Fuchsian and both components of $\Omega(\rho)$ are moderately bent, then $\mathrm{Ad}\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$.

Roundness

- If β is a measured lamination on a hyperbolic surface X, Epstein and Markovic defined its **roundness** to be

$$||\beta|| = \sup\{i(a,\beta) : a \text{ is an open geodesic arc of length } 1 \text{ on } X \text{ which is transverse to } \beta\}.$$

- If β is a curve with weight b, then $i(a,\beta)$ is simply b times the number of intersections of a with β. So, in this case, the supremum is achieved by the geodesic arc of length one which intersects β the most.
- We used techniques developed by Bridgeman, Canary and Yarmola to prove:

  Theorem (BCS): If ρ is quasifuchsian and $||\beta^\nu|| < .73$, then $\Omega^\nu(\rho)$ is moderately bent.
- For comparison, if ρ is quasifuchsian, $||\beta^\nu|| < 4.24$ for both ν. Moreover, this theorem cannot be true for any bound bigger than .97.

Our open sets

- **Corollary:** If ρ is quasifuchsian, but not Fuchsian, and $||\beta^+|| < .73$ **or** $||\beta^-|| < .73$, then $[\rho]$ is not a critical point of the entropy function h on $QF(S)$.
- **Corollary:** If ρ is quasifuchsian, but not Fuchsian, and $||\beta^+|| < .73$ **and** $||\beta^-|| < .73$, then $\mathrm{Ad}\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2, \mathbb{C})$.
- Since $||\beta^+||$ and $||\beta^-||$ both vary continuously over $QF(S)$, both corollaries give rise to open neighborhoods of the Fuchsian locus.

More general Lie groups

Corollary (BCS): If G is a complex Lie group with Lie algebra $\mathfrak{g}$, then there is an open set U in $\mathrm{Hom}(\pi_1(S), \mathsf{G})$, so that if $\rho \in U$, then $\mathrm{Ad}\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{g}$.

The proof involves a more general criterion for when $(\rho, \vec{v})$ gives rise to a proper affine action $F_{(\rho,\vec{v})}$ (due to the same list of people) and work of Sambarino which shows that the set of $(\rho, \vec{v})$ verifying this criterion is open.

We make use of the principal embedding $\tau : \mathsf{SL}(2, \mathbb{C}) \to \mathsf{G}$ and show that if ρ is (the lift of) a quasifuchsian representation and $(\rho, \vec{v})$ satisfies the criterion, then $(\tau \circ \rho, D\bar{\tau}(\vec{v}))$ satisfies the criterion where $\bar{\tau} : \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2, \mathbb{C})) \to \mathrm{Hom}(\pi_1(S), \mathsf{G})$ is given by $\bar{\tau}(\rho) = \tau \circ \rho$.

Group manifolds

- The group manifold of a semi-simple Lie group G is the quotient
$$\Big(\mathsf{G} \times \mathsf{G} \setminus \mathrm{Diag}(G)\Big)/\mathsf{G}$$
- We may identify the group manifold with G and it inherits a pseudo-Riemannian metric from the Killing form which is invariant under both left and right multiplication. The group manifold of $\mathsf{PSL}(2,\mathbb{R})$ is 3-dimensional anti-de Sitter space
- **Corollary (BCS):** If G is a simple complex Lie group, then there is an open set of representations into $\mathsf{G} \times \mathsf{G}$ which are discrete and faithful in each factor and acts properly on G by left/right multiplication.
- Danciger, Guéritaud and Kassel established this when $\mathsf{G} = \mathsf{PSL}(2,\mathbb{C})$.
- The key tool in the proof is the properness criterion of Benoist and Kobayashi.

Variation of complex length under bending

- Let $L_\gamma : \mathrm{PSL}(2,\mathbb{C}) \to \mathbb{C}/\langle 2\pi i\rangle$ be the complex translation length, so $\ell_\gamma = \mathrm{Re}(L_\gamma)$.
- Given two oriented geodesics g and h in $\mathbb{H}^3$, let $\sigma(g,h)$ be the (unsigned) complex distance between g and h.
- **Theorem (Kourouniotis):** Suppose that ρ is quasifuchsian and β is a simple closed curve on S with with weight $b > 0$ and let $\vec{v}$ be the infinitesmal bending deformation of ρ along β. If $\gamma \in \pi_1(S)$ and γ intersects β transversely

$$DL_\gamma(\vec{v}) = -bi \sum_{x\in\alpha\cap\beta} \cosh(C_x, B_x)$$

.

- Let $h : \tilde{S} \to \mathbb{H}^3$ be a ρ-equivariant map, let $\tilde{x}$ be a lift of x to $\tilde{S}$, then let $\tilde{\gamma}$ and $\tilde{\beta}$ be the lifts of γ and β passing through $\tilde{x}$ and orient them so that $\tilde{\gamma}$ crosses $\tilde{\beta}$ from left to right. Let C_x and B_x be the oriented geodesics in $\mathbb{H}^3$ with the same endpoints as $h(\tilde{\gamma})$ and $h(\tilde{\beta})$.

Interpreting the terms in the formula

- One may compute that

$$\mathrm{Im}\Big(\cosh(C_x, B_x)\Big) = \frac{-2\mathrm{Im}([c_-, c_+, b_-, b_+]}{\Big|[c_-, c_+, b_-, b_+] - 1\Big|^2}.$$

  where $(c_-, c_+) = C_x$ and $(b_-, b+) = B_x$ as oriented geodesics.

- So in order to show that $\ell_\gamma(\vec{v}) < 0$ it suffices to show that for all $x \in \alpha \cup \beta$, we have

$$\mathrm{Im}([c_-, c_+, b_-, b_+]) > 0.$$

- If μ is a general measured lamination, then we generalize Kourouniotis' formula to obtain an integral of a function all of whose outputs have the form $-i \cosh(C_x, B_x)$.

Linking this to moderately bent

We can normalize so that $(b_-, b+) = (0, \infty)$, the closed upper half plane intersects $\Lambda(\rho)$ only at 0 and ∞. If $\Omega^+(\rho)$ is moderately bent we have the following picture: