Contemporary Mathematics

Pushing the boundary

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ABSTRACT. We give a brief survey of recent results concerning the boundaries of deformation spaces of Kleinian groups.

1. Introduction

The goal of the deformation theory of Kleinian groups is to classify and parameterize the space $AH(\pi_1(M))$ of all (marked) hyperbolic 3-manifolds homotopy equivalent to a fixed compact 3-manifold M. The interior $MP(\pi_1(M))$ of $AH(\pi_1(M))$ is well-understood, due to work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston and others. In this paper, we will survey some recent results concerning the boundary of $MP(\pi_1(M))$. We will attempt to largely limit ourselves to the results which were touched on in our talk at the Ahlfors-Bers Colloquium, so, by necessity, many exciting results will be left out.

The components of $MP(\pi_1(M))$ are enumerated by topological data and each component is parameterized by analytic data. We will begin by describing the parameterization of $MP(\pi_1(M))$. We will then survey some recent work on the "bumping" and "self-bumping" of components of $MP(\pi_1(M))$, by Anderson, Bromberg, Canary, Holt, McCullough and McMullen.

The Bers-Sullivan-Thurston Density Conjecture predicts that $AH(\pi_1(M))$ is the closure of its interior. Thurston's Ending Lamination Conjecture provides a conjectural classification of the points in $AH(\pi_1(M))$ in terms of topological and geometrical data. We will discuss Brock and Bromberg's pioneering work on the Density Conjecture and Minsky's recent announcement of the solution (in collaboration with Brock, Canary and Masur) of the Ending Lamination Conjecture for hyperbolic 3-manifolds with freely indecomposable fundamental group. In these sections, we will be rather sketchy, as other recently written surveys exist of this work and neither subject was dealt with at length in our talk. In fact, Minsky's announcement took place 6 months after the Ahlfors-Bers Colloquium.

Thurston's Ending Lamination Conjecture suggests that geometrically finite hyperbolic 3-manifolds are dense in the boundary of $MP(\pi_1(M))$. This suggestion turns out to be correct, for all compact hyperbolizable 3-manifolds, a fact whose proof combines work of Canary, Culler, Hersonsky, McMullen and Shalen.

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We will also discuss tameness results for manifolds in the boundary of $AH(\pi_1(M))$ and the role of tameness in our understanding of the deformation theory of Kleinian groups.

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2. Definitions

We will assume throughout that M **is a compact, oriented, hyperbolizable 3-manifold with non-empty boundary.** In order to simplify matters, we will also assume that ∂M **contains no tori**. All the results we describe have analogues for manifolds with toroidal boundary components and more generally for pared manifolds.

Let $\mathcal{D}(\pi_1(M)) \subset Hom(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ denote the set of all discrete faithful representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbf{C})$. Then,

$$AH(\pi_1(M)) = \mathcal{D}(\pi_1(M))/\mathrm{PSL}_2(\mathbf{C})$$

where $\mathrm{PSL}_2(\mathbf{C})$ acts by conjugation on $\mathcal{D}(\pi_1(M))$. $AH(\pi_1(M))$ sits inside the character variety

$$X(M) = Hom(\pi_1(M)), \mathrm{PSL}_2(\mathbf{C}))//\mathrm{PSL}_2(\mathbf{C})$$

which is the algebro-geometric quotient of $Hom(\pi_1(M)), \mathrm{PSL}_2(\mathbf{C}))$ (see Morgan-Shalen [**51**] for details). We define $MP(\pi_1(M))$ to be the interior of $AH(\pi_1(M))$ in $X(\pi_1(M))$.

Given $\rho \in \mathcal{D}(\pi_1(M))$, $N_\rho = \mathbf{H}^3/\rho(\pi_1(M))$ is a hyperbolic 3-manifold and there exists a homotopy equivalence $h_\rho : M \to N_\rho$, called the *marking* of N_ρ, such that $(h_\rho)_* = \rho$ where we think of ρ as giving an identification, well-defined up to conjugation, of $\pi_1(M)$ with $\pi_1(N_\rho)$.

Alternatively, we may view $AH(\pi_1(M))$ as the set of pairs (N, h) where N is an oriented hyperbolic 3-manifold and $h : M \to N$ is a homotopy equivalence. Two pairs (N_1, h_1) and (N_2, h_2) are equivalent if there is an orientation-preserving isometry $j : N_1 \to N_2$ such that $j \circ h_1$ is homotopic to h_2.

Similarly, the Teichmüller space $\mathcal{T}(F)$ of a closed surface F is the set of pairs (S, h) where S is a Riemann surface and $h : F \to S$ is an orientation-preserving homeomorphism, where two pairs (S_1, h_1) and (S_2, h_2) are equivalent if there is a conformal map $j : S_1 \to S_2$ such that $j \circ h_1$ is homotopic to h_2.

In a topological vein, let $\mathcal{A}(M)$ consist of the set of pairs (M', h') where M' is a compact, oriented irreducible 3-manifold and $h' : M \to M'$ is a homotopy equivalence, where two pairs (M_1, h_1) and (M_2, h_2) are equivalent if there is an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that $j \circ h_1$ is homotopic to h_2. We think of $\mathcal{A}(M)$ as the set of all marked, oriented, irreducible, compact 3-manifolds homotopy equivalent to M (If M has toroidal boundary components, we would further insist that elements of $\mathcal{A}(M)$ be atoroidal, so that they would all be hyperbolizable.)

It will be useful to consider the conformal extension of a hyperbolic 3-manifold N_ρ. The *domain of discontinuity* $\Omega(\rho)$ is the largest open subset of $\widehat{\mathbf{C}}$ on which $\rho(\pi_1(M))$ acts properly discontinuously. The *limit set* $\Lambda(\rho)$ is the complement in $\widehat{\mathbf{C}}$ of $\Omega(\rho)$. The *conformal boundary* is the quotient $\partial_c N_\rho = \Omega(\rho)/\rho(\pi_1(M))$. Then

$$\widehat{N}_\rho = N_\rho \cup \partial_c N_\rho = (\mathbf{H}^3 \cup \Omega(\rho))/\rho(\pi_1(M))$$

is the *conformal extension* of N_ρ. It has a complete hyperbolic structure on its interior and a conformal structure on its boundary.

We say that ρ (or N_ρ) is *convex cocompact* if $\widehat{N}_\rho$ is compact. More generally, ρ (or N_ρ) is *geometrically finite* if $\widehat{N}_\rho$ is homeomorphic to $\widehat{M} - \widehat{P}$ where $\widehat{M}$ is a compact 3-manifold and $\widehat{P}$ is a finite collection of disjoint annuli and tori in $\partial\widehat{M}$. Geometrically finite hyperbolic 3-manifolds are the best understood class of hyperbolic 3-manifolds.

3. The parameterization of $MP(\pi_1(M))$

In this section we review the parameterization of $MP(\pi_1(M))$ which was completed in the 1960's and 1970's through work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurston. A more extensive treatment from this viewpoint is given in [**22**]. Bers [**8**] wrote an excellent survey of much of this theory from a more analytic viewpoint.

If one combines Marden's Stability Theorem [**39**] and results of Sullivan [**58**] one sees that $\rho \in MP(\pi_1(M))$ if and only if M is convex cocompact. Therefore, we can define a map $\Theta : MP(\pi_1(M)) \to \mathcal{A}(M)$ by setting $\Theta(\rho) = [(\widehat{N}_\rho, h_\rho)]$. Marden's Isomorphism Theorem [**39**] implies that $\rho_1, \rho_2 \in MP(\pi_1(M))$ lie in the same component of $MP(\pi_1(M))$ if and only if $\Theta(\rho_1) = \Theta(\rho_2)$. Thurston's Geometrization Theorem (see [**54**]) implies that Θ is surjective. Therefore, components of $MP(\pi_1(M))$ are in a one-to-one correspondence with the space $\mathcal{A}(M)$ of all marked, oriented, atoroidal, irreducible 3-manifolds homotopy equivalent to M.

Let B be a component of $MP(\pi_1(M))$, $\rho_0 \in B$ and $N = N_{\rho_0}$. If $\rho \in B$, then there exists an orientation-preserving homeomorphism $j : \widehat{N} \to \widehat{N}_\rho$ such that $j_* : \pi_1(\widehat{N}) \to \pi_1(\widehat{N}_\rho)$ induces the same identification, well-defined up to conjugation, as $\rho \circ \rho_0^{-1}$. This homeomorphism gives rise to a point

$$(\partial_c N_\rho, j|_{\partial\widehat{N}}) \in \mathcal{T}(\partial_c N).$$

One may apply work of Ahlfors, Bers [**7**], Kra [**36**] and Maskit [**40**] to see that

$$B \cong \mathcal{T}(\partial_c N)/Mod_0(N)$$

where $Mod_0(N)$ is the group of (isotopy classes of) homeomorphisms of $\widehat{N}$ which are homotopic to the identity. (This makes sense since the homeomorphism j is really only well-defined up to an element of $Mod_0(N)$.) Maskit [**40**] proved that $Mod_0(N)$ acts freely and properly discontinuously on $\mathcal{T}(\partial_c N)$, so B is always a manifold. We summarize this discussion in the following theorem:

Parameterization Theorem: *If M is a hyperbolizable compact oriented 3-manifold with no torus boundary components, then $MP(\pi_1(M))$ is homeomorphic to the disjoint union*

$$\bigsqcup_{(M',h') \in \mathcal{A}(M)} \mathcal{T}(\partial M')/\mathrm{Mod}_0(M')$$

If M has incompressible boundary, then $Mod_0(M')$ is always trivial and $MP(\pi_1(M))$ is a union of topological balls. McCullough [**43**] proved that if $Mod_0(M)$ is finitely generated, then M is "almost incompressible" and $Mod_0(M)$ is abelian. Canary and McCullough [**22**] have completely characterized when $MP(\pi_1(M))$ has infinitely many components. Assuming that M has no toroidal boundary components, then $MP(\pi_1(M))$ has infinitely many components unless M has incompressible boundary or M is homeomorphic to a handlebody, a boundary connected sum of two I-bundles or is obtained by attaching a single handle to an I-bundle.

Examples: 1) If $M = S \times I$, then $\mathcal{A}(M)$ has a single element, and

$$MP(M) = QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S)$$

is called *quasifuchsian* space. (Historically, this was the first quasiconformal deformation space to be completely understood, through Bers' work on simultaneous uniformization [**5**].)

2) If M is a handlebody, then again $\mathcal{A}(M)$ has a single element, and

$$MP(\pi_1(M)) \cong \mathcal{T}(\partial M)/Mod_0(M)$$

is known as *Schottky space*, and $Mod_0(M)$ is infinitely generated.

3) We will give a more complicated family $\{M_n\}$ of examples such that $MP(\pi(M_n))$ has $(n-1)!$ components. Let $n \geq 3$. We first form a 2-complex X_n which embeds in $\mathbf{R}^3$. Begin with $S^1 \times [0,1]$ and to each curve $S^1 \times \{\frac{j}{n}\}$ (with $j = 1, \ldots, n$) attach a once-punctured surface of genus j. Let M_n be the regular neighborhood of X_n in $\mathbf{R}^3$. M_n is an example of a book of I-bundles. Notice that one may also think of M_n as being constructed from a solid torus V, which is a regular neighborhood of $S^1 \times I$, by attaching I-bundles along parallel longitudinal annuli in ∂V.

If $\tau \in S_n$ (the permutation group of $\{1, \ldots, n\}$), then we may form a homotopy equivalent, but not necessarily homeomorphic, 3-manifold M_τ. Let X_τ be constructed from $S^1 \times [0,1]$ by attaching a once-punctured surface of genus $\tau(j)$ to $S^1 \times \{\frac{j}{n}\}$ (for each $j = 1, \ldots, n$). Then M_τ is simply a regular neighborhood of X_τ. If you collapse $S^1 \times I$ to a circle, then X_n and X_τ become homeomorphic 2-complexes, so M_τ and M_n are homotopy equivalent. It turns out that M_τ is homeomorphic to M_n (by an orientation-preserving homeomorphism) if and only if τ is a power of the cyclic permutation $(123 \cdots n)$. In fact, $\mathcal{A}(M_n)$ is in a one-to-one correspondence with $S_n/\mathbf{Z}_n$, so $MP(\pi_1(M_n))$ has $(n-1)!$ components (see Lemma 3.2 in Anderson-Canary [**2**]).

Remarks: There are also conjectural parameterizations of a component of $MP(\pi_1(M))$ by more geometric data. One expects that the bending laminations on the ends of the convex core parameterize $MP(\pi_1(M))$. Bonahon-Otal [**10**] and LeCuire [**38**] have given complete descriptions of which bending laminations can occur. Keen and Series, see for example [**34**], have extensively studied this proposed parameterization in a variety of special cases. One also expects that the conformal structure on the boundary of the convex core provides a parameterization analogous to the one given by the Parameterization Theorem above, but much less is known about this conjecture.

4. Bumping of deformation spaces

We will say that two components of $MP(\pi_1(M))$ *bump* if they have intersecting closures. The phenomenon of bumping was first discovered by Anderson and Canary [**2**] who showed that if M_n is the book of I-bundles constructed in example 3 above, then any two components of $MP(\pi_1(M_n))$ bump. In particular, this shows that topological type does not vary continuously on $AH(\pi_1(M_n))$. Holt [**29**] further showed that, in these same examples, there exists points simultaneously in the closure of all components of $MP(\pi_1(M_n))$

Anderson, Canary, and McCullough [**4**] gave a complete characterization of when two components of $MP(\pi_1(M))$ bump in the case that M has incompressible boundary. Roughly, two components bump if and only if their corresponding homeomorphism types differ by rearranging the way a collection of submanifolds are glued along primitive solid torus components of the characteristic submanifold[1] $\Sigma(M)$ of M. A solid torus component V of the characteristic submanifold of M is *primitive* if each component of $V \cap \partial M$ is an annulus such that the inclusion map into V is a homotopy equivalence.

To give a feeling for this characterization, we construct a new manifold M_n' similar to M_n such that the closures of any two components of $MP(\pi_1(M_n'))$ are disjoint. Let V be a solid torus and let $\{A_1, \dots, A_n\}$ be a family of incompressible, parallel, disjoint annuli in the boundary of V such that the inclusion of A_i into V is not a homotopy equivalence, i.e. the core curve of A_i wraps more than once around the core of V. We form M_n' by attaching $F_j \times I$ to A_j, where F_j is a once-punctured surface of genus j, along $\partial F_j \times I$. Again, $\mathcal{A}(M_n')$ may be identified with $S_n/\mathbf{Z}_n$. However, in this case no two components of $MP(\pi_1(M_n'))$ bump. As a hint at why one cannot make the components bump, note that V is not primitive in M_n', so one cannot construct a hyperbolic structure on the interior of M_n' such that the core curve of V is homotopic into a cusp.

We now develop the formalism to allow us to state our bumping criterion precisely. Given two 3-manifolds M_1 and M_2 with nonempty incompressible boundary, a homotopy equivalence $h\colon M_1 \to M_2$ is a *primitive shuffle* if there exists a finite collection $\mathcal{V}_1$ of primitive solid torus components of $\Sigma(M_1)$ and a finite collection $\mathcal{V}_2$ of solid torus components of $\Sigma(M_2)$, so that $h^{-1}(\mathcal{V}_2) = \mathcal{V}_1$ and so that h restricts to an orientation-preserving homeomorphism from the closure of $M_1 - \mathcal{V}_1$ to the closure of $M_2 - \mathcal{V}_2$. (Recall that $\Sigma(M_i)$ denotes the characteristic submanifold of M_i.)

If M is a compact, hyperbolizable 3-manifold with nonempty incompressible boundary, we say that two elements $[(M_1, h_1)]$ and $[(M_2, h_2)]$ of $\mathcal{A}(M)$ are *primitive shuffle equivalent* if there exists a primitive shuffle $s\colon M_1 \to M_2$ such that $[(M_2, h_2)] = [(M_2, s \circ h_1)]$. In section 7 of [**4**] it is established that primitive shuffle equivalence gives an equivalence relation on $\mathcal{A}(M)$ and we let $\widehat{\mathcal{A}}(M)$ be the quotient of $\mathcal{A}(M)$ by this equivalence relation.

THEOREM 4.1 (Anderson-Canary-McCullough [**4**]). *Let M be a compact, hyperbolizable 3-manifold with nonempty incompressible boundary, and let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be two elements of $\mathcal{A}(M)$. The associated components of $MP(\pi_1(M))$ have intersecting closures if and only if $[(M_2, h_2)]$ is primitive shuffle equivalent to $[(M_1, h_1)]$.*

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[1] The characteristic submanifold of a compact, irreducible 3-manifold with incompressible boundary is a minimal collection $\Sigma(M)$ of disjoint essential Seifert fibered spaces and I-bundles in M such that every essential annulus and torus in M is homotopic into $\Sigma(M)$. In the case of a hyperbolizable 3-manifold, all the Seifert fibered components of $\Sigma(M)$ are either solid tori or thickened tori. The characteristic submanifold was introduced by Jaco-Shalen [**32**] and Johannson [**33**]. For a discussion of the characteristic submanifold in the setting of hyperbolic 3-manifolds see [**22**].

One immediate consequence of this theorem is a topological enumeration of the components of the closure of $MP(\pi_1(M))$.

COROLLARY 4.2. *If M has incompressible boundary, then the components of the closure of $MP(\pi_1(M))$ are in one-to-one correspondence with $\widehat{\mathcal{A}}(M)$.*

We also note that if two components of $MP(\pi_1(M))$ bump then $AH(\pi_1(M))$ is not a manifold.

Holt further showed that if a collection of components of $MP(\pi_1(M))$ all bump one another, then there is a point in the closure of all the components.

THEOREM 4.3 (Holt [**30**]). *Let M be a compact hyperbolizable 3-manifold with non-empty incompressible boundary. If $\{B_1, \ldots, B_n\}$ are components of $MP(\pi_1(M))$ and $\Theta(B_i)$ is primitive shuffle equivalent to $\Theta(B_j)$ for all i and j, then $\bigcap \overline{B_i}$ is non-empty.*

Remark: If M is allowed to have toroidal boundary components, then $MP(\pi_1(M))$ is the space of geometrically finite, marked hyperbolic 3-manifolds all of whose cusps have rank two. All of the theorems in this section remain true in this setting. If M has incompressible boundary, the sets $\mathcal{A}(M)$ and $\widehat{\mathcal{A}}(M)$ have infinitely many elements if and only if M has *double trouble* (i.e. there is a thickened torus component W of the characteristic submanifold of M such that $W \cap \partial M$ has at least 3 components), see Canary-McCullough [**22**]. In particular, the closure of $MP(\pi_1(M))$ can have infinitely many components.

5. Self-bumping

More recently, it has been discovered that individual components of $MP(\pi_1(M))$ may self-bump. A component B of $MP(\pi_1(M))$ is said to *self-bump* if there exists a point $\rho \in \partial B$ such that if V is any sufficiently small neighborhood of ρ, then $B \cap V$ is disconnected. McMullen [**45**] used Anderson and Canary's construction and the theory of complex projective structures on surfaces to show that quasifuchsian space self-bumps.

THEOREM 5.1 (McMullen [**45**]). *If S is a closed surface and $M = S \times I$, then $MP(\pi_1(M))$ self-bumps.*

In a remarkable breakthrough Bromberg and Holt [**17**] proved that if M contains a primitive essential annulus[2], then every component of $MP(\pi_1(M))$ self-bumps. They conjecture that if M contains no primitive essential annuli, then no component of $MP(\pi_1(M))$ self-bumps.

THEOREM 5.2 (Bromberg-Holt [**17**]). *Let M be a compact hyperbolizable 3-manifold. If M contains a primitive essential annulus, then every component B of $MP(\pi_1(M))$ self-bumps.*

Notice that Bromberg and Holt's result applies even if M has compressible boundary. Moreover, it implies that $AH(\pi_1(M))$ is not a manifold if M contains a primitive essential annulus.

[2] A properly embedded annulus A is the image of an embedding of a closed annulus into M such that $A \cap \partial M = \partial A$. An annulus A is primitive and essential if $\pi_1(A)$ maps onto a maximal infinite cyclic subgroup of $\pi_1(M)$ and A is not properly homotopic into the boundary of M.

COROLLARY 5.3 (Bromberg-Holt [**17**]). *Let M be a compact hyperbolizable 3-manifold. If M contains a primitive essential annulus, then $AH(\pi_1(M))$ is not a manifold.*

Of course, one expects that $AH(\pi_1(M))$ is a rather exotic object, but the bumping results only indicated that it is not a manifold. There are some very intriguing pictures of slices of the space of projective structures on a surface produced by Komori, Sugawa, Wada and Yamashita which may be viewed at:

http://www.kusm.kyoto-u.ac.jp/complex/Bers/

These pictures of this related space give some idea of the "fractal nature" of $AH(\pi_1(M))$. See also the related work on the space of projective structures by Komori-Sugawa [**37**], Miyachi [**50**], Ito [**31**] and Bromberg-Holt [**18**].

6. Bers-Sullivan-Thurston Density Conjecture

Bers [**6**] conjectured that every hyperbolic manifold which "belongs" in the boundary of a Bers slice actually lies in the boundary of a Bers slice. Recall that if S is a closed surface and $M = S \times I$, then $MP(\pi_1(M)) = QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S)$. A *Bers slice* is a subset of the form $\{\sigma\} \times \mathcal{T}(S)$ (or $\mathcal{T}(S) \times \{\sigma\}$) for some fixed $\sigma \in \mathcal{T}(S)$. An element of $AH(\pi_1(M))$ "belongs" in the boundary of a Bers Slice if $\partial_c N_\rho$ contains exactly one surface homeomorphic to S. It "belongs" in the boundary of $\{\sigma\} \times \mathcal{T}(S)$ or $\mathcal{T}(S) \times \{\sigma\}$ if the conformal structure on that surface is equivalent to σ.

In a stunning breakthrough, Bromberg [**16**] used the cone manifold techniques developed by Craig Hodgson and Steve Kerckhoff (see [**28**]) to prove Bers' original conjecture for hyperbolic 3-manifolds without cusps. (A hyperbolic 3-manifold N_ρ is said to be *without cusps* if $\rho(\pi_1(M))$ contains no parabolic elements.)

Sullivan [**58**] and Thurston [**60**] generalized Bers' original density conjecture. They conjecture that every hyperbolic 3-manifold with finitely generated fundamental group is a limit of geometrically finite hyperbolic 3-manifolds.

Bers-Sullivan-Thurston Density Conjecture: *$AH(\pi_1(M))$ is the closure of $MP(\pi_1(M))$.*

Brock and Bromberg [**12**] strengthened Bromberg's techniques to prove that if M has incompressible boundary, $\rho \in AH(\pi_1(M))$ and N_ρ is without cusps, then ρ lies in the closure of $MP(\pi_1(M))$. For a survey article on this very important work see Brock-Bromberg [**13**].

7. Thurston's Ending Lamination Conjecture

Thurston's Ending Lamination Conjecture (see [**60**]) provides a conjectural classification of hyperbolic 3-manifolds with finitely generated fundamental group.

Thurston's Ending Lamination Conjecture: *If M is a compact hyperbolizable 3-manifold, then a hyperbolic 3-manifold in $AH(\pi_1(M))$ is determined by its (marked) homeomorphism type and its ending invariants (which encode the asymptotic geometry of its ends.)*

We will not explicitly define ending invariants here. We recommend that the reader see Minsky's survey article [**46**]. In March 2002, Minsky announced the solution of Thurston's Ending Lamination Conjecture in the case that M has incompressible boundary.

THEOREM 7.1 (Minsky [**48**], Brock-Canary-Minsky [**15**]). *If M is a compact hyperbolizable 3-manifold with incompressible boundary, then Thurston's Ending Lamination Conjecture holds for $AH(\pi_1(M))$.*

An excellent survey of this result is given by Minsky in [**49**]. In outline, the proof uses the ending invariants to construct a model for the manifold which is then proven to be bilipschitz to the actual manifold. The key tools in the construction of the model manifold are provided by Masur and Minsky's analysis of the curve complex of a surface [**41, 42**]. One then applies rigidity results for quasiconformal maps, e.g. Sullivan's rigidity theorem [**57**], to complete the result.

One nearly immediate consequence of this result and results in the literature (for example, Ohshika [**52**]) is a full proof of the Bers-Sullivan-Thurston Density Conjecture for manifolds with incompressible boundary.

COROLLARY 7.2. *If M is a compact hyperbolizable 3-manifold with incompressible boundary, then $AH(\pi_1(M))$ is the closure of $MP(\pi_1(M))$.*

Since the work of Anderson, Canary and McCullough [**4**] gave an enumeration of the components of the closure of $MP(\pi_1(M))$ we immediately obtain an enumeration of the components of $AH(\pi_1(M))$. In particular, if M is allowed to have a toroidal boundary component and has double trouble, then $AH(\pi_1(M))$ has infinitely many components.

COROLLARY 7.3. *If M is a compact hyperbolizable 3-manifold with incompressible boundary, the components of $AH(\pi_1(M))$ are in one-to-one correspondence with $\widehat{\mathcal{A}}(M)$.*

Another corollary of our result is that freely indecomposable torsion-free Kleinian groups which are topologically conjugate are also quasiconformally conjugate. More formally,

COROLLARY 7.4. *If M is a compact hyperbolizable 3-manifold with incompressible boundary, $\rho_1, \rho_2 \in AH(\pi_1(M))$ and there exists a homeomorphism $\phi : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $\rho_1 = \phi \circ \rho_2 \circ \phi^{-1}$, then there exists a quasiconformal homeomorphism $\psi : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $\rho_1 = \psi \circ \rho_2 \circ \psi^{-1}$.*

The existence of a quasiconformal homeomorphism conjugating ρ_1 to ρ_2 is equivalent to the existence of a bilipschitz homeomorphism $h : N_{\rho_1} \to N_{\rho_2}$ such that $h_* : \pi_1(N_{\rho_1}) \to \pi_1(N_{\rho_2})$ is conjugate to the identification given by $\rho_2 \circ \rho_1^{-1}$.

It should be pointed out that Thurston's Ending Lamination Conjecture does not provide a conjectural parameterization of $AH(\pi_1(M))$ as the data in the classification does not vary continuously. We observed in section 4 that the topological type does not vary continuously and it is also the case that the ending invariants do not vary continuously, see Brock [**11**] and Minsky [**47**]. This leaves us with the following wide-open question.

Question: Is there a "nice" parameterization of $AH(\pi_1(M))$?

8. Density of cusps

The Bers-Sullivan-Thurston Density Conjecture predicts that geometrically finite hyperbolic 3-manifolds are dense in $AH(\pi_1(M))$. Thurston's Ending Lamination Conjecture further suggests that geometrically finite hyperbolic manifolds are also dense in the boundary of $MP(\pi_1(M))$. In fact, it is natural to think of geometrically finite hyperbolic 3-manifolds with cusps as "rational points" in the boundary of MP. This analogy is especially evocative in the case of punctured torus groups. Let T be the punctured torus. The space $AH(\pi_1(T), \pi_1(\partial T))$ of punctured torus groups is the set of (conjugacy classes of) discrete faithful representations $\rho : \pi_1(T) \to \mathbf{PSL}_2(\mathbf{C})$ such that every non-trivial element of $\rho(\pi_1(\partial T))$ is parabolic. The interior of $AH(\pi_1(T), \pi_1(\partial T))$ is

$$QF(T) \cong \mathcal{T}(T) \times \mathcal{T}(T) = \mathbf{H}^2 \times \mathbf{H}^2$$

which is the space of quasifuchsian punctured tori. We identify $\partial\mathbf{H}^2$ with $\mathbf{R} \cup \{\infty\}$. As a precursor to the full proof of Thurston's Ending Lamination conjecture for 3-manifolds with incompressible boundary, Minsky proved:

Theorem 8.1 (Minsky [**47**]). *$AH(\pi_1(T), \pi_1(\partial T))$ is identified with $\overline{\mathbf{H}^2} \times \overline{\mathbf{H}^2} - \Delta$ where Δ consists of the diagonal elements in $\partial\mathbf{H}^2 \times \partial\mathbf{H}^2$.*

A hyperbolic manifold in $AH(\pi_1(T), \pi_1(\partial T))$ is geometrically finite if and only if it is identified with a point in $\overline{\mathbf{H}^2} \times \overline{\mathbf{H}^2}$ which has both coordinates lying in either $\mathbf{H}^2$ or $\mathbf{Q} \cup \infty$.

The first proof that geometrically finite groups are dense in a boundary is due to McMullen [**44**] and it takes place in the setting of a Bers slice.

Theorem 8.2 (McMullen [**44**]). *Geometrically finite manifolds are dense in the boundary of a Bers Slice. Moreover, "maximal cusps" are dense in the boundary of a Bers slice.*

In this restricted context, a "maximal cusp" in the boundary of a Bers slice in $MP(\pi_1(S \times I))$ is a geometrically finite representation $\rho \in AH(\pi_1(S \times I))$ whose conformal boundary $\partial_c N_\rho$ has one component homeomorphic to S and all other components are thrice-punctured spheres.

In general, N_ρ is a *maximal cusp* if its conformal extension $\widehat{N}_\rho$ is homeomorphic to $R - P$ where R is a compact 3-manifold and P is a maximal collection of disjoint, incompressible, non-parallel annuli and tori in ∂R. In particular, maximal cusps are geometrically finite. McMullen also established that maximal cusps are dense in the boundary of Schottky space, although he never wrote up this result. Recall that Schottky space of genus k is $MP(\pi_1(H_k))$ where H_k is the handlebody of genus k.

Theorem 8.3 (McMullen). *Maximal cusps are dense in the boundary of Schottky space of genus $k \geq 2$.*

Canary, Culler, Hersonsky and Shalen generalized McMullen's techniques to show that maximal cusps are dense in the boundary of any component B of $MP(\pi_1(M))$ such that the associated (marked) manifold $\Theta(B)$ has connected boundary.

Theorem 8.4 (Canary-Culler-Hersonsky-Shalen [**20**]). *Let M be a compact hyperbolizable 3-manifold with no toroidal boundary components. If $\rho \in \partial MP(\pi_1(M))$ and its domain of discontinuity $\Omega(\rho)$ is empty, then ρ may be approximated by maximal cusps. Moreover, if B is a component of $MP(\pi_1(M))$ and $\Theta(B)$ has connected boundary, then maximal cusps are dense in ∂B.*

This line of research was completed by Canary and Hersonsky, who proved that geometrically finite hyperbolic 3-manifolds are always dense in the boundary of $MP(\pi_1(M))$. Again, their work makes central use of the machinery developed by McMullen [**44**].

Theorem 8.5 (Canary-Hersonsky [**21**]). *Let M be a compact hyperbolizable 3-manifold with no toroidal boundary components. Geometrically finite hyperbolic 3-manifolds are dense in the boundary of $MP(\pi_1(M))$.*

More generally, if $N = \mathbf{H}^3/\Gamma$ is a geometrically finite hyperbolic manifold, then geometrically finite hyperbolic manifolds are dense in the boundary $\partial QC(\Gamma)$ of its quasiconformal deformation space. In other language, if (M, P) is a pared 3-manifold, then geometrically finite hyperbolic 3-manifolds are dense in the boundary of $MP(\pi_1(M), \pi_1(P))$, the space of geometrically finite hyperbolic 3-manifolds whose relative compact cores are homotopy equivalent to (M, P).

Historical note: McMullen's proof that maximal cusps are dense in the boundary of Schottky space was motivated by a question of Culler and Shalen. Culler, Shalen and their co-authors used McMullen's theorem about the boundary of Schottky space as part of an extensive program to study volumes of hyperbolic 3-manifolds. In particular, it was used to prove a quantitative version of the Margulis lemma for free Kleinian groups.

Theorem 8.6 (Anderson, Canary, Culler, Shalen [**3**]). *Let Γ be a Kleinian group contained in the closure of Schottky space of genus k and freely generated by elements $\{\gamma_1, \ldots, \gamma_k\}$. If $z \in \mathbf{H}^3$, then*

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d(z,\gamma_i(z))}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \ldots, k\}$ such that $d(z, \gamma_i(z)) \geq \log(2k - 1)$.

Here are some examples of the applications of this Margulis lemma to volumes of hyperbolic 3-manifolds.

Theorem 8.7 (Culler-Hersonsky-Shalen [**24**]). *The smallest volume orientable hyperbolic 3-manifold has first Betti number at most 2. In particular, if N is a hyperbolic 3-manifold and* $\mathrm{rank}(H_1(N))$ *is at least 3, then*

$$vol(N) \geq .94689.$$

Theorem 8.8 (Culler-Shalen [**26**]). *If N is an orientable hyperbolic 3-manifold and* $\mathrm{rank}(H_1(N)) \geq 2$*, then*

$$vol(N) \geq .34$$

Theorem 8.9 (Anderson-Canary-Culler-Shalen [**3**]). *If N is an orientable hyperbolic 3-manifold,* $\mathrm{rank}(H_1(N)) \geq 4$ *and $\pi_1(N)$ does not contain the fundamental group of a closed surface of genus 2, then*

$$vol(N) \geq 3.08$$

Remark: Theorem 8.6 is a generalization of Culler and Shalen's original result (see [**25**]) which applied when $k = 2$. Przeworski [**55**] has improved the lower bound in Theorem 8.7 to 1.105, and Agol [**1**] has improved the lower bound in Theorem 8.8 to .887.

9. Marden's Tameness Conjecture

It seems unlikely that one can attack Bers' Density Conjecture or Thurston's Ending Lamination Conjecture in the compressible boundary setting, without first resolving:

Marden's Tameness Conjecture: *Every hyperbolic 3-manifold with finitely generated fundamental group is topologically tame, i.e. homeomorphic to the interior of a compact 3-manifold.*

Bonahon proved Marden's conjecture for hyperbolic 3-manifolds with freely indecomposable fundamental group. This seminal result underlies almost all subsequent work on these manifolds.

Theorem 9.1 (Bonahon [**9**]). *If M has incompressible boundary and $\rho \in AH(\pi_1(M))$, then N_ρ is topologically tame.*

Marden's Tameness Conjecture is known to imply a variety of conjectures about the geometry and dynamics of hyperbolic 3-manifolds, including Ahlfors' Measure Conjecture.

Theorem 9.2 (Canary [**19**]). *If $\rho \in AH(\pi_1(M))$ and N_ρ is topologically tame, then Ahlfors' Measure Conjecture holds, i.e. either the limit set $\Lambda(\rho)$ has measure zero or the domain of discontinuity $\Omega(\rho)$ is empty. If $\Omega(\rho) = \emptyset$, then $\rho(\pi_1(M))$ acts ergodically on $\partial\mathbf{H}^3$.*

Thurston [**59**] originally proved Marden's Tameness Conjecture for many hyperbolic 3-manifolds with freely indecomposable fundamental group which are limits of geometrically finite hyperbolic 3-manifolds. There have been a series of such results in the freely decomposable case, see, for example, Ohshika [**53**], Canary-Minsky [**23**], Evans [**27**] and Kleineidam-Souto [**35**]. The best current result is due to Brock, Bromberg, Evans and Souto and had its genesis at the Ahlfors-Bers Colloquium. (We recall that M is a compression body if it has a boundary component S such that the inclusion map induces a surjection of $\pi_1(S)$ onto $\pi_1(M)$). In particular, this implies that $\pi_1(M)$ is a free product of surface groups and cyclic groups.)

Theorem 9.3 (Brock-Bromberg-Evans-Souto [**14**]). *If ρ lies in the boundary of the interior of $AH(\pi_1(M))$ and either M is not homotopy equivalent to a compression body or $\Omega(\rho)$ is non-empty, then N_ρ is topologically tame.*

So we are now roughly in the same situation in the compressible boundary setting as we were in the incompressible setting, before Bonahon's breakthrough. I would also like to draw attention to a beautiful paper of Souto [**56**] which develops new criteria which imply topological tameness.

One hopes that Minsky's program to prove the Ending Lamination Conjecture can be implemented in the setting of topologically tame hyperbolic 3-manifolds[3].

[3] In August 2003, Brock, Canary and Minsky announced the solution of Thurston's Ending Lamination Conjecture for topologically tame hyperbolic 3-manifolds

A proof of Marden's Tameness Conjecture and the desired generalization would complete Thurston's Ending Lamination Conjecture.

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