Abelian varieties

Bhargav Bhatt

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Chapter 1

Abelian varieties

1.1 Notation

A variety over a field k is a geometrically reduced and separated scheme of finite type. We shall use that varieties are generically smooth. For a map $f : X \to S$ of schemes and an S-scheme T, we write $X_T \to T$ for the base change of f to T. When $T = \mathrm{Spec}(\kappa(s))$ for $s \in S$, we also write X_s instead.

1.2 Group schemes: definitions and basic properties

Fix a base scheme S.

Definition 1.2.1. An S-*group scheme* is an S-scheme G equipped with maps $m : G \times G \to G$, $i : G \to G$, and $e : S \to G$ satisfying the following:

1. m is associative, i.e., the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "{(m,\mathrm{id})}"] \arrow[d, "{(\mathrm{id},m)}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

   commutes.

2. e provides left and right identities for m, i.e., the diagrams

$$\begin{tikzcd} G \arrow[r, "{(e,\mathrm{id})}"] \arrow[dr, equal] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "{(\mathrm{id},e)}"] \arrow[dr, equal] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

   commute.

3. i provides left and right inverses for m, i.e., the diagrams

$$\begin{tikzcd} G \arrow[r, "{(i,\mathrm{id})}"] \arrow[d] & G \times G \arrow[d, "m"] \\ S \arrow[r, "e"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "{(\mathrm{id},i)}"] \arrow[d] & G \times G \arrow[d, "m"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

   commute.

If m is symmetric, then we say that G is a *commutative group scheme*. Given two S-group schemes G, H, a *homomorphism* $G \to H$ of S-group schemes is a map of S-schemes that commutes with m, i, and e.

unctorPtsGrp **Remark 1.2.2** (Functor of points perspective)**.** Recall that there is a fully faithful Yoneda embedding

$$\mathrm{Sch/S}^{op} \to \mathrm{PShv(Sch/S)} \quad \text{via} \quad X \mapsto h_X := \mathrm{Hom}_S(-, X).$$

The elements of $h_X(T) = X(T)$ for an S-scheme T are called the T-valued points of X; when $T = S = \mathrm{Spec}(k)$ for a field k, then $X(T)$ is exactly the k-rational points of X. The representable presheaf h_X is thus often called the *functor of points* of X. Using that the functor $X \mapsto h_X$ preserves finite products, one then checks the following: given an S-scheme G, specifying the structure of an S-group scheme on G is the same as specifying a group structure on the presheaf h_G, i.e., a factorization

$$\mathrm{Sch/S}^{op} \to \mathrm{Groups} \xrightarrow{forget} \mathrm{Sets}$$

of h_G. In practice, it is often much easier to specify the group structure on h_G than to write down explicit formulas for m, i, and e.

Remark 1.2.3. Using the functor of points, one can perform similar operations with group schemes as one performs with groups. For example, if $g \in G(S)$, then we can define the (left) translation by g map $T_g : G \to G$ by simply asking that for any S-scheme T, the induced map $T_g(T) : G(T) \to G(T)$ be left translation by (the image in $G(T)$ of) g. Alternately, one uses the following formula to define T_g

$$G \xrightarrow{(g,\mathrm{id})} G \times G \xrightarrow{m} G.$$

Similarly, one may define a "conjugation by g" map $c_g : G \to G$, (normal) subgroup schemes, etc..

Remark 1.2.4. If $G := \mathrm{Spec}(A) \to S := \mathrm{Spec}(R)$ is a morphism of affine schemes, then specifying an S-group scheme structure on G is equivalent (by definition) to endowing A with the structure of a *Hopf algebra* in the category of R-algebras; the resulting map $m^* : A \to A \otimes_R A$ is often called the *comultiplication*.

We give some fundamental examples.

Example 1.2.5 (The additive group)**.** Assume $S = \mathrm{Spec}(R)$. The *additive group* is the S-group scheme given by $\mathbf{G}_{a,S} := \mathrm{Spec}(R[t])$; we often write $\mathbf{G}_a$ instead if the base S is clear. The group scheme structure is determined by the formulas

$$m^*(t) = t \otimes t \quad \text{and} \quad i^*(t) = -t \quad \text{and} \quad e^*(t) = 0.$$

It is an exercise to see that this satisfies the axioms for a group scheme. To understand what this means, note that by Remark 1.2.2, this endows $\mathbf{G}_{a,S}(T)$ with a group structure functorially in T. But $\mathbf{G}_{a,S}(T) = \mathcal{O}(T)$. Unwinding definitions, one checks that the group structure on $\mathcal{O}(T)$ defined this way is the obvious one.

ex:MultGroup **Example 1.2.6** (The multiplicative group)**.** Assume $S = \mathrm{Spec}(R)$. The *multiplicative group* is the S-group scheme given by $\mathbf{G}_{m,S} = \mathrm{Spec}(R[t, t^{-1}])$; the additional structure is determined by the formulas

$$m^*(t) = t \otimes 1 + 1 \otimes t \quad \text{and} \quad i^*(t) = t^{-1} \quad \text{and} \quad e^*(t) = 1.$$

It is an exercise to check that this satisfies the axioms for a group scheme. Again, these formulas translate to something very natural: for any S-scheme T, we have $\mathbf{G}_{m,S}(T) = \mathcal{O}(T)^*$, and this has an obvious group structure functorially in T, which corresponds to the one determined by the preceding formulas.

Example 1.2.7 (General linear group)**.** Assume $S = \mathrm{Spec}(R)$. Let $G := \mathrm{GL}_{n,S}$, i.e.,

$$G = \mathrm{Spec}(R[x_{ij}]_{1 \le i,j \le n}[\frac{1}{\det(x_{ij})}]).$$

We then have $G(T) = \mathrm{GL}_n(\mathcal{O}(T))$ for any S-scheme T. This has an obvious group structure functorially in T given by multiplication of matrices; the group scheme $\mathrm{GL}_{n,S}$ obtained this way is called the *general linear group*. We encourage the reader to make the formula for m^* explicit for $n = 2$.

More generally, if V is any vector bundle on S, one can define (by glueing) an S-affine S-group scheme $\mathrm{GL}(V)$ characterized as follows: for any $f : T \to S$, we have $\mathrm{GL}(V)(T) = \mathrm{Aut}_T(f^*V)$. Taking determinants of matrices locally defines a homomorphism $\mathrm{GL}(V) \to \mathbf{G}_{m,S}$ of group schemes.

Example 1.2.8 (The constant group)**.** For any set X, write $\underline{X} := \sqcup_{x \in X} X$ for the constant S-scheme attached to S. The functor $X \mapsto \underline{X}$ commutes with finite products. Consequently, if G is an ordinary group, then $\underline{G}$ is naturally an S-group scheme. We encourage the reader to describe the functor of points of $\underline{G}$.

The preceding examples were all smooth. This need not always be true:

Example 1.2.9 (Roots of unity)**.** Let $S = \mathrm{Spec}(R)$, and fix an integer n. Let $G = \mu_n := \mathrm{Spec}(R[x]/(x^n - 1))$. Then for any R-scheme T, we have

$$\mu_n(T) = \{\zeta \in \mathcal{O}(T)^* \mid \zeta^n = 1\}.$$

The right hand side has an evident group structure functorially in T, so μ_n is naturally a group scheme. In fact, the natural map $\mu_n \to \mathbf{G}_m$ is a closed immersion of group schemes and is compatible with the group law, so μ_n can be viewed as a closed subgroup scheme in $\mathbf{G}_m$. However, μ_n is not smooth over R if n is not invertible in R. For example, if $R = k$ is a field of characteristic p and $n = p$, then $k[x]/(x^p - 1) = k[x]/((x - 1)^p)$ is a non-reduced ring.

Example 1.2.10 (The group scheme α_p)**.** Let $S = \mathrm{Spec}(R)$, and assume $p = 0$ in R. Let $G = \alpha_p := \mathrm{Spec}(R[x]/(x^p))$. Then, for any R-scheme T, we have natural identifications

$$\alpha_p(T) = \{\epsilon \in \mathcal{O}(T) \mid \epsilon^p = 0\}.$$

As R has characteristic p, the right hand is naturally a group under addition, and thus α_p becomes a group scheme. The natural inclusion $\alpha_p \subset \mathbf{G}_a$ realizes α_p as a closed subgroup scheme of $\mathbf{G}_a$.

Exercise 1.2.11. Let G/S be a group scheme. Using the functor of points, show that $G \to S$ is separated if and only if the identity section $e : S \to G$ is a closed immersion. (Hint: show that the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times G \arrow[d, "m \circ (\mathrm{id}, i)"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

is Cartesian.) Using this criterion, describe a nonseparated group scheme.

Exercise 1.2.12. Let $f : G \to H$ be a morphism of S-group schemes. Assume $H \to S$ is separated. Define a kernel for f using the functor of points, and show that it is also (representable by) a group scheme. Compute explicitly the kernel of the following homomorphisms:

1. The map $\mathbf{G}_{m,S} \to \mathbf{G}_{m,S}$ determined by $t \mapsto t^n$ on the functor of points.

2. The map $\mathbf{G}_{a,S} \to \mathbf{G}_{a,S}$ given by $x \mapsto x^p$ on the functor of points (here S has characteristic p).

3. The map $\mathbf{G}_{a,S} \to \mathbf{G}_{a,S}$ given by $x \mapsto x^p - x$ on the functor of points (here S has characteristic p).

upSchReduced **Exercise 1.2.13.** Let G be a group scheme over a field k that is locally of finite type. If G is geometrically reduced, then show that G is smooth. Give an example of reduced group schemes that are not geometrically reduced. (Hint: for an imperfect field k, consider the kernel of $\mathbf{G}_{a,k}^2 \to \mathbf{G}_{a,k}$ of the form $(x, y) \mapsto x^p + \alpha y^p$ for $y \in k$ general.)

1.3 Abelian varieties: definitions and basic properties

def:AbSch **Definition 1.3.1.** An *abelian scheme over* S or an *abelian* S*-scheme* is a group scheme A/S such that the structure map $A \to S$ is proper and smooth with geometrically connected fibers. When $S = \mathrm{Spec}(k)$ is a field, we say that A is an *abelian variety* over k.

One thinks of an abelian S-scheme $A \to S$ as a flat family of abelian varieties $A_s \to \mathrm{Spec}(\kappa(s))$ parametrized by $s \in S$.

VarAlternate **Remark 1.3.2.** Smooth and geometrically connected varieties over a field are geometrically integral, so an abelian scheme has geometrically integral fibers. Moreover, as geometrically reduced group schemes are smooth by Example 1.2.13, the smoothness assumption on the structure map $A \to S$ in Definition 1.3.1 can be weakened to flatness if we require the fibres to also be geometrically reduced.

Remark 1.3.3. This definition is compatible with base change, i.e., if $T \to S$ is a map of schemes and A is an abelian scheme over S, then $A_T := A \times_S T$ is naturally an abelian scheme over T: the relevant structures (the group law, the inversion map, the identity section) over T arise via base change from the corresponding structures over S.

Example 1.3.4. When $S = \mathrm{Spec}(k)$ is a field, then an abelian variety of dimension 1 over k is the same thing as an elliptic curve over k. Taking products, we obtain examples of higher dimensional varieties.

Remark 1.3.5. (Abelian varieties over $\mathbf{C}$) When $k = \mathbf{C}$, an abelian variety A/k has an analytification A^{an} which is a compact complex manifold equipped with a group structure in the category of compact complex manifolds[1], i.e., A^{an} is a compact complex Lie group. In fact, the compactness forces such groups to be commutative (thus justifying the name *abelian* varieties):

Proof sketch. For any $t \in A^{an}$, conjugating by t is a group automorphism c_t of A^{an}, and the assocation $t \mapsto c_t$ induces an action of A^{an} on its Lie algebra $V := T_e(A^{an})$. This action is classified by a holomorphic map $A^{an} \to \mathrm{End}(V)$; but A^{an} is compact and $\mathrm{End}(V)$ is a complex vector space, so the map must be constant by the maximal principle in complex analysis[2]. Thus, $c_t : A^{an} \to A^{an}$ acts as the identity on the tangent space. Using the exponential for Lie groups, it follows that c_t acts as the identity in an open neighbourhood of the origin in A^{an}. Any such open neighbourhood generates A^{an} as a Lie group as A^{an} is connected, so c_t must be the constant map with value $c_t(e) = e \in A^{an}$. In other words, the group structure on A^{an} is commutative. Using GAGA, it follows the same must be true for A too. $\square$

Thus, A^{an} may be regarded as a commutative compact complex Lie group. It is a basic fact that all such groups are of the form $\mathbf{C}^g/\Lambda$, where $\Lambda \simeq \mathbf{Z}^{2g} \subset \mathbf{C}^g$ is a lattice; most such tori are not algebraic, and the the lattices corresponding to the algebraic ones were classified by Riemann. A consequence of this theory is that, as topological groups, we can describe A^{an} completely: there is a homeomorphism $A^{an} \simeq (S^1)^{2g}$. In particular, the n-torsion $A^{an}[n]$ is isomorphic to $(\mathbf{Z}/n)^{2g}$.

We shall prove analogues of the preceding analytic facts purely algebraically next.

1.3.1 Rigidity properties

The following rigidity result (due to Weil) is crucial in setting up the theory of abelian varieties, and is also useful elsewhere.

[1] This follows from the fact that the analytification functor $X \mapsto X^{an}$ commutes with fibre products, and thus the group law on A induces one on A^{an}.

[2] This principle states that any holomorphic function on a domain D in $\mathbf{C}^n$ that attains its maximum at a point of D must be constant. We apply it to suitable open subsets of A^{an} that give charts: the compactness of A^{an} forces each co-ordinate of $\mathrm{End}(V)$ to define a function on A^{an} that must attain a maximum.

prop:Rigid **Proposition 1.3.6** (Rigidity)**.** *Let $f : X \to S$ be a proper flat morphism with $\kappa(s) \simeq H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$. Then $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism. In particular, if $T \to S$ is an affine morphism, then any map $X \to T$ is constant, i.e., it factors over f.*

The condition on the cohomology of the fibers is ensured if the fibers are geometrically integral (or just geometrically reduced and geometrically connected).

Proof. We only give the argument when S is noetherian. The general case can be deduced from this one using "noetherian approximation" techniques.

First, assuming we have shown the first part, the second part follows immediately from the fact that any S-map $X \to T$ factors uniquely over the canonical "affinization" map $X \to \underline{\mathrm{Spec}}(f_*\mathcal{O}_X)$.

The assertion $\mathcal{O}_S \simeq f_*\mathcal{O}_X$ is part of the "cohomology and base change" package, but we give a direct argument. We may assume $S = \mathrm{Spec}(R)$ is the spectrum of a noetherian local ring $(R, \mathfrak{m})$. By the formal functions theorem and faithful flatness of completions, it is enough to show that $R/\mathfrak{m}^n \to H^0(X, \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X)$ is an isomorphism. We shall show more generally that for any finite length R-module M, the natural map $\eta_M : M \to F(M) := H^0(X, f^*M)$; we will induct on the length $\ell(M)$ of M.

If $\ell(M) = 1$, then $M \simeq R/\mathfrak{m}$, so the the claim is true by assumption on the fibres of f. If $\ell(M) > 1$, then we can find a short exact sequence

$$0 \to K \to M \to Q \to 0$$

with $\ell(K)$ and $\ell(Q)$ strictly smaller than $\ell(M)$. As $F(-)$ is left-exact by the flatness of f, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\eta_K"] & M \arrow[r] \arrow[d, "\eta_M"] & Q \arrow[r] \arrow[d, "\eta_Q"] & 0 \\
0 \arrow[r] & F(K) \arrow[r] & F(M) \arrow[r] & F(Q) &
\end{tikzcd}$$

with exact rows. Both η_K and η_Q are isomorphisms by induction. A diagram chase then implies the same for the middle one. ☐

gidClassical **Corollary 1.3.7.** *Say S is a noetherian scheme. Let $f : X \to S$ be a proper flat morphism with $\kappa(s) \simeq H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$, and let $g : Y \to S$ be a separated morphism. Let $\pi : X \to Y$ be an S-morphism such that $\pi_s : X_s \to Y_s$ is a constant map for some $s \in S$, i.e., it factors over the structure map $X_s \to \mathrm{Spec}(\kappa(s))$. Then π is constant over the connected component of $s \in S$.*

In practice, this will be applied with $Y = Y_0 \times S$ being a constant family. In this case, the lemma simply says that one cannot collapse a single fibre of a flat family of integral proper varieties without collapsing all nearby fibers too.

Proof. We may assume S is connected and affine.

We first show that if $\pi|_{X_V}$ is constant over some affine open neighbourhood $V \subset S$ of s. Let $U \subset Y$ be an affine open subset containing the point $\pi_s(X_s)$. Then $\pi^{-1}(U) \subset X$ is an open subset containing X_s. By properness of f, there exists some affine open neighbourhood $s \in V \subset S$ such that $X_V := f^{-1}(V) \subset \pi^{-1}(U)$. Applying Proposition 1.3.6 to the the first square diagram

$$\begin{tikzcd}
X_{V_t} \arrow[r, "\pi|_{X_V}"] \arrow[d] & U_V \arrow[r] \arrow[d] & U \subset Y \arrow[d] \\
V \arrow[r, equal] & V \arrow[r] & S
\end{tikzcd}$$

shows that $\pi|_{X_V}$ is constant; note that U_V is affine as U, S, and V are affine.

By repeating the previous argument, it is enough to check π_t is constant for all $t \in S$. Let $W \subset S$ be the set of all points $t \in S$ with this property. We know that $s \in W$. The argument in the previous paragraph shows that W is open. It is thus enough to show that W is closed under specialization. Detecting specializations[3] using valuation rings, we may then reduce to the case where $S = \mathrm{Spec}(V)$ for a discrete valuation ring V with uniformizer $t \in V$, and $s \in S$ is the generic point. In this case, X is reduced: $\mathcal{O}_X$ embeds into $\mathcal{O}_X[\frac{1}{t}]$ by flatness of X/V, and the latter is reduced by assumption on the generic fibre. Moreover, the subset $\pi(X) \subset Y$ is closed as $X \to S$ is proper and $Y \to S$ is separated; replacing Y with $\pi(X)$ (endowed with its reduced structure), we may assume that π is surjective, and thus Y is also proper over S. Now, by assumption, $\pi(X_s) \in Y_s$ gives a point in the generic fibre of $Y \to S$. The closure of this point gives a section $S \to Y$ of $Y \to S$ by properness. We must check that the induced map $X \to S \to Y$ coincides with π. But this is true over $X_s \subset X$ by construction, and thus must be true everywhere by the scheme-theoretic density of $X_s \subset X$. □

r:AbVarMapHom **Corollary 1.3.8.** *Let A and B be abelian schemes over some noetherian base scheme S. Let $f : A \to B$ be an S-map. Then $f = T_y \circ h$ for $y = f(e_A) \in B(S)$ with $h : A \to B$ being a homomorphism.*

In other words, f is the composition of a homomorphism $A \to B$ with a translation on B.

Proof. By replacing f with $T_{i(y)} \circ f$, we may assume $f(e_A) = e_B$, i.e., f preserves the identity elements. We must check f is a homomorphism. In other words, we want to check that two maps $A \times A \to B$ given by $f \circ m_A$ and $m_B \circ (f, f)$ coincide; equivalently, we must check that the map

$$g : A \times A \to B \quad g = f \circ m_A \circ i_B \circ m_B \circ (f, f)$$

is constant with value e_B. For this, we may assume S is connected. We now apply Corollary 1.3.7 with $X \to S$ being $\mathrm{pr}_1 : A \times A \to A$, $Y \to S$ being $\mathrm{pr}_2 : B \times A \to A$ (i.e., $Y_0 = B$ and $Y = Y_0 \times S$), $s \in S$ being any point contained in $S \simeq e_A(S) \in A$, and $\pi = g$. The hypothesis that π_s is constant is verified in our case precisely because f preserves the identity element. The conclusion is that g is constant over the nonempty connected component of A. But Proposition 1.3.6 implies that A is connected since S is connected, so g is constant. □

Corollary 1.3.9. *Let $f : A \to S$ be a proper smooth morphism with a section $e : S \to A$. Then there exists at most one structure of an abelian S-scheme on A having e as the identity section.*

Proof. The identity map $A \to A$ carries e to itself. If we endow the source and target with arbitrary abelian S-scheme structures having e as the identity, then the identity map is a homomorphism by Corollary 1.3.8. This formally implies that all such abelian S-scheme structures on A coincide. □

Proposition 1.3.10. *Let A be an abelian scheme over a base S. Then A is commutative.*

Proof via rigidity. We may assume S is a connected noetherian scheme by standard arguments. The inversion map $i : A \to A$ preserves identity elements, so Corollary 1.3.8 implies i is a homomorphism, which implies that A is commutative (by the same argument as for ordinary groups, applied via the functor of points). □

Proof via nonexistence of global functions. We give a second proof that is closer in spirt to the complex analytic proof. For simplicity, we restrict to the case where $S = \mathrm{Spec}(k)$ is a field.

Let $m : A \times A \to A$ be the multiplication, and let $s : A \times A \to A \times A$ be the involution that switches the two factors. We must show that $m = m \circ s$. An equality of morphisms between two k-schemes can be checked after base change to the algebraic closure, so we may assume k is algebraically closed.

As k is algebraically closed and both $A \times A$ and A are varieties, it is enough to check the claim on k-valued points by the Nullstellensatz. Thus, fix some $t \in A(k)$. We must show that conjugation by t acts trivially on A. As A is integral and c_t fixes $e \in A(k)$, it is enough to show that c_t acts trivially on the local ring $\mathcal{O}_{A,e}$.

[3]We are using the following: if S is a noetherian scheme, and $x \rightsquigarrow y$ is a specialization of points of S, then there exists a map $f : \mathrm{Spec}(V) \to S$ with V being a discrete valuation ring such that f carries the generic point to x and the closed point to y. To prove this, replacing S with $\overline{\{x\}}$, one may assume S is an integral scheme with x being the generic point. Blowing up the closed subscheme $\overline{\{y\}} \subset S$ and using the discrete valuation defined by the exceptional divisor on the blowup provides the desired V.

If $\mathfrak{m} \subset \mathcal{O}_{A,e}$ denotes the maximal ideal, then $\mathcal{O}_{A,e}$ injects into its $\mathfrak{m}$-adic completion, so it is enough to show that c_t acts trivially on $V_n := \mathcal{O}_{A,e}/\mathfrak{m}^n$ for each $n \geq 1$.

Fix an integer $n \geq 0$. Then the conjugation action can be viewed as a map $c(k) : A(k) \to \mathrm{Aut}(V_n) := \mathrm{GL}(V_n)(k)$. Assume for the moment that we have lifted this construction to a morphism $c : A \to \mathrm{GL}(V_n)$ of k-group schemes. We can then conclude by observing that $\mathrm{GL}(V_n)$ is affine, while A is a proper variety (and hence has no non-constant global functions).

It remains to define the conjugation action c. This is an exercise in thinking about the functor of points, and follows from the fact that the formation of V_n commutes with base change. More precisely, for any k-algebra R, write $A_R = A \otimes_k R$ for the abelian scheme over R defined by A via base change. The identity section $e_R : \mathrm{Spec}(R) \to A_R$ is defined via base change from e. Let $\mathcal{I}_{e_R} \subset \mathcal{O}_{A_R}$ denote the ideal sheaf of e_R. Then $\mathcal{I}_{e_R}^n \subset \mathcal{O}_{A_R}$ defines the n-fold infinitesimal neighbourhood of the closed subscheme $e_R(\mathrm{Spec}(R)) \subset \mathcal{A}_R$; this subscheme is finite over R, and its co-ordinate ring (viewed as an R-algebra) identifies naturally with $V_n \otimes_k R := \mathcal{O}_{A,e}/\mathfrak{m}^n \otimes_k R$. It follows that the same recipe used to define $c(k)$ enables one to define $c(R)$ for any k-algebra R in a manner that is compatible as R changes, and thus we obtain the promised map c. □

1.3.2 Differential properties

GroupSchOmega

Proposition 1.3.11. *Let $f : G \to S$ be a group scheme with identity section $e : S \to G$. Then there is a canonical isomorphism $\Omega^1_{G/S} \simeq f^* e^* \Omega^1_{G/S}$. In particular, if S is the spectrum of a field, then $\Omega^1_{G/S}$ is free.*

Proof. For any S-scheme T and $g \in G_T(T)$, the translation by g map $T_g : G_T \to G_T$ is an isomorphism, and thus provides a canonical isomorphism $T_g^* \Omega^1_{G_T/T} \simeq \Omega^1_{G_T/T}$. Applying this to $T = G$ and $g \in (G \times G)(G)$ being the diagonal map Δ (and unwinding definitions), one obtains a canonical isomorphism

$$m^* \Omega^1_{G/S} \simeq \mathrm{pr}_2^* \Omega^1_{G/S}$$

on $G \times G$. Alternately, this also follows by identifying both sides with $\Omega^1_{\mathrm{pr}_1}$ via the commutative diagram

$$\begin{tikzcd}
G \times G \arrow[dr, "(\mathrm{id},m)", "\simeq"'] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & \\
& G \times G \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & G \arrow[d, "f"] \\
& G \arrow[r, "f"] & S
\end{tikzcd}$$

using that both squares are cartesian. Restricting to $G \xrightarrow{(\mathrm{id},e)} G \times G$, this gives the desired canonical isomorphism

$$\Omega^1_{G/S} \simeq f^* e^* \Omega^1_{G/S}$$

of sheaves on G. □

Remark 1.3.12. Proposition 1.3.11 does *not* imply group schemes G/k over a field k are smooth as the rank of $\Omega^1_{G/k}$ might exceed the dimension of G. For example, if $G = \mu_p$ and k has characteristic p, then $\Omega^1_{G/k}$ is free of rank 1, while G has dimension 0.

Remark 1.3.13. Assume S is affine. The isomorphism from Proposition 1.3.11 yields (by adjunction) a map $e^* \Omega^1_{G/S} \to H^0(G, \Omega^1_{G/S})$. This map is injective with left-inverse provided by restricting to the identity section. Moreover, the construction shows that its image is exactly the "translation invariant" 1-forms, i.e., those $\omega \in H^0(G, \Omega^1_{G/S})$ such that $T_g^* \omega_T = \omega_T \in H^0(G_T, \Omega^1_{G_T/T})$ for any $g \in G(T)$ for any S-scheme T.

Example 1.3.14. Let $G = \mathbf{G}_{m,S}$ with co-ordinate $t \in H^0(G, \mathcal{O}_G)$, as in Example 1.2.6. Then $e^* \Omega^1_{G/S}$ is a free $\mathcal{O}_S$-module of rank 1, identified with I_e/I_e^2, where $I_e \subset \mathcal{O}_G$ is the ideal sheaf of the zero section. The generator $t-1 \in I_e$ thus defines a translation invariant 1-form in $H^0(G, \Omega^1_{G/S})$. Unwinding definitions, one checks that this 1-form is $\frac{dt}{t}$.

bVarRatCurve **Corollary 1.3.15.** *Let A/k be an abelian variety over a field k. Any map $f : \mathbf{P}^1 \to A$ is constant. In particular, abelian varieties are not rational (or even rationally connected).*

Proof. We may assume k is algebraically closed. Assume f is nonconstant. Then its image $C := f(\mathbf{P}^1_k) \subset A$ is a unirational (possibly singular) irreducible curve. Replacing our given $\mathbf{P}^1$ with the normalization of C, we may assume f is birational onto its image. Choose a point $c \in C(k)$ in the smooth locus, so c lifts uniquely to $\mathbf{P}^1(k)$. Then $T_c(\mathbf{P}^1) \to T_c(C)$ is an isomorphism. As $C \subset A$ is a closed subscheme, the induced map $T_c(\mathbf{P}^1) \to T_{f(c)}(A)$ is injective. On the other hand, pullback of forms gives a map $f^* : H^0(A, \Omega^1_{A/k}) \to H^0(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1/k})$. The right hand side is 0, so the map is 0. By Proposition 1.3.11, it follows that $f^*\Omega^1_{A/k} \to \Omega^1_{\mathbf{P}^1/k}$ is the 0 map. Taking fibers at c then contradicts the injectivity established earlier, thus proving the claim. $\square$

hemeRatCurve **Remark 1.3.16.** The conclusion of Corollary 1.3.15 is valid for any commutative k-group scheme A, not merely abelian varieties. Indeed, the proof above only uses the smoothness of A, and not the properness. To reduce to the smooth case, we simply observe that once we reduce to $k = \overline{k}$, any map $\mathbf{P}^1 \to A$ factors uniquely over $A_{red} \subset A$; one then observes that A_{red} is a smooth k-group scheme as $k = \overline{k}$. If we ever need this, we shall gvie more details.

Exercise 1.3.17. Let X be a smooth projective surface over a field k, and let A/k be an abelian variety. Any rational map $X \dashrightarrow A$ is to a morphism $X \to A$. (In fact, the same holds true for X of any dimension, see Moonen's notes.)

em:GrpSchTgt **Lemma 1.3.18.** *Let G/k be a finite type group scheme over a field k. Then there is a canonical isomorphism $T_{(e,e)}(G \times G) \simeq T_e(G) \times T_e(G)$. The multiplication map $m : G \times G \to G$ induces a map $T_{(e,e)}(G \times G) \to T_e(G)$ that coincides with the addition map under the previous isomorphism. A similar statement holds true for G^n for any $n \geq 1$.*

Proof. Recall that for a k-scheme X with $x \in X(k)$, there is a canonical identification[4]

$$T_x(X) \simeq \{f : \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \to X \mid f|_{\mathrm{Spec}(k)} = x\}.$$

It formally follows that if Y is another k-scheme with $y \in Y(k)$, then canonical map gives an isomorphism

$$T_{(x,y)}(X \times Y) \xrightarrow{\simeq} T_x(X) \times T_y(Y),$$

which gives the first part; here one must think through the compatibility of the preceding isomorphism with the k-vector space structure on either side. For the second part, consider the inclusions

$$i_1 : G \xrightarrow{(e,\mathrm{id})} G \times G \quad \text{and} \quad i_2 : G \xrightarrow{(\mathrm{id},e)} G.$$

Under the preceding identification of $T_{(e,e)}(G \times G)$, this produces maps

$$i_{1,*}, i_{2,*} : T_e(G) \to T_e(G) \times T_e(G),$$

[4] To construct this identification, it is enough to do in the affine case. Thus, given a k-algebra R and a maximal ideal $\mathfrak{m}$ with $k = \kappa(\mathfrak{m})$, we must identify the k-linear dual of $\mathfrak{m}/\mathfrak{m}^2$ with the set k-algebra of maps $R \to k[\epsilon]/(\epsilon^2)$ lifting $R \to R/\mathfrak{m} \simeq k$. Both sides are compatible with localization, so we may assume R is local with $\mathfrak{m}$ being the maximal ideal. Now given a map $R \to k[\epsilon]/(\epsilon^2)$ as above, the induced map on cotangent spaces then gives a k-linear functional on $\mathfrak{m}/\mathfrak{m}^2$ (as the cotangent space of the target is canonically trivialized by ϵ). Conversely, given a nonzero functional $\lambda : \mathfrak{m}/\mathfrak{m}^2 \to k$, we have a natural identification $(R/\mathfrak{m}^2)/\ker(\lambda) \simeq k[\epsilon]/(\epsilon^2)$, where ϵ corresponds to $1 \in \mathrm{im}(\lambda) = k$. This yields a map in the other direction, and one checks that it is an inverse to the previous construction. In terms of this description, the k-vector space structure of $T_x(X)$ is described the k^*-action on $k[\epsilon]/(\epsilon^2)$ by scaling ϵ (corresponding to scalar multiplication on the tangent space) and the k-algebra map $k[\epsilon_1]/(\epsilon_1^2) \times_k k[\epsilon_2]/(\epsilon_2^2) \to k[\epsilon]/(\epsilon^2)$ given by $\epsilon_i \mapsto \epsilon$ (corresponding to addition of tangent vectors).

and one checks by composing with the two projections that these coincide with the obvious inclusion of the factors. The map $m_* : T_{(e,e)}(G \times G) \to T_e(G)$ can then be described as

$$m_*(a, b) = m_*((a, 0) + (0, b)) = m_*(i_{1,*}(a) + i_{2,*}(b)) = m_* i_{1,*}(a) + m_* i_{2,*}(b) = a + b,$$

as wanted. For $n \geq 1$, one proceeds inductively. $\square$

tnSurjective **Proposition 1.3.19.** *Let A/S be an abelian scheme. For any integer n, write $[n] : A \to A$ for the multiplication by n map. If n is invertible on S, then $[n]$ is finite étale and surjective. In particular, for any algebraically closed field k over S, the group $A(k)$ is n-divisible for any n invertible on S.*

For n invertible on S, it formally follows that the kernel $A[n]$ of $[n]$ is a finite étale S-group scheme: it is the base change of $[n] : A \to A$ along $e : S \to A$. We shall later see that $A[n]$ is a twisted form of $\underline{\mathbf{Z}/n}^{2g}$.

Proof. Standard properties about étale morphisms reduce us to checking this assertion in each fibre, i.e., we may assume $S = \mathrm{Spec}(k)$ for an algebraically closed field k. As A is proper and connected, any étale map is automatically finite (as quasi-finite proper maps are finite) and thus surjective (as the image is closed by properness and open by étaleness), so it suffices to check étaleness. Since A is a smooth k-variety, the map $[n]$ is étale if and only if it is an isomorphism on tangent spaces at closed points. Using suitable translations, it is enough to check the claim at the tangent space at e, i.e., we want $[n]_* : T_e(A) \to T_e(A)$ to be an isomorphism. But this map is the map on tangent spaces induced by

$$A \xrightarrow{\Delta} A^n \xrightarrow{m_n} A,$$

where m_n is the "sum of all co-ordinates" map. Applying Lemma 1.3.18 inductively then shows that $[n]_*$ coincides with multiplication by n. As n is invertible on S, this is an isomorphism. $\square$

Remark 1.3.20. In Proposition 1.3.19, when n is not invertible on S, the map $[n]$ is never étale: the proof above shows that $[n]_*$ is multiplication by n on tangent spaces, and thus not invertible by hypothesis. Nevertheless, $[n]$ is still finite surjective; this shall be proven later using intersection theory. More precisely, we shall need to understand the behaviour of line bundles on abelian varieties.

Chapter 2

Cohomology and base change: review

Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of noetherian schemes and a quasi-coherent sheaf F on X. Then there is a natural base change map

$$g^* R^i f_* F \to R^i f'_* g'^* F.$$

If $S' \to S$ corresponds to a map $A \to B$ of rings, then the above map is simply

$$H^i(X, A) \otimes_A B \to H^i(X_B, F_B).$$

This map is not an isomorphism in general. In fact, we give an example where this fails next with f being smooth and projective, and F being a line bundle.

Example 2.0.1. Let (E, e) be an elliptic curve over a field k. Choose a 1-parameter family $\{L_t\}_{t \in T}$ of non-trivial line bundles on E that degenerate to the trivial bundle. Concretely, we take T to be the spectrum of the local ring $\mathcal{O}_{E,e}$; the line bundle $\mathcal{O}_{E \times E}(\Delta) \otimes pr_1^* \mathcal{O}_E(e)$ on $E \times E$ restricts to give a line bundle L on $X := E \times T$, viewed as a T-scheme via the projection $f : X \to T$. If η and s denote the special and generic points of T, then $L_\eta \in \mathrm{Pic}(X_\eta)$ is a non-trivial degree 0 line bundle, while L_s is the trivial bundle. We claim that $R^0 f_* L = H^0(X, L)$ is 0 while $H^0(X_s, L_s)$ is 1-dimensional. The second assertion is clear as $X_s \simeq E$ and L_s is the trivial bundle. For the first, we simply remark as that the flatness of π, we have injection of sections $H^0(X, L) \hookrightarrow H^0(X_\eta, L_\eta)$. Now L_η is a non-trivial degree 0 line bundle on the elliptic curve X_η, and thus has no global sections.

The theme of this chapter is to understand the relationship between the sheaf $R^i f_* F$ and the function $s \mapsto H^i(X_s, F_s)$ on S.

2.1 Basic theorems on coherent cohomology

Theorem 2.1.1 (Finiteness theorem)**.** *Let $f : X \to S$ be a proper morphism of noetherian schemes, and let F be a coherent sheaf on X. Then $R^i f_* F$ is coherent for all i.*

$\boxed{\texttt{FFT}}$ **Theorem 2.1.2** (Formal functions theorem)**.** *Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of noetherian schemes, let $I \subset A$ be an ideal, and let F be a coherent sheaf on X. Then the natural maps give an isomorphisms*

$$H^i(X, F) \otimes_A \widehat{A} \simeq \widehat{H^i(X, F)} \simeq \lim H^i(X, F/I^n),$$

where all completions are I-adic.

Proof sketch. Let $\mathfrak{U}$ be a finite cover of X by affines, and let $C^\bullet(\mathfrak{U}, F)$ be the associated Cech complex. Then the vanishing of cohomology of affines ensures that

$$H^i(X,F) \simeq H^i(C^\bullet(\mathfrak{U},F)) \quad \text{and} \quad H^i(X,F/I^n) \simeq H^i(C^\bullet(\mathfrak{U},F/I^n)) \simeq H^i(C^\bullet(\mathfrak{U},F)/I^n)$$

for all n. Thus, it is enough to show the following: if $K^\bullet$ is a complex of A-modules with $H^i(K)$ finitely generated for all i, then the natural map gives an isomorphism

$$H^i(K) \otimes_A \widehat{A} \simeq \widehat{H^i(K)} \simeq \lim H^i(K/I^n K).$$

Both these are proven using the Artin-Rees lemma. ☐

prop:FlatBC

Proposition 2.1.3 (Flat base change)**.** *Let $f : X \to S$ be a qcqs morphism of schemes, and let F be a quasi-coherent sheaf on X. Let $g : T \to S$ be any map, and consider the fiber product square:*

$$\begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

Then there is a natural base change isomorphism

$$g^* R^i f_* F \simeq R^i f'_* g'^* F$$

Proof. We may assume $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ are affine. Our goal is to show that

$$H^i(X,F) \otimes_A B \simeq H^i(X_B, F_B)$$

via the natural map. We give an argument when X is separated, and the general case is similar. Let $\mathfrak{U}$ be a finite cover of X by affines, and let $C^\bullet(\mathfrak{U},F)$ be the associated Cech complex. Then we have:

1. $H^i(X,F) \simeq H^i(C^\bullet(\mathfrak{U},F))$ by the vanishing of the cohomology of affines as X is separated.
2. $H^i(X_B,F_B) \simeq H^i(C^\bullet(\mathfrak{U}_B,F_B)) \simeq H^i(C^\bullet(\mathfrak{U},F) \otimes_A B) \simeq H^i(C^\bullet(\mathfrak{U},F)) \otimes_A B$, where the first isomorphism exists for the same reason as above, the second as Cech complexes are compatible with base change, and the third because the functor $- \otimes_A B$ is exact (and thus commutes with taking cohomology of a complex).

Combining the above gives the claim. ☐

Remark 2.1.4. Proposition 2.1.3 often allows us to reduce a cohomological statement about a general morphism $X \to S$ to one where the base S is a complete noetherian local ring. In this case, Theorem 2.1.2 often allows us to reduce further to the case of artinian local rings. The artinian case can then be analysed explicitly using exact sequences relating an artinian local ring to a field.

Proposition 2.1.5 (Detecting vanishing fibrally)**.** *Let $f : X \to S$ be proper morphism of noetherian schemes, and let F be a coherent sheaf on X that is flat over S. Then $R^i f_* F = 0$ for all $i \geq i_0$ if and only if $H^i(X_s, F_s) = 0$ for all $i \geq i_0$.*

Proof. We may assume $S = \mathrm{Spec}(A)$ is affine. We want to show that $H^i(X,F) = 0$ for all $i \geq i_0$ exactly when $H^i(X_s,F_s) = 0$ for all $i \geq 0$.

For $\Leftarrow$, assume $H^i(X_s,F_s) = 0$ for all $i \geq i_0$. It is enough to show that the stalks of $H^i(X,F)$ at closed points are zero. By flat base change, we may assume A is complete noetherian local with maximal ideal $\mathfrak{m}$ corresponding to the closed point $s \in \mathrm{Spec}(A)$. By the finiteness and formal functions theorems, we have $H^i(X,F) \simeq \lim H^i(X_s, F/\mathfrak{m}^n)$. We know that $H^i(X_s, F/\mathfrak{m}) = 0$. Using the filtration of $R/\mathfrak{m}^n$ given by powers of $\mathfrak{m}$, and that R-flatness of R, it follows from the LES that $H^i(X, F/\mathfrak{m}^n) = 0$ for all $n \geq 0$, which proves the claim. (Note: this argument works separately for each degree i.)

For $\Rightarrow$, assume $H^i(X, F) = 0$ for $i \geq i_0$. Let $\mathcal{U}$ be an affine open cover of X, and $C^\bullet(\mathcal{U}, F)$ be the associated Cech complex. Then $C^\bullet(\mathcal{U}, F)$ is a bounded complex of flat A-modules by assumption on F. Moreover, for any A-algebra B, we have $H^i(C^\bullet(\mathcal{U}, F) \otimes_A B) \simeq H^i(X_B, F_B)$ as the formation of Cech complexes commutes with base change. Now, as $C^\bullet(\mathcal{U}, F)$ is a bounded complex of flat A-modules, it is homotopically flat[1] there is a spectral sequence,

$$E_2^{p,q} : \mathrm{Tor}^A_{-p}(H^q(C^\bullet(\mathcal{U}, F)), B) \Rightarrow H^{p+q}(C^\bullet(\mathcal{U}, F) \otimes_A B).$$

It then immediately follows that if $H^i(C^\bullet(\mathcal{U}, F)) = 0$ for $i \geq 0$, the same must be true for $H^i(C^\bullet(\mathcal{U}, F) \otimes_A B)$. Taking B to be the residue fields of A then proves the proposition. □

Proposition 2.1.6 (Projection formula)**.** *Let $f : X \to S$ be a qcqs morphism of schemes. Let E be a vector bundle on S and let F be a quasi-coherent sheaf on X. Then there is a natural isomorphism*

$$E \otimes_{\mathcal{O}_S} R^i f_* F \simeq R^i f_*(f^* E \otimes_{\mathcal{O}_X} F).$$

Proof. There is a natural map from the left to the right given by "cup products"; these can be explicitly constructed using Cech complexes or by observing that f_* is lax symmetric monoidal. So it suffices to prove the isomorphism locally on S, and we can then reduce to the case of trivial bundles, which is clear[2]. □

Proposition 2.1.7 (Kunneth)**.** *Let $S := \mathrm{Spec}(k)$ be the spectrum of a field. Let X and Y be qcqs k-schemes. Let F (resp. G) be a quasi-coherent sheaf on X (resp. Y). Then there is a natural isomorphism*

$$\oplus_{i+j=n} H^i(X, F) \otimes_k H^j(Y, G) \simeq H^n(X \times Y, F \boxtimes G).$$

Proof. We give an argument when X and Y are separated, and the general case is similar. Let $\mathcal{U}$ and $\mathcal{V}$ be finite covers of X and Y by affines. Then taking products gives an affine open cover $\mathcal{U} \times \mathcal{V}$ of $X \times Y$. We have an obvious isomorphism

$$C^\bullet(\mathcal{U}, F) \otimes_k C^\bullet(\mathcal{V}, G) \simeq C^\bullet(\mathcal{U} \times \mathcal{V}, F \boxtimes G),$$

so the claim follows from the usual Kunneth formula for complexes. □

2.2 Relating the cohomology of a family to the fibers

We make the following provisional[3] definition:

Definition 2.2.1. For a commutative ring A, a complex $M^\bullet$ of A-modules is called *perfect* if each M^i is a finite projective A-module, and $M^i = 0$ for $i \notin [-n, n]$ for some n.

n:CechPerfect **Lemma 2.2.2.** *Let A be a noetherian ring. Let $K^\bullet$ be a complex of A-modules satisfying:*

1. *Each K^i is flat, and $K^i = 0$ for $i \notin [a, b]$ for some fixed integers $a \leq b$.*
2. *Each $H^i(K)$ is finitely generated.*

Then there exists a perfect A-complex $M^\bullet$ and a quasi-isomorphism[4] $M^\bullet \to K^\bullet$ with $M^i = 0$ for $i \notin [a, b]$.

Proof. See Mumford. □

[1] This means that tensoring with it preserves quasi-isomorphisms, and hence can be used to compute derived tensor products.
[2] One must identify the natural map with the obvious map as well, but we leave this to the reader.
[3] Once derived categories are introduced, we shall call a complex perfect if it is isomorphic in the derived category to one satisfying the conditions of the previous definition.
[4] This is a map of complexes that induces an isomorphism on all cohomology groups.

lem:QisFlat **Lemma 2.2.3.** *Let $f : M^\bullet \to K^\bullet$ be a quasi-isomorphism between bounded complexes of flat modules over a commutative ring R. Then for any A-module B, the map $M^\bullet \otimes_A B \to K^\bullet \otimes_A B$ is also a quasi-isomorphism.*

Proof. Exercise. □

rwardPerfect **Proposition 2.2.4.** *Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism with A noetherian. Let F be a coherent sheaf on X that is flat over A. Then there exists a perfect A-complex $M^\bullet$ such that for any A-algebra B, there is a natural isomorphism*

$$H^i(M^\bullet \otimes_A B) \simeq H^i(X_B, F_B)$$

for all i.

Proof. Consider the Cech complex $C^\bullet(\mathfrak{U}, F)$ attached to a finite cover $\mathfrak{U} = \{U_i\}_{i=1,\dots,n}$ of X by affines $U_i \subset X$. Then we have:

1. By the vanishing of the coherent cohomology of affine schemes, we have $H^i(C^\bullet(\mathfrak{U}, F)) \simeq H^i(X, F)$. In particular, each $H^i(C^\bullet(\mathfrak{U}, F))$ is a finitely generated A-module, and vanishes outside $[0, n]$.

2. Each $C^i(\mathfrak{U}, F)$ is a flat A-module by assumption on F.

3. The formation of $C^\bullet(\mathfrak{U}, F)$ commutes with base change, i.e., if B is any A-algebra, then

$$C^\bullet(\mathfrak{U}, F) \otimes_A B \simeq C^\bullet(\mathfrak{U}_B, F_B)$$

   via the natural map.

Using (1) and (2) and applying Lemma 2.2.2 to $C^\bullet(\mathfrak{U}, F)$ gives a perfect A-complex equipped with a quasi-isomorphism $M^\bullet \to C^\bullet(\mathfrak{U}, F)$. To finish, we must show that the quasi-isomorphism $M^\bullet \to C^\bullet(\mathfrak{U}, F)$ remains a quasi-isomorphism after applying $- \otimes_A B$. This follows from Lemma 2.2.3. □

shforwardQis **Remark 2.2.5.** In the setup of Proposition 2.2.4, by Lemma 2.2.3, we are free to replace $M^\bullet$ with any quasi-isomorphic perfect complex without affecting the conclusion.

cor:SC **Corollary 2.2.6** (Semicontinuity)**.** *Fix X, A, and F as in Proposition 2.2.4. Then:*

1. *For each $i \geq 0$, the function $s \mapsto \dim H^i(X_s, F_s)$ on $\mathrm{Spec}(A)$ is upper semicontinuous, i.e., the sets $\{s \in S \mid \dim H^i(X_s, F_s) \geq k\}$ are closed for any integer k. In particular, the value of this function can only go up under specialization.*

2. *The function $s \mapsto \chi(X_s, F_s) = \sum_i (-1)^i \dim H^i(X_s, F_s)$ is locally constant.*

3. *Assume that $H^i(X_s, F_s) = 0$ for some fixed i and all $s \in \mathrm{Spec}(A)$. Then $H^i(X, F) = 0$ and the natural map gives an isomorphism*

$$H^{i-1}(X, F) \otimes_A \kappa(s) \simeq H^{i-1}(X_s, F_s)$$

   for all $s \in \mathrm{Spec}(A)$.

4. *Assume A is reduced, and fix an integer i. Then the function $s \mapsto H^i(X_s, F_s)$ is constant if and only if $H^i(X, F)$ is a finite projective A-module and $H^i(X, F) \otimes_A \kappa(s) \simeq H^i(X_s, F_s)$ for all $s \in \mathrm{Spec}(A)$. If either condition are satisfied, then $H^{i-1}(X, F) \otimes_A \kappa(s) \to H^{i-1}(X_s, F_s)$ is an isomorphism as well.*

Proof. All assertions are equally valid for the cohomology groups of the fibers of any perfect A-complex $M^\bullet$.

1. Choose a perfect complex $M^\bullet$ of A-modules as in Proposition 2.2.4. By shrinking the base further, we may assume each M^i is finite free. The differential $d^i : M^i \to M^{i+1}$ is thus a matrix with entries in A. Our goal is to show that the function

$$s \mapsto \dim(\ker(d^i \otimes_A \kappa(s))/\mathrm{im}(d^{i-1} \otimes_A \kappa(s)))$$

is upper semicontinuous. As all dimensions involved can take on only finitely many values (between 0 and the largest rank of the M^i's), it is enough to show that the functions

$$k_i(s) := \dim(\ker(d^i \otimes_A \kappa(s))) \quad \text{and} \quad r_i(s) = -\dim(\mathrm{im}(d^{i-1} \otimes_A \kappa(s)))$$

are upper semicontinuous. By rank-nullity, we have $r_i(s) = \mathrm{rank}(M^{i-1}) - k_{i-1}(s)$, so it is enough to show that the claim for k_i for all i. But

$$\{s \in \mathrm{Spec}(A) \mid k_i(s) \geq c\} = \{s \in \mathrm{Spec}(A) \mid \mathrm{rank}(d^i \otimes_A \kappa(s)) < \mathrm{rank}(M^i) - c\}.$$

The right hand side is the vanishing locus of the set of $(\mathrm{rank}(M^i) - c)^2$-minors of d^i, and is thus closed.

Remark 2.2.7. We pause to give a possibly more enlightening proof for the last statement of the claim, i.e., the values of the function can only go up under specialization. By detecting specializations using valuation rings, we may assume A is a discrete valuation ring with fraction field K, residue field k, and uniformizer $t \in A$. We must show that $\dim_K H^i(X_K, F_K) \leq \dim_k H^i(X_k, F_k)$. Choose a finite complex $M^\bullet$ of finite projective A-modules as in Proposition 2.2.4. We must show that $\dim H^i(M^\bullet \otimes_A K) \leq \dim H^i(M^\bullet \otimes_A k)$. As $A \to K$ is flat, we have $H^i(M^\bullet \otimes_A K) = H^i(M^\bullet)[\frac{1}{t}]$. On the other hand, tensoring the exact sequence

$$0 \to A \xrightarrow{t} A \to k \to 0$$

with $M^\bullet$ gives a short exact sequence of complexes (as each M^i is flat). Taking cohomology of this sequence gives a short exact sequence of $k = A/t$-modules

$$0 \to H^i(M^\bullet)/t \to H^i(M^\bullet \otimes_A k) \to H^{i+1}(M^\bullet)[t] \to 0.$$

It thus suffices to check $\dim_K H^i(M^\bullet)[\frac{1}{t}] \leq \dim_k H^i(M^\bullet)/t$. This holds true for any finitely generated A-module in place of $H^i(M^\bullet)$, and can be easily seen using the classification of such A-modules.

2. We may assume that A is a local ring. Our goal is to show that the function $s \mapsto \chi(X_s, F_s)$ is constant. Choose a complex $M^\bullet$ as in Proposition 2.2.4. As A is local, each M^i is a finite free A-module of some rank r_i. We claim that for any $s \in \mathrm{Spec}(A)$, we have

$$\chi(X_s, F_s) = \sum_i (-1)^i r_i.$$

The right side is clearly independent of s, so this would prove the required statement. To see this formula, write $k = \kappa(s)$ for simplicity. Then the left side is $\sum_i (-1)^i \dim_k H^i(M^\bullet \otimes_A k)$ by construction. As $M^i \otimes_A k$ is a k-vector space of dimension r_i, we are reduced to showing the following: for any finite complex $N^\bullet$ of finite dimensional k-vector spaces, we have

$$\sum_i (-1)^i \dim_k H^i(N^\bullet) = \sum_i (-1)^i \dim_k N^i.$$

We leave this assertion to the reader as an exercise in linear algebra.

3. Fix a prime ideal $\mathfrak{m} \subset A$ with residue field k. Our goal is to show that

$$H^{i-1}(X, F) \otimes_A k \to H^{i-1}(X, F/\mathfrak{m})$$

is an isomorphism; the vanishing of $H^i(X, F)$ will be deduced en route. For this, by flat base change, we may assume that A is complete noetherian local with maximal ideal $\mathfrak{m}$. We shall show more generally for any A-module M, the natural map

$$H^{i-1}(X, F) \otimes_A M \to H^{i-1}(X, F \otimes_A M)$$

is an isomorphism. Using a presentation of M, it is enough to check that the functor

$$M \mapsto H^{i-1}(X, F \otimes_A M)$$

is right-exact. As F is A-flat, this is implied by the vanishing of the functor

$$M \mapsto H^i(X, F \otimes_A M)$$

and the long exact sequence[5]. To show this vanishing, by the formal functions theorem, it suffices handle the case where M is killed by $\mathfrak{m}^n$ for some $n \geq 0$. The case $n = 1$ follows by our assumption since any such M is a finite product of copies of k. The general case then follows by filtering M by the $\mathfrak{m}$-adic filtration and using the attached long exact sequences. Taking $M = A$ also gives $H^i(X, F) = 0$, as wanted.

4. We explain a simple proof when A is a dvr, and leave the rest to the references. So assume A is a dvr with fraction field K, uniformizer t, and residue field k. Choose a perfect A-complex $M^\bullet$ as in Proposition 2.2.4. Assuming that $\dim_K(H^i(M^\bullet)[\frac{1}{t}]) = \dim_k(H^i(M^\bullet \otimes_A k))$, we shall show:

   (a) $H^i(M^\bullet)$ is free,

   (b) $H^i(M^\bullet)/t \simeq H^i(M^\bullet \otimes_A k)$, and

   (c) $H^{i-1}(M^\bullet)/t \simeq H^{i-1}(M^\bullet \otimes_A k)$.

   This clearly proves the claim. Tensoring the exact sequence

   $$0 \to A \xrightarrow{t} A \to k \to 0$$

   with $M^\bullet$ gives a short exact sequence of complexes (as each M^i is flat). Taking cohomology of this sequence gives a short exact sequence of $k = A/t$-modules

   $$0 \to H^i(M^\bullet)/t \to H^i(M^\bullet \otimes_A k) \to H^{i+1}(M^\bullet)[t] \to 0.$$

   We then get

   $$\dim_K(H^i(M^\bullet)[\frac{1}{t}]) \leq \dim_k H^i(M^\bullet/t) \leq \dim_k(H^i(M^\bullet \otimes_A k)),$$

   where the first inequality is a general fact about finitely generated A-modules, and the second follows from the SES. Our hypothesis ensures that the outer terms are equal, and hence all terms are equal, say to some integer $r \geq 0$. It follows that

   $$H^i(M^\bullet)/t \simeq H^i(M^\bullet \otimes_A k),$$

   proving (b). Choosing r generators of the this k-module and lifting to A, we get a surjection $A^r \to H^i(M^\bullet)$. This map is bijective after reduction modulo t by construction; it is also bijective after inverting t as $\dim_K(H^i(M^\bullet)[\frac{1}{t}]) = r$. It is then easy to see that this map is an isomorphism, which proves (a). Finally, (c) follows (a) and the exact sequence used above shifted one degree down.

□

[5] If a functor between abelian categories carries short exact sequences to right exact sequences, then it also carries right exact sequences to right exact sequences.

Chapter 3

Line bundles

3.1 The seesaw and cube theorems

:LineTrivial

Lemma 3.1.1. *A line bundle M on a proper geometrically integral variety Y over a field k is trivial if and only if $H^0(Y, M)$ and $H^0(Y, M^{-1})$ are nonzero*

Proof. The "only if" direction is clear, and for the "if" direction we observe that if $s : \mathcal{O}_Y \to M$ and $t : M \to \mathcal{O}_Y$ are nonzero maps, then the compositions st and ts are both nonzero (as Y is geometrically integra), and thus isomorphisms (as Y is proper), implying that both s and t were isomorphisms. $\square$

InvertibleBC

Proposition 3.1.2. *Let $f : X \to S$ be a proper flat morphism of noetherian schemes. Fix $L \in \mathrm{Pic}(X)$. Then there exists a unique locally closed subscheme $Z \subset S$ such that:*

1. *The pushforward $(f_Z)_* L_Z$ is invertible on Z.*
2. *If $T \to S$ is an S-map, then $T \to S$ factors (necessarily uniquely) over Z exactly when $(f_T)_* L_T$ is invertible. In this case, $(f_T)_* L_T$ is pulled back from $(f_Z)_* L_Z$.*

In the sequel, we shall refer to the conjunction of (1) and (2) above for the pair (f_Z, L) as saying that $(f_Z)_* L_Z$ is invertible and of formation compatible with base change.

Proof. The universal property in (2) characterizes Z, and also shows that its formation is compatible with base change (provided it exists). We may thus work locally on S to find such a Z. In fact, the underlying set is clear from (2): $|Z| = \{s \in S \mid \dim H^0(X_s, L_s) = 1\} \subset S$. This set is constructible by Corollary 2.2.6 (1). To find the desired scheme structure, fix some $s \in |Z|$. As $|Z|$ is locally closed, the intersection $|Z| \cap U$ is closed for a suitably small affine open neighbourhood $s \in U \subset S$. We shall equip this closed set with a natural closed subscheme structure satisfying the analog of (1) and (2) for the base change f_U. In particular, these structures patch together via the universal property to produce the required scheme structure on Z.

First, we make a preliminary construction. Assume $S = \mathrm{Spec}(A)$. By shrinking S further, we may choose a finite complex $K^\bullet$ of finite free A-modules as in Proposition 2.2.4, so $H^0(K^\bullet \otimes_A B) = H^0(X_B, L_B)$ for all A-algebras B. Setting

$$Q = \mathrm{coker}\big((K^1)^\vee \xrightarrow{d^\vee} (K^0)^\vee\big),$$

this translates to a functorial identification $\mathrm{Hom}_A(Q, B) \simeq H^0(X_B, L_B)$ for all A-algebras B. In particular, it follows that the formation of Q commutes with localization on A.

Now, as we have fixed $s \in |Z| \subset S$, we have $\dim H^0(X_s, L_s) = 1$, so $\mathrm{Hom}_A(Q, \kappa(s)) = (Q \otimes_A \kappa(s))^\vee$ is 1-dimensional. Then Nakayama shows that Q is a cyclic A-module, at least after shrinking S further around s. Write $Q = A/I$ for some ideal $I \subset A$. We then claim that setting $Z = V(I)$ solves the problem, i.e., satisfies (1) and (2). For (1), we must show that $\mathrm{Hom}_A(Q, A/I)$ is an invertible A/I-module, which is clear as $Q = A/I$. To check (2), we shall show the following:

(∗) For any A-algebra B, the B-module $\mathrm{Hom}_A(Q, B)$ is invertible exactly when $IB = 0$.

Once either of these conditions is satisfied, the resulting map $\mathrm{Hom}_A(Q, A/I) \otimes_{A/I} B \to \mathrm{Hom}_A(Q, B)$ is an isomorphism as $Q = A/I$, which proves the rest of (2).

To prove (∗), note that $\mathrm{Hom}_A(Q, B) = B[I] = \{b \in B \mid I \cdot b\}$. If this B-module is invertible and $IB \neq 0$, then, after passing to an open cover of $\mathrm{Spec}(B)$, we would obtain a free B-module of rank 1 that is annihilated by a nonzero ideal of B, which is absurd. □

thm:Seesaw **Theorem 3.1.3** (Seesaw theorem)**.** *Let $f : X \to S$ be a proper flat morphism of noetherian schemes with geometrically integral fibers. Fix $L \in \mathrm{Pic}(X)$. Then:*

1. *The set $Z := \{s \in S \mid L|_{X_s} \text{ is trivial}\}$ is closed in S.*
2. *We have $L|_{Z_{red}} \simeq f^*_{Z_{red}} M$ for some $M \in \mathrm{Pic}(Z_{red})$.*
3. *There exists a unique closed subscheme structure on Z such that $L|_Z$ is pulled back from $\mathrm{Pic}(Z)$, and the map $Z \to S$ is universal with this property, i.e., if $T \to S$ is some map with $L_T \in \mathrm{Pic}(X_T)$ pulled back from $\mathrm{Pic}(T)$, then the structure map $T \to S$ factors uniquely over Z.*

In particular, if L is trivial on all the fibers of f and S is reduced, then L is pulled back from S.

Proof. 1. By Lemma 3.1.1, we have

$$Z = \{s \in S \mid H^0(X_s, L_s) \geq 1\} \cap \{s \in S \mid H^0(X_s, L_s^{-1}) \geq 1\},$$

which is closed by Corollary 2.2.6 (1).

2. For this part, we may replace S with Z_{red} to assume L is trivial on all the fibers. Thus, $H^0(X_s, L_s)$ is 1-dimensional for all s by our hypothesis on the fibers. As the base is reduced, Corollary 2.2.6 (4) implies that $M = f_*L$ is a line bundle, and the map $M_s \to H^0(X_s, L_s)$ is an isomorphism for all s. There is a pullback $f^*M \to L$ defined by adjunction, and it is an isomorphism after restriction to each fiber X_s by construction (as each line bundle is trivial on the fibers, and the map on global sections is an isomorphism). Any such map must be an isomorphism: if a map between finite projective modules over a noetherian ring is an isomorphism on each fiber, it is an isomorphism by Nakayama.

3. We first claim that L is pulled back from S if and only if f_*L and f_*L^{-1} are invertible and of formation compatible with base change. Indeed, the "only if" implication follows from the projection formula as $f_*\mathcal{O}_X \simeq \mathcal{O}_S$ by Proposition 1.3.6. Conversely, if f_*L and f_*L^{-1} are invertible and of formation compatible with base change, then the adjunction map $f^*f_*L \to L$ is map between line bundles that is an isomorphism on the fibers by our base change compatibility assertion (as in (2)), and thus an isomorphism by Nakayama.

   Consider the maximal locally closed subscheme $W \subset S$ such that $(f_W)_*L_W$ and $(f_W)_*L_W^{-1}$ are both invertible and of formation compatible with base change, as provided by Proposition 3.1.2 applied to L and L^{-1}. By the previous paragraph, the map $W \to S$ satisfies the universal property formulated in (2). It thus remains to check that $W = Z$ as subsets of S; indeed, this will imply that $W \to X$ is a closed immersion as $Z \subset X$ is closed by (1). But $W = Z$ is also clear from the universal property in (3) and the definition of Z, so we are done.

□

Remark 3.1.4 (Seesaw via Picard schemes)**.** For X/S as in Theorem 3.1.3, consider the presheaf $\mathrm{Pic}_{X/S}$ on Sch/S defined by $T \mapsto \mathrm{Pic}(X_T)/f^*\mathrm{Pic}(T)$. If f is assumed to be projective and admits a section, then a non-trivial theorem of Grothendieck shows that $\mathrm{Pic}_{X/S}$ is representable by a separated S-group scheme. If one is willing to use this result, then Theorem 3.1.3 admits a direct proof: the line bundle L defines a section $[L] : S \to \mathrm{Pic}_{X/S}(T)$, and the desired closed subscheme $Z \subset S$ is simply the pullback of the 0-section of $\mathrm{Pic}_{X/S}$ along $[L]$.

Remark 3.1.5. It follows from the universal property in Theorem 3.1.3 (2) that the formation of Z itself commutes with base change, i.e., if $g : S' \to S$ is any map, then the scheme-theoretic inverse image $Z' := g^{-1}(Z) \subset S'$ satisfies the universal property in Theorem 3.1.3 (2) for the pair $(f_{S'}, L_{S'})$.

Remark 3.1.6. Theorem 3.1.3 owes its name to the special case X is a product of two proper geometrically integral varieties over a field k, and f is a projection. In this case, the theorem implies that line bundles on the product that are trivial on the fibers of one of the projection maps are pulled back from the base of the projection, thus evoking a "seesaw" image.

Remark 3.1.7. It is tempting to ask the following question: if X and Y are proper and geometrically integral varieties over k, and $L \in \mathrm{Pic}(X \times Y)$ is trivial on $X \times \{y\}$ and $\{x\} \times Y$ for a single pair of points $(x, y) \in X(k) \times Y(k)$, then is L is trivial? The answer is no. For example, take an elliptic curve (E, e), and let $D \subset X \times Y$ be the divisor given by $\Delta - pr_1^{-1}(e) - pr_2^{-1}(e)$. Then $D|_{\{e\}\times E}$ and $D|_{E\times\{e\}}$ are trivial, but D is not trivial: if it were trivial, then for any $x \in E(k)$, we would see that $D|_{E\times\{x\}} \simeq \mathcal{O}_E(x - e)$ is the trivial line bundle on E, which is not possible unless $x = e$ as the map $E(k) \to \mathrm{Pic}(E)$ given by $y \mapsto \mathcal{O}(y - e)$ is injective.

thm:Cube

Theorem 3.1.8. *Let S be a connected noetherian scheme, and let $X \to S$ and $Y \to S$ be two proper flat morphisms with geometrically integral fibers. Fix $L \in \mathrm{Pic}(X \times_S Y)$ Assume the following:*

1. *There exist sections $e_X \in X(S)$ and $e_Y \in Y(S)$ such that the pullback of L along $f_X : X \xrightarrow{(\mathrm{id}_X, e_Y)} X \times_S Y$ and $f_Y : Y \xrightarrow{(e_X, \mathrm{id}_Y)} X \times_S Y$ is trivial.*
2. *There exists a point $s \in S$ with L_s trivial.*

Then L is pulled back from S.

The idea of the proof is deformation theoretic: using (1) and deformation theory, one "spreads out" the conclusion of (2) to arbitrary artinian thickenings of s, and, in the limit, to a formal neighbourhood of s.

Proof. Write $P = X \times_S Y$ for the product. Let $Z \subset S$ be the maximal closed subscheme with L_Z pulled back from Z universally, as in Theorem 3.1.3. We must show $Z = S$. We have $s \in Z$ by (2), so $Z \neq \emptyset$. As Z is closed, it is enough to show Z is also open (as S is connected). Moreover, to prove openness, it is enough to check stability under generalizations: a closed subset of a noetherian affine scheme that is closed under generalizations has to be open. We may thus assume $S = \mathrm{Spec}(R)$ for a noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k. Relabelling, write $s \in S$ for the closed point, and write $I \subset \mathfrak{m}$ for the ideal of Z. We want to show $I = 0$. If not, then there exists a smaller ideal $J \subset I$ with $I/J \simeq k$ as an R-module: we may simply take J to be the preimage of any codimension 1 subspace of $I/\mathfrak{m}I$ (as the latter is nonzero if $I \neq 0$ by Nakayama). Let $W := \mathrm{Spec}(R/J)$, so we have a strict containment $Z \subsetneq W$ of closed subschemes of S. By definition of Z, we know that L_Z is pulled back from Z, and hence trivial (as Z is local). We shall show that L_W is also trivial, contradicting the maximality of Z, thus proving the theorem.

As L_Z is trivial, we can choose some $s \in H^0(P_Z, L_Z)$ giving an isomorphism $\mathcal{O}_{P_Z} \overset{s}{\simeq} L_Z$. Pulling back the exact sequence

$$1 \to k \to R/J \to R/I \to 1$$

to P and tensoring with L gives an exact sequence

$$1 \to L_s \to L_W \to L_Z \to 1$$

of sheaves. Hence, the obstruction to lifting s to an element $\tilde{s} \in H^0(P_Z, L_Z)$ is an element $\xi = \xi(s) \in H^1(P_s, L_s)$. We shall show that $\xi = 0$. This will imply the theorem. In fact, we claim the stronger statement that $\xi = 0$ exactly when L_W is trivial. Indeed, if $\xi = 0$, then the map s lifts to $\tilde{s}$ that necessarily trivializes L_W (by Nakayama). Conversely, if L_W is trivial, then $\xi = 0$ as the map $H^0(P_W, L_W) \to H^0(P_Z, L_Z)$ identifies with the map $R/J \to R/I$ by Proposition 1.3.6, and hence is surjective.

It remains to show $\xi = 0$. By functoriality of forming ξ from s, the pullback class $\xi_X := f_X^*(\xi) \in H^1(X_s, f_X^*(L_s))$ measures the obstruction to trivializing $f_X^*(L_W)$ on X_W, and similarly for $\xi_Y := f_Y^*(\xi) \in H^1(Y_s, f_Y^*(L_s))$. But $f_X^*(L)$ and $f_Y^*(L)$ are themselves pulled back from S by hypothesis, and hence trivial as S is local. It follows that $\xi_X = \xi_Y = 0$. It is therefore enough to check that pullback gives in injective map

$$H^1(P_s, L_s) \to H^1(X_s, f_X^* L_s) \times H^1(Y_s, f_Y^* L_s).$$

But, after fixing an isomorphism $L_s \simeq \mathcal{O}_{P_s}$, this identifies with the pullback

$$H^1(P_s, \mathcal{O}_{P_s}) \to H^1(X_s, \mathcal{O}_{X_s}) \times H^1(Y_s, \mathcal{O}_{Y_s}),$$

which is even bijective by the Kunneth formula. □

cor:cube

Corollary 3.1.9 (Theorem of the cube)**.** *Fix a base scheme S. Let $X \to S$ and $Y \to S$ be proper flat maps with geometrically integral fibers, and let Z be any connected finite type S-scheme. Let $L \in \mathrm{Pic}(X \times Y \times Z)$. If there exist sections $x \in X(S)$, $y \in Y(S)$, and $z \in Z(S)$ with $L|_{\{x\} \times Y \times Z}$, $L|_{X \times \{y\} \times Z}$, and $L|_{X \times Y \times \{z\}}$ being trivial, then L is trivial.*

Proof. Our hypotheses ensure that Theorem 3.1.8 applies to the projection $X \times Y \times Z \to Z$, so we learn that L is pulled back from Z. On the other hand, L is trivial on the section $\{x\} \times \{y\} \times Z \subset X \times Y \times Z$ of the projection by assumption, and hence L must be trivial. □

3.2 The theorem of the square and applications

cor:CubeAbVar

Lemma 3.2.1. *Fix a connected base scheme S. Let $\pi : A \to S$ be an abelian scheme, and let Z be any S-scheme. Fix maps $f, g, h : Z \to A$. Then for any $L \in \mathrm{Pic}(A)$, there exists an isomorphism*

$$(f+g+h)^* L \simeq (f+g)^* L \otimes (f+h)^* L \otimes (g+h)^* L \otimes f^*(L^{-1}) \otimes g^*(L^{-1}) \otimes h^*(L^{-1}) \otimes \pi^* e^* L$$

Here we follow the convenient that if $a, b : Z \to A$ are two maps, then $a+b$ is the composite $Z \xrightarrow{(a,b)} A \times A \xrightarrow{m} A$.

Proof. It is enough to handle the universal case where $Z = A \times A \times A$, and f, g, h are the 3 projections. Set $m = f+g+h$, $m_{fg} = f+g$, $m_{gh} = g+h$, and $m_{fh} = f+h$. Write $e : A \times A \times A \to A$ for the constant map. It is enough to show that the line bundle

$$M := m^* L^{-1} \otimes m_{fg}^* L \otimes m_{gh}^* L \otimes m_{fh}^* L \otimes f^* L^{-1} \otimes g^* L^{-1} \otimes h^* L^{-1} \otimes \pi^* e^* L$$

is trivial on $A \times A \times A$. It is easy to see that the following pairs of maps induce the same maps on composition with $A \times A \xrightarrow{(\mathrm{id}, \mathrm{id}, e \circ \pi)} A \times A \times A$:

$$m, m_{fg} \quad \text{and} \quad m_{gh}, g \quad \text{and} \quad m_{fh}, f \quad \text{and} \quad e \circ \pi, h.$$

It then immediately follows that $M|_{A \times A \times \{e\}}$ is the trivial bundle. By symmetry, we also M also restricts to the trivial bundle over $\{e\} \times A \times A$ and $A \times \{e\} \times A$. Corollary 3.1.9 then implies that M is trivial. □

MultNLB

Corollary 3.2.2. *Let A/S be an abelian scheme. Then for any integer n and $L \in \mathrm{Pic}(A)$, we have*

$$[n]^* L \simeq L^{\frac{n^2+n}{2}} \otimes [-1]^* L^{\frac{n^2-n}{2}} \otimes \pi^* e^* L^{-n^2}.$$

In particular, if L is symmetric (i.e., $L \simeq [-1]^* L$), then $[n]^* L \simeq L^{n^2}$ up to line bundles pulled back from S.

Given any line bundle M, the bundle $M \otimes [-1]^* M$ is symmetric.

Proof. One proves this separately for positive and negative n by induction. We give the argument for $n = 2$. Applying Lemma 3.2.1 to $Z = A$ with $f = [1]$, $g = [1]$, and $h = [-1]$, we obtain

$$L \simeq [2]^*L \otimes [0]^*L \otimes [0]^*L \otimes L^{-1} \otimes L^{-1} \otimes [-1]^*L^{-1} \otimes \pi^*e^*L,$$

which, as $[0] = e \circ \pi$, simplifies to give

$$[2]^*L \simeq L^3 \otimes [-1]^*L \otimes \pi^*e^*L^{-3},$$

as wanted. □

square **Corollary 3.2.3** (Theorem of the square). *Let A/S be an be an abelian scheme. For all $L \in \mathrm{Pic}(A)$ and $x, y \in A(S)$, we have an isomorphism*

$$t_{x+y}^*L \otimes L \simeq t_x^*L \otimes t_y^*L$$

up to line bundles pulled back from S.

Proof. Write c_x and c_y for the constant maps with values x and y respectively. Apply Lemma 3.2.1 with $f = c_x$, $g = c_y$ and $h = \mathrm{id}$ to get

$$t_{x+y}^*L \simeq c_{x+y}^*L \otimes t_y^*L \otimes t_x^*L \otimes c_x^*L^{-1} \otimes c_y^*L^{-1} \otimes L^{-1} \otimes c_e^*L.$$

This gives the desired formula up to line bundles pulled back from S. □

phiL **Remark 3.2.4** (The ϕ_L construction)**.** Corollary 3.2.3 implies that for each line bundle L on A, the map of presheaves

$$\phi_L : A \to \mathrm{Pic}_{A/S}$$

defined on points by $\phi_L(x) = t_x^*L \otimes L^{-1}$ is a homomorphism. The kernel of this map classifies those points $x : T \to A$ such that the line bundle $t_x^*(L_T) \otimes L_T^{-1}$ on $A \times T$ is pulled back from T. We shall give a different description of this kernel in Corollary 3.2.6.

3.2.1 The Mumford bundle and $K(L)$

ss:KofL

Let L be a line bundle on an abelian scheme A/S.

Definition 3.2.5. The Mumford bundle attached to L is

$$\Lambda(L) := m^*(L) \otimes pr_1^*L^{-1} \otimes pr_2^*L^{-1} \in \mathrm{Pic}(A \times A).$$

Applying the Seesaw theorem to the first projection $pr_1 : A \times A \to A$, we obtain a maximal closed subscheme $K(L) \subset A$ such that $\Lambda(L)|_{K(Z)\times A}$ is universally pulled back[1] from $K(L)$.

If $x : T \to A$ is a map, then pullback $\Lambda(L)|_{T\times A}$ can be identified as $t_x^*(L_T) \otimes pr_1^*x^*(L) \otimes L_T$ on $T \times A$, where $L_T = pr_2^*(L)$. Thus, we have:

KofLAlternate **Corollary 3.2.6.** *A map $x : T \to A$ factors through $K(L)$ exactly when $t_x^*L_T \otimes L_T^{-1}$ on $T \times A$ is pulled back from T.*

Using this, we claim:

Lemma 3.2.7. *The subscheme $K(L) \subset A$ is a subgroup scheme.*

Proof. Let $x, y \in A(T)$ be two scheme-theoretic points. We must show that if $t_x^*(L_T) \otimes L_T^{-1}$ and $t_y^*(L_T) \otimes L_T^{-1}$ are pulled back from T, the so is $t_{x+y}^*(L_T) \otimes L_T^{-1}$. But this is immediate from Corollary 3.2.3. □

[1] In fact, we have $\Lambda(L)|_{K(L)\times A} \simeq \mathcal{O}_{K(L)\times A}$: this follows by pulling back along the section $K(L) \xrightarrow{(\mathrm{incl},e)} K(L) \times A$ to pr_1. This is the main reason to introduce the $pr_1^*L^{-1}$ term in the definition of $\Lambda(L)$. If we had not done this, then $K(L)$ would be the same scheme, but we would lose the triviality.

KofLRestrict **Lemma 3.2.8.** *Assume $S = \mathrm{Spec}(k)$ for an algebraically closed field k.*

1. *The maximal connected reduced subscheme $B := (K(L)^0)_{red}$ is an abelian subvariety of A.*

2. *The line bundle $L|_B \otimes [-1]^*L|_B$ is trivial.*

In particular, over any base scheme S, if L is relatively ample for A/S, then $K(L)$ is finite over S.

Proof. For (1), observe that B is a proper connected reduced variety such that $B(k) \subset A(k)$ is a subgroup. As k is algebraically closed, this implies that B is an abelian variety. Write $M = L|_B$. It is immediate from the definitions that $\Lambda(L)|_{B\times B} \simeq \Lambda(M)$. Now $\Lambda(L)|_{K(L)\times A}$ is trivial, and hence $\Lambda(M)$ is also trivial on $B \times B$. Pulling back along $(1, -1) : B \to B \times B$ shows that $M \otimes [-1]^*M$ is trivial on B, giving (2).

For the last assertion: as $K(L) \subset A$ is closed, the map $K(L) \to S$ is proper. Moreover, the formation of $K(L)$ commutes with base change. Thus, to show finiteness, we may assume S is a geometric point. Adopting the notation of (1), it is enough to show $\dim(B) = 0$. As L is ample, so is M, and hence the same holds true for $M \otimes [-1]^*M$. But the latter is trivial on B by (2). The claim follows as there are no positive dimensional connected projective variety where the trivial bundle is ample. $\square$

Remark 3.2.9. As $\Lambda(L)^{-1} = \Lambda(L^{-1})$, we have an equality $K(L) = K(L^{-1})$ of subgroup schemes of A. In particular, the finiteness of $K(L)$ does not force ampleness of L. We shall see later that the noneffectivity of L is the only obstruction here.

3.2.2 Projectivity of abelian varieties

The following proposition ensures that any map non-finite map out of an abelian variety arises essentially by collapsing a non-trivial abelian subvariety.

MapOutOf **Proposition 3.2.10.** *Let k be an algebraically closed field. Let $f : A \to Y$ be a map of k-varieties with A an abelian variety. For each $a \in A(k)$, write F_a for the connected component of $f^{-1}(f(a))$, given its reduced structure. Then F_0 is an abelian subvariety of A, and $F_a = a + F_0 \subset A$ for any $a \in A(k)$.*

Proof. Fix some $a \in A(k)$, and consider the map $\phi : A \times F_a \to Y$ defined by restriction $A \times A \xrightarrow{m} A \xrightarrow{f} Y$. Now $\phi(\{0\} \times F_a)$ is simply the closed point $f(a)$. By Corollary 1.3.7, it follows that ϕ factors over the projection $pr_1 : A \times F_a \to A$ via a map $\overline{\phi} : A \to Y$. Using the section $A \xrightarrow{b\mapsto(b,a)} A \times F_a$, it follows that $\overline{\phi}(b) = \phi(b, a) = f(b + a)$ for any $b \in A(k)$. This gives

$$f(b - a + F_a) =: \phi(b - a, F_a) = \overline{\phi}(b - a) = f(b - a + a) = f(b).$$

Taking $a = 0$ gives $f(b+F_0) = f(b)$, so $b+F_0 \in F_b$ for all $b \in A(k)$. Taking $b = 0$ gives $f(-a+F_a) = f(0)$, so $-a + F_a \subset F_0$ for all $a \in A(k)$, and hence $F_a \subset a + F_0$ for all $a \in A(k)$. Combining the previous two sentences shows $F_a = a + F_0$ for all $a \in A(k)$.

As F_0 is proper, geometrically reduced and geometrically connected, by Remark 1.3.2, it remains to check $F_0(k) \subset A(k)$ is closed under the group operation. But we have already seen that $a + F_a = F_0$ for all $a \in A(k)$. If $a \in F_0(k)$, then $F_a = F_0$, so this gives $a + F_0 = F_0$, and hence $F_0(k)$ is closed under the group operation. $\square$

Effective divisors on an abelian variety are effectively semiample.

EffectSA **Proposition 3.2.11.** *Let A/k be an abelian variety, and let D be an effective divisor on A. Then the line bundle $L = \mathcal{O}_A(D)$ is semiample. More precisely, the linear system $|2D|$ is basepoint free.*

Proof. We may assume k is algebraically closed. We must show that for each $a \in A(k)$, there exists some $E \in |2D|$ such that $a \notin E$. Let $U \subset A$ denote the dense open set $-a + (A - D)$. Then $U \cap [-1]^*(U)$ is not empty, so we can choose some $b \in A(k)$ with $b, -b \in -a+(A-D)$. This means $a+b \in A-D$ and $a-b \in A-D$; equivalently, we have $a \notin -b + D$ and $a \notin b + D$. But this can also be written as $a \notin T^*_{-b}(D) \cup T^*_b(D)$. Now the divisor $E = T^*_{-b}(D) + T^*_b(D)$ belongs to the linear system $|2D|$ by Corollary 3.2.3, and we just checked $a \notin E$, as wanted. $\square$

KofLSA **Remark 3.2.12.** Let A be an abelian variety over an algebraically closed field k, and $L \in \mathrm{Pic}(A)$ an effective line bundle. Proposition 3.2.11 gives a morphism $f : A \to \mathbf{P}(H^0(A, L^2))$ such that $f^*\mathcal{O}(1) = \mathcal{L}^2$. Proposition 3.2.10 gives an abelian subvariety $F_0 \subset A$ in the fibre over $f(0)$. On the other hand, we also obtain the subgroup scheme $K(L) \subset A$ as in §3.2.1. We claim that F_0 coincides with $B := (K(L)^0)_{red}$.

To show $F_0 \subset B$, fix some $x \in F_0(k)$. As F_0 is connected, it is enough to check that $x \in K(L)(k)$, i.e., we have $T_x^*(L) \simeq L$. Fix a nonzero section $s \in H^0(A, L)$ corresponding to an effective divisor $D \subset A$. The section s^2 gives a hyperplane $H \subset \mathbf{P}^n$ which pulls back to the divisor $2D$. As $x \in F_0$, we have $f \circ T_x = f$, so $T_x^*(2D)$ and $2D$ are the *same* divisor. As this is an equality of divisors and not merely divisor classes, we must also have $T_x^*(D) = D$. Passing to associated line bundles shows $T_x^* L \simeq L$.

To show $B \subset F_0$, set $M := L|_B$. Then M^2 is globally generated (as L^2 is so). We shall check that M^2 is trivial. This implies that the composition $B \to A \xrightarrow{f} \mathbf{P}(H^0(A, L^2))$ is the constant map; its image is necessarily $f(0)$, which would prove that $B \subset F_0$. To check triviality of M^2, note that Lemma 3.2.8 implies that $M^{-1} \simeq [-1]^*M$, and hence $M^{-2} \simeq [-1]^*M^2$. As $[-1]$ is an automorphism, it follows that M^{-2} is also globally generated. Lemma 3.1.1 then implies that M^2 is trivial.

KofLHofD **Remark 3.2.13.** Let (A, L, D) as in Remark 3.2.12. The proof of $F_0 \subset B$ in Remark 3.2.12 shows something stronger: we have

$$F_0 \subset H(D) := \{x \in A(k) \mid t_x^* D = D\},$$

where the equality is an equality of divisors (and not merely divisor classes). In particular, we have

$$K(L)^\circ(k) = B(k) \subset F_0 \subset H(D).$$

In particular, any open set in A containing $H(D)$ also contains $K(L)^\circ$.

We get the promised characterization of ampleness of L in terms of $K(L)$:

mpleCritKofL **Corollary 3.2.14.** *Let A/k be an abelian variety, and let $L \in \mathrm{Pic}(A)$ be an effective line bundle. If $K(L)$ is finite, then L is ample.*

Proof. We may assume k is algebraically closed. Let $f : A \to \mathbf{P}^n$ be the map defined by L^2 by Proposition 3.2.11. It is enough to show that the fibers of f are finite. In fact, as all fibers are translates of each other, it suffices to show that F_0 is finite. But Remark 3.2.12 tells us that $F_0 = B \subset K(L)$, which is finite by hypothesis, so we are done. $\square$

neComplement **Lemma 3.2.15.** *Let X be a separated noetherian scheme. Let $U \subset X$ be a dense affine open subset. Then each generic point of $X - U$ has codimension 1 in X. In other words, $X - U$ is a union of Weil divisors in X.*

Proof. We give a proof when X is normal; see [?, Tag 0BCQ] for the general case. For any $x \in X$, the the base change $U_x := U \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) \subset X_x := \mathrm{Spec}(\mathcal{O}_{X,x})$ is a dense affine open subset by the separatedness of X and the density of U. Now if x is a generic point of $X - U$, then $X_x - U_x$ is a single closed point, so U_x is the punctured spectrum of the local scheme X_x. If x had codimension ≥ 2 in X, then it would follow from normality that $H^0(X_x, \mathcal{O}_{X_x}) \simeq H^0(U_x, \mathcal{O}_{U_x})$. But both U_x and X_x are affine, so this forces $U_x = X_x$, which is impossible as $x \notin U$. Thus, x has codimension ≤ 1; in fact, the codimension is exactly 1 as U_x is nonempty. $\square$

tiveFieldExt **Exercise 3.2.16.** Let X/k be a scheme of finite type. Assume that $X_{\overline{k}}$ is projective. Show that X is projective.

Theorem 3.2.17. *Abelian varieties are projective.*

Proof. By Exercise 3.2.16, we may assume k is algebraically closed. Let $U \subset A$ be an affine open subset containing e. Let $D = A - U$, so D is an effective Weil (and hence Cartier) divisor on A by Lemma 3.2.15. Let $L = \mathcal{O}_A(D)$ and consider $H(D) = \{x \in A(k) \mid t_x^* D = D\}$ as in Remark 3.2.13. Any $x \in H(D)$ carries U to itself under translation (by definition). As $e \in U$, it follows that $x \in U$, and thus $H(D)$ is contained in

U. Since $K(L)^\circ(k) \subset H(D)$, it follows that $K(L)^\circ$ is also contained in U (see Remark 3.2.13). But $K(L)^\circ$ is proper and connected, while U is affine, so $K(L)^\circ = \{*\}$, which proves that $K(L)$ is finite, and thus L is ample by Corollary 3.2.14. $\square$

Alternative proof. By Exercise 3.2.16, we may assume k is algebraically closed. Let $U \subset A$ be any nonempty affine open subset. Let $D = A - U$, so D is an effective Weil (and hence Cartier) divisor on A by Lemma 3.2.15. Let $L = \mathcal{O}_A(D)$. By Proposition 3.2.11 the line bundle L^2 is globally generated; write $f : A \to \mathbf{P}^m$ for the associated morphism. Then the divisor $2D$ is defined by a section of L^2, and hence arises as the pullback of a hyperplane $H \subset \mathbf{P}^m$ under f. In particular, for any closed point $x \in \mathbf{P}^m - H$, the preimage $f^{-1}(x)$ is contained in $U := A - D$. But $f^{-1}(x)$ is proper and U is affine, so $f^{-1}(x)$ must be finite. Proposition 3.2.10 then ensures that all fibres of f are finite, and thus f is finite. But this immediately implies that L^2, and hence L, is ample. $\square$

Corollary 3.2.18. *Let A be an abelian variety of dimension g. Then A cannot be embedded into $\mathbf{P}^{2g-1}$.*

Remark 3.2.19. Note that any projective variety of dimension g can be embedded into $\mathbf{P}^{2g+1}$; a better version of the argument below improves $2g-1$ to $2g$ when $g \geq 3$ in the first assertion, thus showing the sharpness of the $2g+1$ bound.

We shall use some intersection theory in the proof, and we summarize what we need. For any coherent sheaf E on $\mathbf{P}^m$, write $c_{tot}(E) = \sum_{i=0}^{\infty} c_i(E)$ for the total Chern class of a sheaf E, viewed as an element of the graded ring $H^{2*}(\mathbf{P}^m)$; here we $H^*(-)$ denotes any Weil cohomology theory. We shall use the following facts:

- The formation of Chern classes is compatible with restriction to subvarieties.
- $c_{tot}(-)$ carries addition in K_0 to multiplication of cohomology classes.
- $c_i(E) = 0$ for $i > \mathrm{rank}(E)$.
- If $h = c_1(\mathcal{O}(1)) \in H^2(\mathbf{P}^m)$ is the hyperplane class, then $h^d|_X \in H^{2d}(X)$ is nonzero if X is a d-dimensional subvariety of $\mathbf{P}^m$.

Proof. Say $i : A \subset \mathbf{P}^m$ is a closed immersion. We have an exact sequence

$$0 \to I/I^2 \to \Omega^1_{\mathbf{P}^m} \to \Omega^1_A \to 0$$

of sheaves on A, where I/I^2 is the conormal bundle of rank $m-g$. Applying c_{tot} to the above sequences gives

$$c_{tot}(\Omega^1_{\mathbf{P}^m}|_A) = c_{tot}(\Omega^1_A) \cdot c_{tot}(I/I^2).$$

Now $\Omega^1_A = \mathcal{O}_A^g$, so $c_{tot}(\Omega^1_A) = 1$. Also, $c_{tot}(I/I^2)$ vanishes in degrees $> \mathrm{rank}(I/I^2) = m-g$. Thus, we get

$$c_i(\Omega^1_{\mathbf{P}^m}|_A) = 0 \quad for \quad i > m-g.$$

On the other hand, the Euler sequence on $\mathbf{P}^m$ is

$$0 \to \Omega^1_{\mathbf{P}^m} \to \mathcal{O}_{\mathbf{P}^m}(-1)^{m+1} \to \mathcal{O}_{\mathbf{P}^m} \to 0.$$

Writing $h = c_1(\mathcal{O}_{\mathbf{P}^m}(1)) \in H^2(\mathbf{P}^m)$ for the hyperplane class, this gives

$$c_{tot}(\Omega^1_{\mathbf{P}^m}) = (1-h)^{m+1}.$$

As the formation of total Chern classes is compatible with restriction to subvarieties, we have the same formula after restriction to A. In particular, as $c_{m-g+1}(\Omega^1_{\mathbf{P}^m}|_A) = 0$, we get

$$h^{m-g+1} = 0 \in H^{2*}(A).$$

As A is a projective variety of dimension g and h is a hyperplane class, we also know that $h^g \neq 0$ on A, so it follows that

$$m - g + 1 > g, \quad so \quad m > 2g-1,$$

as wanted. $\square$

3.2.3 Torsion subgroups

We begin by calculating the degree of multiplication by n.

orsionDegree **Theorem 3.2.20.** *Let A/S be an abelian scheme of relative dimension g. Then for any integer n, the multiplication map $[n] : A \to A$ is finite flat of degree n^{2g}. In particular, for any algebraically closed S-field k, the abelian group $A(k)$ is divisible, and its n-torsion $A(k)[n]$ is finite.*

We may assume S is a geometric point (exercise!).

Proof that $[n]$ is finite flat. It is enough to check that the fibers are finite schemes. Let L be an ample line bundle A Set $M := L \otimes [-1]^*L$. so M is symmetric and ample. Then $[n]^*M \simeq M^{n^2}$ by Corollary 3.2.2. Let $X \subset A$ be a fibre of $[n]$ (viewed as a reduced proper variety). But then $M^{n^2}|_X \simeq ([n]^*M)|_X$ would be trivial. As M^{n^2} is ample, this forces X to be 0-dimensional, as wanted. □

To proceed further, we need the notion of a degree for a coherent sheaf on a projective variety:

DegreeSheaf **Construction 3.2.21.** Let X be an irreducible projective variety of dimension g over a field k. For a line bundle L and coherent sheaf F on X, the function $P_{F,L}(n) = \chi(F \otimes L^n)$ is a polynomial in n. Write $\frac{d_L(F)}{g!}$ for the coefficient of n^g in this polynomial, and write $\deg(L) = d_L(\mathcal{O}_X)$. We shall use the following facts

1. $F \mapsto d_L(F)$ is additive in short exact sequences. Indeed, the polynomials $P_{F,L}(n)$ behave additively in F.

2. For any integer k, we have an equality $P_{F,L}(kn) = P_{F,L^k}(n)$ of polynomials, and thus an equality $\deg(L^k) = k^g \deg(L)$.

3. If L is ample, then $P_F(L)$ has degree $\dim(F)$ (i.e., the coefficient of $n^{\dim(F)}$ is nonzero when $F \neq 0$). This is proven by induction on $\dim(F)$. Using (2), we may assume L is very ample. The case $\dim(F) = 0$ is clear: $P_F(n)$ is simply the sum of the dimensions of the (finitely many) stalks of F and is clearly independent of n. In general, using (1) and standard exact sequences relating F to its restriction to irreducible components, we may assume $\dim(F) = g$ (so F has irreducible support) and that F is torsionfree (as the torsion has smaller dimensional support). Now if $g > 0$, as F is torsionfree, we can find an exact sequence
$$0 \to F \otimes L^{-1} \to F \to F \otimes \mathcal{O}_{Z(s)} \to 0.$$
It follows that $P_F(n) - P_F(n-1) = P_{F\otimes \mathcal{O}_{Z(s)}}(n)$. By induction, $P_{F\otimes \mathcal{O}_{Z(s)}}(n)$ is a polynomial $h(x)$ of degree $\dim(F) - 1$. It follows that $P_F(n) = \sum_{j=0}^n h(j)$ is a polynomial degree $\dim(F)$.

4. $d_L(F) > 0$ if $F \neq 0$ and $\dim(F) = g$: this follows from (3) as $P_F(n) = \dim H^0(X, F \otimes L^n) \geq 0$ for $n \gg 0$ is positive.

5. $d_L(F) = 0$ if $\dim(F) < g$: this follows from (3).

This notion behaves well with respect to finite morphisms.

eFiniteCover **Proposition 3.2.22.** *Adopt the notation of Construction 3.2.21. Then:*

1. *$d_L(F) = \mathrm{rank}(F) \cdot \deg(L)$.*

2. *If $f : Y \to X$ is a finite surjective map with $\dim(Y) = g$, then $\deg(f^*L) = \deg(L) \cdot \deg(f)$.*

Proof. For (1): we can choose an exact sequence
$$0 \to I^{\mathrm{rank}(F)} \to F \to Q \to 0$$
of coherent sheaves where $I \subset \mathcal{O}_X$ is an ideal, and Q is torsion: if $U \subset X$ is the affine open complement of an ample divisor $H \in |L^k|$, then we have an inclusion $\mathcal{O}_U^{\mathrm{rank}(F)} \subset F|_U$ with a torsion cokernel, so we get the

above sequence by "clearing denominators" and may thus take $I = \mathcal{O}_X(-kH)$ for $k \gg 0$. As Q is torsion, $d_L(Q) = 0$. It follows that

$$\mathrm{rank}(F) \cdot d_L(I) = d_L(I^{\mathrm{rank}(F)}) = d_L(F).$$

Applying a similar argument to the exact sequence

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0$$

then proves (1).

For (2): by the projection formula, we have

$$P_{\mathcal{O}_Y, f^*L}(n) = \chi(Y, f^*L^n) = \chi(X, f_*\mathcal{O}_Y \otimes L^n) = P_{f_*\mathcal{O}_Y, L}(n),$$

for all n, so $P_{Y,f^*L} = P_{f_*\mathcal{O}_Y,L}$ as polynomials. Using (1), this gives

$$\deg(f^*L) := d_{f^*L}(\mathcal{O}_Y) = d_L(f_*\mathcal{O}_Y) = \deg(L) \cdot \mathrm{rank}(f_*\mathcal{O}_Y) = \deg(L) \cdot \deg(f),$$

as wanted. ☐

We can now finish the degree calculation.

Proof of Theorem 3.2.20. Let L be an ample line bundle on A. By replacing L with $L \otimes [-1]^*L$, we may assume L is symmetric. Applying Proposition 3.2.22 to $[n] : A \to A$ gives

$$\deg([n]^*L) = \deg(L) \cdot \deg([n]).$$

On other hand, as L is symmetric, we have $[n]^*L = L^{n^2}$. As $\deg(L) \neq 0$, it is enough to show that $\deg(L^k) = k^g \cdot \deg(L)$. Write $Q(n) = \chi(L^n)$ and $R(n) = \chi(L^{kn})$. These are both polynomials degree g in n, and we have $Q(kn) = R(n)$. In particular, the leading coefficient of R is k^g times the leading coefficient of Q, which gives the claim by definition of degrees. ☐

We can now analyze the torsion subgroups.

Theorem 3.2.23. *Let A/k be an abelian variety of dimension g, and fix an integer n.*

1. *If n is invertible on k, then $[n]$ is finite étale of degree n^{2g}. Moreover, the group scheme $A[n]$ is isomorphic with the constant group scheme $\underline{(\mathbf{Z}/n)^{2g}}$ when k is algebraically closed.*
2. *If $\mathrm{char}(k) = p$, then there exists some integer $0 \le i \le g$ such that $A[p^m](k) \simeq (\mathbf{Z}/p^m)^i$ for all $m \ge 0$.*

The integer i appearing in (2) is called the *p-rank* of A.

Proof. Proposition 1.3.19 shows that $[n]$ is finite étale while Theorem 3.2.20 shows its degree is n^{2g}, so the first half of (1) is clear. For the second, note that the category finite étale k-algebras is equivalent to the category of finite sets as k is algebraically closed. As $A[n]$ is a finite étale k-group scheme, it follows that $A[n]$ is canonically identified with the constant k-group scheme attached to the finite abelian group $A[n](k)$. The latter has cardinality n^{2g} as $[n]$ is finite étale of degree n^{2g}. Moreover, for each $m \mid n$, the m-torsion subgroup of $A[n](k)$ has cardinality m^{2g} by the same reasoning. Elementary group theory then shows that $A[n](k)$ must be isomorphic to $(\mathbf{Z}/n)^{2g}$.

For (2), consider first the case $m = 1$. The analysis Proposition 1.3.19 showed that $[p]^*$ induces the 0 map on cotangent spaces as e. We have also seen that $H^0(A, \Omega^1_{A/k}) \simeq e^*\Omega^1_{A/k}$ and that $\Omega^1_{A/k}$ is free. It follows that $[p]^*$ induces the zero map $[p]^*\Omega^1_{A/k} \to \Omega^1_{A/k}$ of sheaves. Applying Lemma 3.2.24 (and Remark 3.2.25) shows that $[p] : A \to A$ factors unique as

$$A \xrightarrow{Frob_{A/k}} A^{(1)} \xrightarrow{V} A$$

for some map V. Now $Frob_{A/k}$ is finite flat and purely inseparable of degree p^g (this holds true for any smooth k-scheme of dimension g). In particular, there is a natural bijection $A[p](k) \simeq A^{(1)}[V](k)$. As $Frob_{A/k}$ and $[p]$ are finite and faithfully flat, the same holds true for V. In particular, V is finite flat of degree $\frac{\deg([p])}{\deg(Frob_{A/k})} = \frac{p^{2g}}{p^g} = p^g$. It follows that $A[V](k)$ is an abelian group killed by p of order $\leq p^g$, and hence must be $(\mathbf{Z}/p)^i$ for some $0 \leq i \leq g$.

To pass to larger m, note that the abelian group $A(k)$ is divisible by Theorem 3.2.20. Hence, we have exact sequences

$$0 \to A[p](k) \subset A[p^m](k) \xrightarrow{p \cdot (-)} A[p^{m-1}](k) \to 0.$$

The left side is $(\mathbf{Z}/p)^i$. By induction, the right side is $(\mathbf{Z}/p^{m-1})^i$. It is then easy to see from the structure theory of abelian groups that the middle term must be $(\mathbf{Z}/p^m)^i$, for its p-torsion is $(\mathbf{Z}/p)^i$.

□

Kerneld **Lemma 3.2.24.** *Let R be a smooth algebra over a perfect field k of characteristic p. Then the kernel of $R \xrightarrow{d} \Omega^1_{R/k}$ is exactly the subring R^p of p-th powers in R.*

Proof. Viewing R as an R^p-algebra, the map $R \xrightarrow{d} \Omega^1_{R/k}$ is an R^p-linear map of finite R^p-modules whose kernel K certainly contains R^p. To show $R^p = K$, we may work étale locally on R^p. Moreover, note that the formation of K commutes with étale localization on R^p: any étale R^p-algebra is of the form S^p for an étale R-algebra S, and, in this case, the map obvious map $R \otimes_{R^p} S^p \to S$ is an isomorphism. We may then reduce to the case $R = k[x_1, ..., x_n]$, where one checks this by direct calculation. □

RelativeFrob **Remark 3.2.25.** For any k-algebra R, write $R^{(1)} = R \otimes_{k, Frob} R$, so $R^{(1)}$ is a k-algebra, and the Frobenius on R induces a k-linear map $Frob_{R/k} : R^{(1)} \to R$ called the *relative Frobenius*. Thus, we have the fundamental diagram

$$\begin{tikzcd}
k \arrow[r, "Frob_k"] \arrow[d, "f"] & k \arrow[d, "f^{(1)}"] \arrow[rdd, bend left, "f"] & \\
R \arrow[r, "Frob_k"] \arrow[rrd, bend right, "Frob_R"] & R^{(1)} \arrow[rd, "Frob_{R/k}"] & \\
& & R.
\end{tikzcd}$$

When R is reduced, the map $Frob_{R/k} : R^{(1)} \to R$ is injective with image exactly $R^p \subset R$. Thus, Lemma 3.2.24 can be reformulated as follows: if R is a smooth k-algebra and $S \to R$ is a map of k-algebras with $\Omega^1_{S/k} \to \Omega^1_{R/k}$ being the 0 map, then $S \to R$ factors uniquely as $S \to R^{(1)} \xrightarrow{Frob_{R/k}} R$. Of course, the analogous statement also holds true for non-affine schemes.

Chapter 4

Group schemes

4.1 Group schemes in characteristic 0

CartierGroup **Theorem 4.1.1** (Cartier)**.** *Let k be a field of characteristic* 0*, and let G/k be a group scheme of finite type. Then G is reduced and, thus, smooth.*

We give de Jong's proof from the Stacks Project. There is also an extremely simple proof by Oort.

ferentialnzd **Lemma 4.1.2.** *Let $k \to R$ be a map of $\mathbf{Q}$-algebras with R noetherian local. Fix some $f \in R$. Assume that the map $R \xrightarrow{df} \Omega^1_{R/k}$ is a direct summand. Then f is a nonzerodivisor on R.*

This is totally false in characteristic p: take $R = k[x]/(x^p)$ and $f = x$.

Proof. Choose a splitting of $R \xrightarrow{df} \Omega^1_{R/k}$, and let $\theta : R \to R$ be the corresponding derivation. In other words, we have $d(a) = \theta(a)df + c(a)$ for $c(a)$ in the kernel of the splitting. Note that $\theta(f) = 1$ by construction. As θ is a derivation, this gives $\theta(f^{n+1}) = (n+1)f^n$ for all $n \geq 0$.

Say $fg = 0$. We shall show that $g \in \cap_n f^n R$, and thus $g = 0$ by Krull's intersection theorem. Applying θ to $fg = 0$ gives $f\theta(g) + g = 0$, so $g \in fR$. By induction, assume we have shown $g \in f^n A$, so $g = f^n h$, and hence $f^{n+1}h = 0$. Applying θ gives

$$f^{n+1}\theta(h) + h \cdot (n+1) \cdot f^n = 0$$

Dividing by $n+1$ shows that $g = f^n h \in f^{n+1}R$, which finishes the proof by induction. □

othnessChar0 **Lemma 4.1.3.** *Let k be a field of characteristic* 0*. Let R be a finite type k-algebra with $\Omega^1_{R/k}$ locally free. Then R is smooth.*

Proof. As k is perfect, it is enough to show that $R_{\mathfrak{m}}$ is regular for each maximal ideal $\mathfrak{m}$. Note that $\Omega^1_{R_{\mathfrak{m}}/k}$ is free of some rank n (by assumption on R) and has fiber given by $\mathfrak{m}/\mathfrak{m}^2$. If $\mathfrak{m}/\mathfrak{m}^2 = 0$, there is nothing to prove (as R is automatically regular by the definition of regularity). If not, then choose some $f \in \mathfrak{m}$ that is nonzero modulo $\mathfrak{m}^2$. By Nakayama $R_{\mathfrak{m}} \xrightarrow{df} \Omega^1_{R_{\mathfrak{m}}/k}$ is a direct summand. Lemma 4.1.2 implies that f is a nonzero divisor, so, by general properties about regular rings, it is enough to show that $S = R_{\mathfrak{m}}/f$ is regular. But we have an exact sequence

$$(f)/(f^2) \xrightarrow{d} \Omega^1_{R_{\mathfrak{m}}/k} \otimes_R S \to \Omega^1_{S/k} \to 0.$$

The left map is split injective as it is the base change to f of $R_{\mathfrak{m}} \xrightarrow{df} \Omega^1_{R_{\mathfrak{m}}/k}$. It follows that $\Omega^1_{S/k}$ is free of rank $n-1$, so we win by induction. □

Proof of Theorem 4.1.1. By Lemma 4.1.3, it is enough to show that $\Omega^1_{G/k}$ is locally free. But this was verified in Proposition 1.3.11, so we are done.

□

4.2 Quotients

Fix a noetherian base ring k. Let G be a finite k-group scheme, and let X/k be a k-scheme. There is an evident notion of a G-action on X that one defines via the functor of points. Such an action is given by a map $act : G \times X \to X$ satisfying suitable actions.

Definition 4.2.1. An action of G on X is *free* if the map

$$G \times X \xrightarrow{(act, pr_2)} X \times X$$

is a closed immersion. If X is separated (as we shall always assume), this is the same[1] as asking that G-action on X is free on the functor of points, i.e., $G(T)$ acting on $X(T)$ has no stabilizers for a k-scheme T.

Given a k-scheme X with a G-action, we also wish to study when a sheaf F on X has a compatible G-action. Roughly, this means one must have transitive system of isomorphisms $\psi_g : F_x \simeq F_{g(x)}$ for each $x \in X$ and $g \in G$. To make this workable, we havet he following:

Definition 4.2.2. Let X be a k-scheme with a G-action. For a k-scheme S and an S-point $g \in G(S)$, write $a_g : X_S \to X_S$ for the induced action, so $a_{hg} = a_h \circ a_g$ for $h, g \in G(S)$. A G-equivariant quasi-coherent sheaf on X is a quasi-coherent sheaf F on X together with specified isomorphisms $a_g^* F_S \xrightarrow{\lambda_g} F_S$ for each scheme-theoretic point $g \in G(S)$ such that, for any pair $g, h \in G(S)$, the map $\lambda_{hg} : a_{hg}^* F_S \simeq F_S$ coincides with

$$a_{hg}^* F_S \simeq a_g^* a_h^* F_S \xrightarrow{a_g^* \lambda_h} a_g^* F_S \xrightarrow{\lambda_g} F_S.$$

In fact, it suffices to specify λ_g for the universal point $g = \mathrm{id}_G \in G(G)$ and to formulate this compatibility for the universal pair of points $(g, h) = (pr_1, pr_2) \in G(G \times G)$.

Example 4.2.3. Let A be an abelian variety over a field k. Let $G \subset A$ be a finite subgroup scheme. Then the translation action of G on A is free.

We shall use the following result about the existence of quotients:

thm:Quot **Theorem 4.2.4.** *Let G be a finite flat k-group scheme acting freely on a flat k-scheme X. Assume that any finite set of points of X are contained in an affine open. Then there exists a universal G-invariant map $\pi : X \to Y$, i.e., the maps $G \times X \xrightarrow{act} X \to Y$ and $G \times X \xrightarrow{pr_2} X \to Y$ coincide. Write $X/G = Y$, call π the quotient, and write $f : G \times X \to X/G$ for the induced map.*

1. *The quotient map $\pi : X \to X/G$ is an fppf G-torsor, i.e., the $X \to X/G$ is faithfully flat, and the map $G \times X \xrightarrow{(act, pr_2)} X \times_{X/G} X$ is an isomorphism.*
2. *The quotient map $\pi : X \to X/G$ is finite flat of degree $\mathrm{rank}(G)$.*
3. *We have $\mathcal{O}_{X/G} \simeq \pi_* \mathcal{O}_X^G := \ker(\pi_* \mathcal{O}_X \xrightarrow{pr_2^* - act^*} f_* \mathcal{O}_{G \times X})$.*
4. *If k is an algebraically closed field, then $X(k)/G(k) \simeq (X/G)(k)$, and $|X|/|G| \simeq |X/G|$.*
5. *If F is a quasi-coherent sheaf on X/G, then $\pi^* F$ is naturally a G-equivariant sheaf on X, and this construction gives an equivalence between quasi-coherent sheaves on X/G and G-equivariant sheaves on X.*
6. *There exists a "norm" map $Nm : \mathrm{Pic}(X) \to \mathrm{Pic}(X/G)$ such that $Nm \circ \pi^*$ is multiplication by the rank of G.*

[1] Indeed, the map $G \times X \to X \times X$ is proper as G is proper over k and X is separated. The freeness on the functor of points ensures this map is a monomorphism. A proper monomorphism is a closed immersion.

7. *Assume now that X is proper. Then X/G is also proper, and for any coherent sheaf F on X/G, we have*
$$\chi(X/G, F) = \mathrm{rank}(G) \cdot \chi(X, \pi^* F).$$

In fact, (1) implies the rest above.

Proof. The first (5) are standard. For (6), we note that if $f^* : A \to B$ is any finite locally free ring map corrsponding to a finite locally free map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes, then there is a norm map $B^* \to A^*$ defined by sending $b \in B^*$ to the determinant of the left-action of b on finite projective A-module B. The composition of this norm map with f^* is multiplication by $[B : A]$. This construction sheafifies in the étale topology in our global situation above to define a map $\pi_* \mathbf{G}_m \to \mathbf{G}_m$ of étale sheaves whose composition with the pullback $\mathbf{G}_m \to \pi_* \mathbf{G}_m$ is multiplication by the rank of G. Now π_* is acyclic on étale sheaves as it is a finite morphism. It follows that $H^i(X/G, \pi_* \mathbf{G}_m) \simeq H^i(X, \mathbf{G}_m)$. Taking $i = 1$ then gives the desired map.

For (7), we refer to Mumford. □

Example 4.2.5. Let A be an abelian variety over a field k. For each integer n, we have a finite subgroup scheme $G := A[n] \subset A$ that acts on A freely via translation. The map $[n] : A \to A$ is G-equivariant, and exhibits the target A as the quotient A/G. Indeed, the map $[n]$ factors uniquely as $A \to A/G \xrightarrow{g} A$ by the universal property of the quotient. Both $\pi : A \to A/G$ and $[n] : A \to A$ are finite flat and surjective, so the same holds true for $g : A/G \to A$. Comparing degrees then shows that g is an isomorphism.

Chapter 5

Dual abelian varieties

5.1 Properties of degree 0 line bundles

In this section, we fix an abelian variety A over an algebraically closed field k. Recall from Remark 3.2.4 that we have a homomorphism

$$\mathrm{Pic}(A) \to \mathrm{Hom}(A, \mathrm{Pic}_{A/k}) \quad \text{via} \quad L \mapsto \phi_L := (x \mapsto t_x^* L \otimes L^{-1}).$$

Thus, $\phi_L = 0$ if and only if for any scheme theoretic point $x : T \to A$, the bundle $t_x^*(L_T) \otimes L_T^{-1}$ is pulled back from T. This motivates the following:

DefPic0 **Definition 5.1.1.** Let $\mathrm{Pic}^0(A) \subset \mathrm{Pic}(A)$ denote the kernel of the map $\mathrm{Pic}(A) \to \mathrm{Hom}(A, \mathrm{Pic}_{A/k})$. In other words, a line bundle L lies in $\mathrm{Pic}^0(A)$ if and only for any $x \in A(k)$, there exists an isomorphism $t_x^* L \simeq L$.

Pic00 **Lemma 5.1.2.** *For any line bundle $L \in \mathrm{Pic}(A)$ and any $x \in A(k)$, the line bundle $t_x^* L \otimes L^{-1}$ lies in $\mathrm{Pic}^0(A)$.*

In other words, ϕ_L can be viewed as a map $\mathrm{Pic}(A)/\mathrm{Pic}^0(A) \to \mathrm{Hom}(A(k), \mathrm{Pic}^0(A))$.

Proof. This follows from the theorem of the square: for any $y \in A(k)$, we have

$$t_y^*(t_x^* L \otimes L^{-1}) \simeq t_{x+y}^* L \otimes t_y^* L^{-1} \simeq t_x^* L \otimes t_y^* L \otimes L^{-1} \otimes t_y^* L^{-1} \simeq t_x^* L \otimes L^{-1},$$

where we use the the theorem in the second isomorphism. ☐

Pic01 **Lemma 5.1.3.** *A line bundle L lies in $\mathrm{Pic}^0(A)$ if and only if $\Lambda(L) := m^* L \otimes pr_1^* L^{-1} \otimes pr_2^* L^{-1}$ is trivial on $A \times A$.*

Proof. If $K(L) \subset A$ denotes the maximal closed subscheme over which $\Lambda(L)$ is pulled back from A via pr_1, then we have seen in Corollary 3.2.6 that $K(L) = A$ exactly that $\phi_L = 0$. It follows that $\phi_L = 0$ exactly when $\Lambda(L)$ is pulled back from A via pr_1. But, in the latter situation, it is easy to see that $\Lambda(L)$ must be trivial using the section $A \xrightarrow{x \mapsto (x,e)} A \times A$ of pr_1. ☐

Pic02 **Lemma 5.1.4.** *If $L \in \mathrm{Pic}^0(A)$, then for any maps $x, y : S \to A$, we have $x^* L \otimes y^* L \simeq (x+y)^* L$ on S. In particular, $[n]^* L \simeq L^n$.*

In contrast, for L ample, we showed in Corollary 3.2.2 that $[n]^* L \simeq L^{n^2}$. Thus, $[n]^*$ behaves linearly on $\mathrm{Pic}^0(A) \subset \mathrm{Pic}(A)$, and quadratically on the ample cone in $\mathrm{Pic}(A)$.

Proof. For $L \in \mathrm{Pic}^0(A)$, Lemma 5.1.3 gives an isomorphism $m^* L \simeq pr_1^* L \otimes pr_2^* L$. Pulling this back along $(x, y) : S \to A \times A$ gives the first part of the claim; the second follows immediately by induction. ☐

Pic03 **Lemma 5.1.5.** *For any $L \in \mathrm{Pic}(A)$, we have $[n]^*L \simeq L^{n^2} \otimes M$ for $M \in \mathrm{Pic}^0(A)$.*

Proof. Corollary 3.2.2 shows that

$$[n]^*L \simeq L^{\frac{n^2+n}{2}} \otimes [-1]^*L^{\frac{n^2-n}{2}} \simeq L^{n^2} \otimes \big(L \otimes [-1]^*L\big)^{\frac{n^2-n}{2}}.$$

It is thus enough to show that $L \otimes [-1]^*L^{-1} \in \mathrm{Pic}^0(A)$ for any line bundle L. Choose a point $x \in A(k)$. Translating by x gives

$$\begin{aligned} t_x^*(L \otimes [-1]^*L^{-1}) &\simeq t_x^*L \otimes [-1]^*t_{-x}^*L^{-1} \\ &\simeq t_x^*L \otimes [-1]^*\big(L \otimes t_{-x}^*L^{-1}\big) \otimes [-1]^*L^{-1}. \end{aligned}$$

As the second term on the right lies in $\mathrm{Pic}^0(A)$, Lemma 5.1.4 simplifies the above expression to give

$$t_x^*(L \otimes [-1]^*L^{-1}) \simeq t_x^*L \otimes L^{-1} \otimes t_{-x}^*L \otimes [-1]^*L^{-1}.$$

Applying the theorem of the square to the first three terms gives

$$t_x^*(L \otimes [-1]^*L^{-1}) \simeq L \otimes [-1]^*L^{-1},$$

as wanted. □

Pic04 **Lemma 5.1.6.** *If $L \in \mathrm{Pic}(A)$ has finite order, then $L \in \mathrm{Pic}^0(A)$.*

Proof. Consider the homomorphism

$$\mathrm{Pic}(A) \xrightarrow{L \mapsto \phi_L} \mathrm{Hom}(A, \mathrm{Pic}_{A/k}).$$

Now if L has finite order, then there is some n such that $\phi_{L^n} = n \cdot \phi_L$ is trivial. But this means that $\phi_L(nx) = n\phi_L(x)$ is trivial for all $x \in A(k)$. As $A(k)$ is n-divisible, it follows that $\phi_L = 0$, as wanted. □

Pic05 **Lemma 5.1.7.** *If S is a connected k-scheme of finite type, then for any $L \in \mathrm{Pic}(S \times A)$ and any points $s, t \in S(k)$, we have $L_s \otimes L_t^{-1} \in \mathrm{Pic}^0(A)$.*

In other words, if two line bundles on A are members of the same connected family, then one lies in Pic^0 exactly when the other does.

Proof. By shrinking S, we may assume $L|_{S \times \{e\}}$ is trivial. Also, by replacing L with $L \otimes pr_1^*L_s^{-1}$, we may assume L_s is trivial for a fixed $s \in S(k)$. We must check that $L_t \in \mathrm{Pic}^0(A)$ for all $t \in S(k)$; equivalently, we must show that $\Lambda(L_t)$ is trivial on $A \times A$ for all $t \in S(k)$. We shall prove this by putting it in a family. Thus, on $S \times A \times A$, consider the bundle $\mu^*L \otimes pr_{12}^*L^{-1} \otimes pr_{13}^*L^{-1}$, where $\mu(s, a, b) = (s, a + b)$. In fact, this is also simply $\Lambda(L)$ if we view L as a line bundle on the abelian S-scheme $A \times S \to S$. The restriction of this line bundle to $\{s\} \times A \times A$ is $\Lambda(L_s)$, and hence is trivial by hypothesis on $s \in S(k)$. The restrictions to $S \times \{e\} \times A$ and $S \times A \times \{e\}$ are trivial simply because $L|_{S \times \{e\}}$ is trivial. The theorem of the cube implies that $\mu^*L \otimes pr_{12}^*L^{-1} \otimes pr_{13}^*L^{-1}$ is trivial. Taking fibers over $t \in S(k)$ then implies that $\Lambda(L_t)$ is trivial, as wanted. □

Pic06 **Lemma 5.1.8.** *Say $L \in \mathrm{Pic}^0(A)$ is not trivial. Then $H^i(A, L) = 0$ for all i.*

Proof. We first observe that $H^0(A, L) = 0$. Indeed, if not, then L is effective. But, as $L \in \mathrm{Pic}^0(A)$, we have $[-1]^*L \simeq L^{-1}$ by Lemma 5.1.4, so L^{-1} is also effective. Lemma 3.1.1 then implies L is trivial, contradicting our assumption.

By induction, hoose the minimal $k > 0$ where we do not yet know $H^k(A, L) = 0$. The composition

$$A \xrightarrow{a \mapsto (a,e)} A \times A \xrightarrow{m} A$$

is the identity, and hence the identity on $H^k(A, L)$ factors over $H^k(A \times A, m^*L)$. As $L \in \mathrm{Pic}^0(A)$, we have $m^*L \simeq pr_1^*L \otimes pr_2^*L$. By Kunneth, we get

$$H^k(A \times A, m^*L) \simeq \oplus_{i+j=k} H^i(A, L) \otimes H^j(A, L).$$

Now the terms for $i = 0$ or $j = 0$ vanish as L has no sections. The remaining terms vanish by induction, so $H^k(A \times A, m^*L) = 0$. As the identity on $H^k(A, L)$ factors through this group, we get $H^k(A, L) = 0$ as well. $\square$

prop:Pic07 **Proposition 5.1.9.** *Assume $L \in \mathrm{Pic}(A)$ is ample. Then $\phi_L : A(k) \to \mathrm{Pic}^0(A)$ is surjective.*

Proof. Fix some $M \in \mathrm{Pic}^0(A)$. Assume towards contradiction that M does not lie in the image of ϕ_L. Consider the line bundle

$$K = \Lambda(L) \otimes pr_1^* M^{-1} \simeq m^*L \otimes pr_1^*(L^{-1} \otimes M^{-1}) \otimes pr_2^* L^{-1}.$$

For each $x \in A(k)$, we have

$$K|_{A \times \{x\}} \simeq t_x^*L \otimes L^{-1} \otimes M^{-1} \quad \text{and} \quad K|_{\{x\} \times A} = t_x^* L \otimes L^{-1}.$$

As M does not lie in the image of ϕ_L, it follows that $K|_{A \times \{x\}}$ is a non-trivial bundle in $\mathrm{Pic}^0(A)$ for all $x \in A(k)$. Lemma 5.1.8 and the formal functions theorem then show that $R^i pr_{2,*}K \simeq 0$ for all i, and hence $H^i(A \times A, K) = 0$ for all i by the Leray spectral sequence for pr_2.

On the other hand, consider the Leray spectral sequence for pr_1. Lemma 5.1.8 and the formal functions theorem again show that $\mathrm{supp}(R^i pr_{1,*}K) \subset K(L)$ for all i. As L is ample, the subscheme $K(L)$ is finite, so $R^i pr_{1,*}K$ is supported on a zero dimensional subscheme of A (and is the direct sum of its stalks). Such sheaves have no cohomology, so the Leray spectral sequence for pr_1 degenerates to give

$$H^i(A \times A, K) \simeq H^0(A, R^i pr_{1,*}K).$$

As the left side is zero for all i, the sheaf appearing on the right must also be zero for all i. Thus, we have $R^i pr_{1,*}K = 0$ for all i. By semicontinuity, this implies $H^i(A, K|_{\{x\} \times A}) = 0$ for all i and all $x \in A(k)$. But taking $x = e$, we have $K|_{\{e\} \times A} \simeq \mathcal{O}_A$, which clearly has a nonzero H^0, giving a contradiction. $\square$

Pic08 **Remark 5.1.10.** Proposition 5.1.9 and Theorem 4.2.4 give a bijection $(A/K(L))(k) \simeq A(k)/K(L)(k) \simeq \mathrm{Pic}^0(A)$. In particular, the group $\mathrm{Pic}^0(A)$ has the the same finiteness features as the set of k-points of an abelian variety. For example, it follows that there are only finitely many n-torsion line line bundles on A (using Lemma 5.1.6).

5.2 Construction of dual abelian variety

cons:DualAV Fix an abelian variety A over a field k. Our goal is to construct the dual abelian variety A^t with the property that there is a natural isomorphism $A^t(k) \simeq \mathrm{Pic}^0(A)$. More precisely, we shall show:

thm:DualAV **Theorem 5.2.1.** *Consider the category of triples (S, L, ι), where S is a k-scheme, $L \in \mathrm{Pic}(S \times A)$ is a line bundle, $\iota : L|_{S \times \{e\}} \simeq \mathcal{O}_S$ is a trivialization, and one has $L|_{\{s\} \times A}$ has degree 0 for all geometric points s of S. This category has a final object $(A^t, \mathcal{P}, \tau)$. Pointing A^t by the natural triple $(\mathrm{Spec}(k), \mathcal{O}_A, \mathrm{std})$ gives a base point $e \in A^t(k)$; the pair (A^t, e) is an abelian variety.*

The universal property identifies $A^t(k)$ with isomorphism clases of pairs (M, ι) where M has degree 0 and ι is a trivialization of M at the origin; as two different ι's differ by a scalar, their difference can be lifted to an automorphism of M, so it follows that $A^t(k)$ is also identified with $\mathrm{Pic}^0(A)$. Thus, Theorem 5.2.1 is endowing $\mathrm{Pic}^0(A)$ with the structure of (the k-points of) an abelian variety. We shall use the description arising from Proposition 5.1.9.

Fix an ample line bundle L on A, so $K(L) \subset A$ is finite. Let $\Lambda(L)$ be the Mumford bundle on $A \times A$. Note that $K(L) \times \{e\}$ is a subgroup scheme of $A \times A$, and hence acts freely on the latter via translation. Our first observation is that this action lifts to $\Lambda(L)$.

LambdaLEquiv **Lemma 5.2.2.** *The line bundle $\Lambda(L)$ is naturally $K(L)$-equivariant for the translation action of $K(L)$ on $A \times A$. More precisely, an equivariant structure is uniquely determined once one fixes an isomorphism $L|_e \simeq k$.*

Proof. For any k-scheme T, wse a subscript of T to denote base change along $T \to \mathrm{Spec}(k)$, so $\Lambda(L)_T = \Lambda(L_T) \in \mathrm{Pic}((A \times A)_T)$ is the corresponding Mumford bundle.

Given a T-valued point $x \in K(L)(T)$, the corresponding automorphism of $(A \times A)_T$ is $t_{x,e}$, where $(x, 0) \in (A \times A)_T(T)$ is the corresponding point. We must thus supply an isomorphism $t_{x,e}^*\Lambda(L) \simeq \Lambda(L)$ for each such x that are compatible with addition on $K(L)$.

To get an isomorphism, observe that we have

$$t_{x,e}^*\Lambda(L_T) = t_{x,e}^*\big(m_T^*L_T \otimes pr_1^*L_T^{-1} \otimes pr_2^*L_T^{-1}\big) = m_T^*t_x^*L_T \otimes pr_1^*t_x^*L_T^{-1} \otimes pr_2^*L_T^{-1}.$$

As $x \in K(L)$, we have some isomorphism $t_x^*L_T \simeq L_T \otimes M_0$ for some line bundle M_0 pulled back from T. Any such choice determines an isomorphism

$$t_{x,e}^*\Lambda(L_T) = \Lambda(L_T) \otimes m_T^*M_0 \otimes pr_1^*M_0^{-1} \simeq \Lambda(L_T),$$

where we use that, since M_0 is pulled back from T, it pulls back the same way along both m and pr_1. To fix this isomorphism, it is enough to fix it after pullback along $i : A_T \times_T \{e\}_T \hookrightarrow (A \times A)_T$: restriction of functions along i is bijective by Kunneth. But we have a canonical isomorphism

$$i^*\Lambda(L)_T \simeq i^*m_T^*L_T \otimes i^*pr_1^*L_T^{-1} \otimes i^*pr_2^*L_T^{-1} \simeq L_T \otimes L_T^{-1} \otimes \underline{V} \simeq \underline{V},$$

where $\underline{V}$ denotes the trivial vector bundle on T with global sections $V = e^*(L)$. Similarly, we have a canonical isomorphism

$$i^*t_{x,e}^*\Lambda(L)_T \simeq t_x^*i^*\Lambda(L)_T \simeq t_x^*\underline{V}.$$

Thus, we fix the isomorphism $t_{x,e}^*\Lambda(L)_T \simeq \Lambda(L)_T$ by requiring that it agree with the standard isomorphism $t_x^*\underline{V} \simeq \underline{V}$ on application of i^*. It is then easy to see that we have the desired transitivity to define a $K(L)$-equivariant structure on $\Lambda(L)$. □

We can now define the pair $(A^t, \mathcal{P}, \iota)$ that shall eventually be shown to have the universal property in Theorem 5.2.1.

Construction 5.2.3 (Construction of the Poincare bundle)**.** Let $\pi : A \to A^t = A/K(L)$ be the quotient, so $A^t \times A \simeq (A \times A)/(K(L) \times \{e\})$. The $K(L)$-equivariance of $\Lambda(L)$ constructed in Lemma 5.2.2 then allows us to descend the line bundle $\Lambda(L)$ to a line bundle $\mathcal{P}$ on $A^t \times A$ via Theorem 4.2.4. Finally, the proof of Lemma 5.2.2 also gives a $K(L)$-equivariant isomorphism $i^*\Lambda(L) \simeq \underline{V}$ where $i : A \times \{e\} \hookrightarrow A \times A$ is the obvious inclusion and $V = e^*(L)$ is the fibre of L at the origin. It follows that we obtain a natural isomorphism $\mathcal{P}|_{A^t \times \{e\}} \simeq \underline{V}$. Fixing a trivialization of V then defines the desired trivialization ι_{univ}.

Proof of Theorem 5.2.1. Fix a triple (S, F, ι) as in the statement. This gives us a line bundle $pr_{23}^*\mathcal{P} \otimes pr_{13}^*F^{-1}$ on $S \times A^t \times A$. Let $\Gamma_S \subset S \times A^t$ be the maximal closed subscheme over which this line bundle pulled back along pr_{13}. We claim that pr_1 induces an isomorphism $\Gamma_S \simeq S$.

Let us see how to use this claim to prove the theorem first. By construction, we know that $M := (pr_{23}^*\mathcal{P} \otimes pr_{13}^*F^{-1})|_{\Gamma_S \times A}$ is pulled back from some $N \in \mathrm{Pic}(\Gamma_S)$. Using the obvious section defined by $\Gamma_S \subset S \times A^t \simeq S \times A^t \times \{e\} \subset S \times A^t \times A$ and the fact that both pr_{23}^*P and pr_{13}^*L come equipped with preferred trivializations on $S \times A^t \times \{e\}$, we conclude that N is the trivial line bundle on Γ_S that comes equipped with a preferred trivialization. In other words, we have a canonical isomorphism $p_{23}^*\mathcal{P}|_{\Gamma_S \times A} \simeq pr_{13}^*F|_{\Gamma_S \times A}$ that is compatible with the given trivializations over $\Gamma_S \times \{e\}$. Now, granting the claim that pr_1 induces an isomorphism $\Gamma_S \simeq S$, it immediately follows that the triple (S, F, ι) is the pullback of $(A^t, \mathcal{P}, \iota_{univ})$ along the map $S \simeq \Gamma_S \xrightarrow{pr_2} A^t$.

We now prove Γ_S maps isomorphically to S via pr_1.

1. Preliminary reductions: As the formation of Γ_S is compatible with base change on S, we immediately reduce to the case where k is algebraically closed, and $S = \mathrm{Spec}(B)$ is an artinian local ring; write $s \in S$ for the closed point. Moreover, as the construction of Γ_S does not involve the choice of trivializations, we can ignore the trivializations from here on. Now the line bundle $F|_{\{s\}\times A}$ has degree 0, and hence also has the form $P|_{\{b\}\times A}$ for some $b \in A^t(k)$ by Proposition 5.1.9. In particular, replacing F with $p_2^* F|_{\{s\}\times A}^{-1}$ does not change the subscheme Γ_S. After making this replacement, we may thus assume that $F|_{\{s\}\times A}$ is the trivial bundle.

2. Freeness of the cohomology of M via pr_{13}: We first claim that $M|_{\{s\}\times A^t\times\{a\}}$ is a degree 0 line bundle on the abelian variety A^t for all $a \in A(k)$: this is true for $a = e$, and thus follows by Lemma 5.1.7. To identify this fibre, note that the pullback line bundle $\pi^*\big(M|_{\{s\}\times A^t\times\{a\}}\big)$ on A is identified with $\Lambda(L)|_{A\times\{a\}} \simeq t_a^* L \otimes L^{-1}$. By the ampleness of L, the set of all $a \in A(k)$ where the latter bundle is trivial is finite (Lemma 3.2.8). The pullback $\mathrm{Pic}(A^t) \xrightarrow{\pi^*} \mathrm{Pic}(A)$ has finite fibres Remark 5.1.10, so there are only finitely many $a \in A(k)$ where $M|_{\{s\}\times A^t\times\{a\}}$ is the trivial line bundle on A^t. By Lemma 5.1.8 and semicontinuity, the support of the coherent sheaf $R^i pr_{13,*}M$ on $S \times A$ is finite. The Leray spectral sequence degenerates to give $H^i(S\times A^t\times A, M) \simeq H^i(S\times A, R^i pr_{13,*}M)$. On the other hand, by the projection formula, we also have $R^i pr_{13,*}M \simeq (R^i pr_{13,*} pr_{23}^* \mathcal{P}) \otimes F^{-1}$. As the support of this sheaf is finite, we can trivialize F in a small neighbourhood of the support of this sheaf to ignore it. We conclude then there is a noncanonical isomorphism
$$H^i(S\times A^t\times A, M) \simeq H^i(S\times A^t\times A, pr_{23}^* P) \simeq B\otimes_k H^i(A^t\times A, P).$$
In particular, these cohomology groups are all free B-modules.

3. The vanishing of most of the pushforwards of M via pr_{12}: For any $a \in A^t(k) \simeq \mathrm{Pic}^0(k)$, the line bundle $M|_{\{s\}\times\{a\}\times A}$ is trivial exactly when $a = e$ is the origin. Thus, the sheaf $R^i pr_{12,*}M$ is supported set-theoretically at $e \in A^t(k)$. Let $R = \mathcal{O}_{A,e}\otimes_k B$ be the local ring at $(s,e) \in S\times A$, so each $R^i pr_{12,*}M$ is can be viewed an R-module. By Proposition 2.2.4, there exists perfect complex
$$K^\bullet := \Big(K^0 \to K^1 \to \dots \to K^g\Big)$$
of R-modules that universally computes (the pullback to R of) $R^i pr_{12,*}M$. Each homology of $K^\bullet$ is an artinian R-module. Now recall the following:

Lemma 5.2.4. *Let $\mathcal{O}$ be a regular local ring of dimension g. Let $M^0 \to M^1 \to \dots \to M^g$ be a perfect complex over $\mathcal{O}$ with each $H^i(M^\bullet)$ being artinian. Then $H^i(M^\bullet) = 0$ for $i \neq g$.*

Proof. Pick the smallest i with $H^i(M^\bullet) \neq 0$. We shall show that $H^{i-g}(M^\bullet \otimes^L_{\mathcal{O}} k)$ is nonzero, which clearly gives a contradiction if $i < g$ as $M^\bullet \otimes^L_{\mathcal{O}} k$ has no cohomology in negative degrees. Consider the canonical exact triangle
$$H^i(M^\bullet)[-i] \to M^\bullet \to \tau^{\geq i+1} M^\bullet.$$
Noting that $-\otimes^L_{\mathcal{O}} k$ carries $D^{\geq j}$ to $D^{\geq j-g}$ by the existence of the Koszul resolution for k, it follows by tensoring above triangle with k and looking at H^{i-g} that
$$\mathrm{Tor}_g^{\mathcal{O}}(H^i(M^\bullet), k) := H^{i-g}(H^i(M^\bullet)\otimes^L_{\mathcal{O}} k) \simeq H^{i-g}(M^\bullet \otimes^L_{\mathcal{O}} k).$$
Now $H^i(M^\bullet, k)$ is a nonzero artinian $\mathcal{O}$-module, so the left side is always nonzero: one can see this by filtering this artinian module in terms of copies of k and using that $\mathrm{Tor}_g^{\mathcal{O}}(-,k)$ is left-exact (as it is the highest left-derived functor of Tor^0). Thus, the right side above is always nonzero, as wanted. ☐

Applying this lemma to $K^\bullet$, viewed as a perfect complex over the regular local ring $\mathcal{O}_{A,e} \subset R$, implies that $H^i(K^\bullet\bullet) = 0$ for $0 \leq i < g$. It follows that $R^i pr_{12,*}M = 0$ for $i < g$, and (the stalk at (s,e) of) $R^g pr_{12,*}M$ is identified with $N = H^g(K^\bullet)$. Taking global sections of $R^g pr_{12,*}M$, using that it's supported set-theoretically at a single point and the Leray spectral sequence, it follows from (2) that the R-module N is free when regarded as a B-module via $B \subset R$.

4. Showing $\Gamma_S \to S$ is a homeomorphism: Γ_S is contained in the support of $\oplus_i R^i pr_{12,*}M$ as M is trivial over Γ_S. It follows from (3) that Γ_S is set-theoretically contained $\{(s,e)\} \subset S \times A^t$. It is also clear that $(s,e) \in \Gamma_S$, so Γ_S is set-theoretically the single point $\{(s,e)\} \subset S \times A^t$, which certainly maps homeomorphically to S (which is set-theoretically also just a single point $\{s\}$).

5. Finding a candidate scheme-structure Γ_S: Let $K^{\bullet,\vee}$ be the complex obtained by dualizing the complex $K^\bullet$ from (3) over the ring R. Then the same reasoning used in (3) shows that there is an exact sequence

$$0 \to K^{g,\vee} \to K^{g-1,\vee} \to ... \to K^{0,\vee} \to Q \to 0,$$

i.e., the complex is exact except on the right, and highest homology group Q is an artinian R-module. By the universality of $K^\bullet$, we know that $\mathrm{Hom}_R(Q,T) \simeq H^0(\mathrm{Spec}(T) \times A, M|_{\mathrm{Spec}(T)\times A})$ for any R-algebra T. Applying this to $T = k$ gives $\mathrm{Hom}_R(Q,k) \simeq H^0(\{s\} \times \{e\} \times A, M|_{\{s\}\times\{e\}\times A}) \simeq H^0(A, \mathcal{O}_A)$, which is a 1-dimensional vector space. It follows by Nakyama that Q is a cyclic R-module, so we can write $Q = R/I$ for some ideal I. We claim that $V(I) = \Gamma_S$. Note that this is the same construction that appeared in Proposition 3.1.2.

6. The containment $\Gamma_S \subset V(I)$: let $J \subset R$ be the ideal defining Γ_S. We must check that $I \subset J$; equivalently, we must show that $\mathrm{Hom}_R(R/I, R/J) \simeq R/J$. But $R/I = Q$, so the left side is $\mathrm{Hom}_R(Q, R/J)$, which calculates H^0 of the pullback of M along $\Gamma_S \times A \subset S \times A^t \times A$. But this restriction is trivial, so $H^0(\Gamma_S \times A, M|_{\Gamma_S\times A}) \simeq H^0(\Gamma_S \times A, \mathcal{O}_{\Gamma_S\times A}) \simeq H^0(\Gamma_S, \mathcal{O}_{\Gamma_S}) \simeq R/J$, as wanted.

7. The containment $V(I) \subset \Gamma_S$: as $\Gamma_S \subset S \times A^t$ is the maximal closed subscheme of $S \times A^t$ over which M is pulled back from Γ_S (and thus trivial, as Γ_S is artinian), we must show that the restriction of M to $\mathrm{Spec}(R/I) \times A \subset S \times A \times A^t$ is trivial. To check this, it suffices to show that the adjunction map

$$\eta : H^0(V(I) \times A, M|_{V(I)\times A}) \otimes_{R/I} \mathcal{O}_{V(I)\times A} \to M$$

is an isomorphism. This is a map between line bundles on $V(I) \times A$, so it suffices to check surjectivity. But surjectivity can be tested after tensoring along $R/I \to k$. Now, after this base change, the line bundle M becomes trivial (as the closed point of $\mathrm{Spec}(R)$ certainly sits in Γ_S). On the other hand, we also have base change for the left hand side by the universal property of $Q \simeq R/I$:

$$H^0(V(I)\times A, M|_{V(I)\times A}) \otimes_{R/I} R/k \simeq \mathrm{Hom}_R(Q, R/I) \otimes_{R/I} R/k \simeq \mathrm{Hom}_R(Q, R/k) \simeq H^0(\{(s,e)\}\times A, M|_{\{(s,e)\}\times A}).$$

Thus, after base change along $R/I \to k$, the map η becomes the standard map

$$H^0(A, \mathcal{O}_A) \otimes_k \mathcal{O}_A \to \mathcal{O}_A,$$

which is certainly an isomorphism.

Thus, with (5) and (6), we have shown $\Gamma_S = V(I)$ where $I \subset R$ was the annihilator of $Q \simeq R/I$. It remains to to show that the composite $B \to R \to R/I$ is an isomorphism.

8. The composite $B \to R \to R/I$ is injective: As $K^{\bullet,\vee}$ resolves Q, it follows any R-linear functor applied to $K^{\bullet,\vee}$ has homology annihilated by I. In particular, I also annihilates the homology of our original complex $K^\bullet = (K^{\bullet,\vee})^\vee$, and thus I annihilates the module $N = H^g(K^\bullet)$. As N was free over $B \subset R$ by (2), it follows that $I \cap B = 0$. In other words, the natural composition $B \to R \to R/I$ is injective.

9. The composite $B \to R \to R/I$ is surjective: it is enough to show surjectivity after reducing modulo the maximal ideal of B. In other words, as the formation of Γ_S (and thus R/I) is compatible with base change, we must show that $k \simeq R/I$ if $S = \mathrm{Spec}(k)$. In this case, we have $M = \mathcal{P}$, so we must check that the maximal closed subscheme $\Gamma \subset A^t$ over which $\mathcal{P}$ is pulled back from Γ coincides with the origin $\mathrm{Spec}(k) \xrightarrow{e} A^t$. By compatibility with base change, the preimage of Γ under $A \xrightarrow{\phi_L} A^t$ coincides with the maximal closed subscheme $\widetilde{\Gamma} \subset A$ such that $\Lambda(L)|_{\widetilde{\Gamma}\times A}$ is pulled back from $\widetilde{\Gamma}$. But this was, by definition, given by $K(L) \subset A$. It follows that $\widetilde{\Gamma} = K(L)$, and thus $\Gamma = K(L)/K(L) \simeq \{e\}$, as wanted.

□

We record two corollaries of the proof.

CohPoincare **Corollary 5.2.5.** *We have $H^i(A^t \times A, \mathcal{P}) = 0$ if $i \neq g$ and $H^g(A^t \times A, \mathcal{P}) \simeq k$. In fact, we have $R^i pr_{1,*}\mathcal{P} = 0$ for $i < g$ an $R^g pr_{1,*}\mathcal{P} = k$.*

Proof. We apply the proof of Theorem 5.2.1 with $S = \mathrm{Spec}(k)$. In the notation of the proof, we are trying to show that $K^\bullet$ is a resolution of k. The proof above already shows that $Q \simeq k$, and thus $K^{\bullet,\vee}$ is a resolution of k (placed in degree g) over R. As R is a regular local ring of dimension g, we can also resolve k by a Koszul complex $M^\bullet$ of dimension. Any two free resolutions of k are homotopy-equivalent, so $M \overset{htpy}{\simeq} K^{\bullet,\vee}$. By duality, we get $M^\vee \overset{htpy}{\simeq} K^\bullet$. As Koszul complexes are self-dual up to a shift, it follows $H^i(M^\vee) = 0$ if $i \neq g$ and $H^g(M^\vee) = k$. The same then holds true for $K^\bullet$. As $K^\bullet$ is a complex of R-modules that universally computes the cohomology of $\mathcal{P}|_{\mathrm{Spec}(R)\times A}$, the claim follows by semicontinuity as $R^i pr_{1,*}\mathcal{P}$ is supported inside $\mathrm{Spec}(R) \subset A^t$. □

CohO **Corollary 5.2.6.** *$H^1(A, \mathcal{O}_A)$ has dimension g, and $H^i(A, \mathcal{O}_A) \simeq \wedge^i H^1(A, \mathcal{O}_A)$ has dimension $\binom{g}{i}$.*

Proof. The proof of Corollary 5.2.5 gives a homotopy-equivalence between $K^\bullet$ the standard Koszul complex resolving k over the regular local ring R. Base changing along $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(R)$ and using the universal property defining $K^\bullet$ then shows that $H^*(A, \mathcal{O}_A)$ is computed by the mod $\mathfrak{m}$ reduction of the Koszul complex on R defining k. The claim now follows from standard properties of the Koszul complex over R: the i-th term is free of rank $\binom{g}{i}$ as it is $\wedge^i$ of the 1-st term, and all differentials are zero modulo $\mathfrak{m}$. □

5.3 Duality

The goal of this section is to explain why the association $A \mapsto A^t$ gives a duality on the category of abelian varieites, i.e., it is contravariantly functorial and satisfies biduality $A \simeq (A^t)^t$.

DualIsogeny **Construction 5.3.1** (The dual map)**.** Let $f : A \to B$ be a homomorphism of abelian varieties over a field k. Then we get an induced map $g : (\mathrm{id}, f) : B^t \times A \to B^t \times B$. If $\mathcal{P}_B \in \mathrm{Pic}(B^t \times B)$ denotes the Poincare bundle, then we get an induced line bundle $g^*\mathcal{P}_B \in \mathrm{Pic}(B^t \times A)$. For each (geometric) point b of B^t, the restriction $g^*\mathcal{P}_B|_{\{b\}\times A}$ is a degree 0 line bundle on A. Moreover, as f is a homomorphism of abelian varieties, the restriction $g^*\mathcal{P}_B|_{B^t\times\{e_A\}} \simeq \mathcal{P}_B|_{B^t\times\{e_B\}}$ comes equipped with a preferred trivialization ι. The triple $(B^t, g^*\mathcal{P}_B, \iota)$ then defines a map $f^t : B^t \to A^t$ via the universal property in Theorem 5.2.1. By definition, we have an identification

$$(f^t, \mathrm{id})^*\mathcal{P}_A \simeq (\mathrm{id}, f)^*\mathcal{P}_B$$

of line bundles on $B^t \times A$. For future reference, we write this line bundle as $Q(f)$.

Proposition 5.3.2 (Functoriality of duality)**.** *Let $f : A \to B$ be a finite surjective homomorphism of abelian varieties over a field k. Then f^t is also finite surjective, and $\deg(f) = \deg(f^t)$.*

Proof. We first explain why f^t is finite (and thus necessarily surjective for dimension reasons). It is enough to show that if $k = \overline{k}$, then $F := (f^t)^{-1}(e) \in A^t(k)$ is finite. By the construction above, F is the set of all $b \in B^t(k)$ with $(g^*\mathcal{P}_B)|_{\{b\}\times A}$ being trivial. But the latter line bundle identifies with $f^*\big(\mathcal{P}_B|_{\{b\}\times B}\big)$. Now $f^* : \mathrm{Pic}(B) \to \mathrm{Pic}(A)$ has a finite kernel by a norm argument[1]. As the map $b \in B^t(k) \mapsto \mathcal{P}|_{\{b\}\times B} \in \mathrm{Pic}^0(B)$ is bijective, it follows that there are only finitely many such b, so F is indeed finite.

[1] As in the proof of Theorem 4.2.4, there is a norm map $Nm : \mathrm{Pic}(A) \to \mathrm{Pic}(B)$ that is a left-inverse to f^* up to multiplication by $\deg(f)$. In particular, the kernel of f^* sits inside the subgroup of all $\deg(f)$-torsion line bundles. As we have already seen all such line bundles lie in $\mathrm{Pic}^0(B)$ (see Lemma 5.1.6), the claim follows from the finiteness of torsion points in $\mathrm{Pic}^0(B)$ (see Remark 5.1.10).

To understand degrees, consider the line bundle $Q \simeq (f^t, \mathrm{id})^* \mathcal{P}_A \simeq (\mathrm{id}, f)^* \mathcal{P}_B$ from Construction 5.3.1. Applying Theorem 4.2.4 (6) to both the isogenies (f^t, id) and (id, f), we learn that

$$\chi(A^t \times A, \mathcal{P}_A) \cdot \deg(f^t) = \chi(B^t \times A, Q) = \deg(f) \cdot \chi(B^t \times B, \mathcal{P}_B),$$

which gives the claim. If g denotes the common dimension of A and B, then $\chi(A^t \times A, \mathcal{P}_A) = (-1)^g$ and $\chi(B^t \times B, \mathcal{P}_B) = (-1)^g$ by Corollary 5.2.5, so the above equality simplifies to give $\deg(f) = \deg(f^t)$. $\square$

Biduality

Theorem 5.3.3 (Biduality)**.** *Let A be an abelian variety over a field k. Then there is a canonical isomorphism $A \to (A^t)^t$ of abelian varieties.*

Proof. Consider the Poincare bundle $\mathcal{P}_A$ on $A^t \times A$. Then $\mathcal{P}_A|_{A^t \times \{a\}}$ is a degree 0 line bundle on A^t (as this is true for $a = e$ by construction, and then follows for any a by Lemma 5.1.7) and $\mathcal{P}_A|_{\{e\} \times A}$ comes equipped with a preferred trivialization ι (namely, the unique one compatible with the trivialization on $\mathcal{P}_{\{e\} \times \{e\}}$ coming from the preferred trivialization of $\mathcal{P}|_{A^t \times \{e\}}$). Thus, the triple $(A, \mathcal{P}_A, \iota)$ defines a map $\alpha : A \to (A^t)^t$ by Theorem 5.2.1. This map preserves the origin, and is thus a homomorphism of abelian varieties. We shall check it is an isomorphism.

Let us first check that α is finite; as $\dim(A) = \dim(A^t) = \dim((A^t)^t)$, finiteness of α automatically implies surjectivity. It is enough to check that over $k = \overline{k}$, the set-theoretic fiber $F = \alpha^{-1}(e) \in A(k)$ is finite. By definition, this is exactly the set F of all $a \in A(k)$ such that $\mathcal{P}_A|_{A^t \times \{a\}}$ is trivial line bundle on A^t. If L denotes a chosen ample line bundle L used to realize $A^t = A/K(L)$ and $\mathcal{P}_A$ as descended from $\Lambda(L)$ along $\pi : A \times A \to (A \times A)/(K(L) \times \{e\}) \simeq A^t \times A$, then for any $a \in F$, the bundle $\Lambda(L)|_{A \times \{a\}}$ is trivial. But this bundle is $t_a^* L \otimes L^{-1}$, and hence this gives $a \in K(L)(k)$. As L is ample, there are only finitely many such a, so F is indeed finite, thus proving that α is finite.

We have shown that $\alpha : A \to (A^t)^t$ is a finite surjective homomorphism of abelian varieties. If $G = \ker(\alpha)$ viewed as a finite subgroup scheme of A, then it follows from degree considerations that $A/G \simeq (A^t)^t$ via α. Our goal is to show $\deg(\alpha) = \mathrm{rank}(G)$ equals 1. For this, consider the induced map $\pi := (\alpha, \mathrm{id}) : A \times A^t \to (A^t)^t \times A$; this map realizes the target a quotient of the source by the free action of $G \times \{e\}$ given by translation. We must show $\deg(\pi) = 1$. Note that we have

$$\chi(A \times A^t, \pi^* F) = \deg(\pi) \cdot \chi((A^t)^t \times A, F)$$

by Theorem 4.2.4 (6). On the other hand, applying the universal property of $(A^t)^t$ gives an identification $\pi^* \mathcal{P}_{A^t} = \mathcal{P}_A$. Taking $F = \mathcal{P}_{A^t}$ then gives

$$\chi(A \times A^t, \mathcal{P}_A) = \deg(\pi) \cdot \chi((A^t)^t \times A, \mathcal{P}_{A^t}).$$

By Corollary 5.2.5, both Euler characteristics showing up above are $(-1)^g$ where $g = \dim(A) = \dim(A^t)$. It follows that $\deg(\pi) = 1$, as wanted. $\square$

Remark 5.3.4. By construction, $\mathcal{P}_A$ comes equipped with a preferred trivializations along $A^t \times \{e\}$. Moreover, there is a unique trivialization along $\{e\} \times A$ compatible with the previous one over $\{e\} \times \{e\}$: existence follows from the functor of points of A^t while uniqueness is clear. Thus, we may view $\mathcal{P}_A$ more symmetrically: it is a line bundle on $A^t \times A$ that comes equipped with preferred and compatible trivializations along $A^t \times \{e\}$ and $\{e\} \times A$. It follows from this, and Lemma 5.1.7, that the restrictions of $\mathcal{P}_A$ to the fibers of either projection give degree 0 line bundles.

Chapter 6

The Fourier-Mukai transform

We work in the setup of schemes of finite type over a field k.

6.1 Reminders on derived categories

Notation 6.1.1. For any k-scheme X of finite type, write $D(X) = D_{qc}(X, \mathcal{O}_X)$ for the quasi-coherent derived category of X. Write $D^b(X)$ for the its bounded version, and $D^b_{coh}(X) \subset D^b(X)$ for the full subcategory spanned by complexes with coherent cohomology. We shall write $(M, N) \mapsto M \otimes^L N$ for the symmetric monoidal structure on $D(X)$.

Proposition 6.1.2. *Let $f : X \to Y$ be a map of finite type k-schemes.*

1. *The right derived functors of pushforward assemble to give an exact functor $Rf_* : D(X) \to D(Y)$. This functor is* lax monoidal, *i.e., there is a natural map $Rf_*(M) \otimes^L Rf_*(N) \to Rf_*(M \otimes^L N)$ (which may not be an isomorphism). If f is proper, then Rf_* preserves D^b_{coh}.*

2. *The left derived functors of pullback assemble to give an exact functor $Lf^* : D(Y) \to D(X)$. This functor is* symmetric monoidal, *i.e., it commutes with tensor products in a natural way. If f is flat (or merely has finite* Tor*-dimension[1]), then Lf^* preserves D^b_{coh}. In particular, this holds whenever Y is smooth.*

3. *Rf_* is right adjoint to Lf^*.*

4. *Projection formula: for any $M \in D(X)$ and $N \in D(Y)$, there is a natural isomorphism $N \otimes^L Rf_*(M) \simeq Rf_*(f^*N \otimes^L M)$ induced by the unit map $N \to Rf_* f^* N$ and the lax monoidal structure of f.*

5. *Base change: given a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

   which is Tor*-independent[2], there is a base change isomorphism*

$$Lg'^* Rf_*(-) \simeq Rf'_* Lg'^*(-)$$

   of functors $D(X) \to D(Y')$.

---

[1]This condition means that the local rings of X have finite Tor-dimension over the corresponding local rings of Y.

[2]This condition means that $\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0$ for all $i > 0$ and compatible points $x \in X$, $y \in Y$, and $x' \in X'$. It holds true if one of f or g is flat, which is the only case we shall use.

6. *Künneth formula: for $i \in \{1, 2\}$, if $f_i : X_i \to Y$ are two maps which are* Tor*-independent (for example, one could be flat) and $M_i \in D(X_i)$, then there is a canonical isomorphism*

$$Rf_{1,*}M_1 \otimes^L Rf_{2,*}M_2 \simeq R\pi_*(Lpr_1^*M_1 \otimes^L Lpr_2^*M_2),$$

where $\pi : X \times_Y X \to Y$ is the structure map.

6.2 Constructing functors via transforms

Construction 6.2.1 (Fourier-Mukai transforms)**.** Let X and Y be k-schemes of finite type. Let $K \in D(X \times_k Y)$. The *integral transforms* or *Fourier-Mukai transforms* Φ_K and Ψ_K attached to K are the functors

$$\Phi_K : D(X) \to D(Y) \quad \text{via} \quad N \mapsto Rpr_{2,*}(Lpr_1^*N \otimes^L K)$$

and

$$\Psi_K : D(Y) \to D(X) \quad \text{via} \quad M \mapsto Rpr_{1,*}(Lpr_2^*M \otimes^L K).$$

We often refer to K as the *kernel* of Φ_K and Ψ_K, and view the above construction as giving functors

$$\Phi : D(X \times_k Y) \xrightarrow{K \mapsto \Phi_K} \mathrm{Fun}(D(X), D(Y)) \quad \text{and} \Psi : D(X \times_k Y) \xrightarrow{K \mapsto \Psi_K} \mathrm{Fun}(D(Y), D(X)),$$

where the target denotes the category of exact k-linear functors.

Example 6.2.2 (The identity)**.** Take $X = Y$, and $K = \mathcal{O}_{\Delta(X)}$ as the structure sheaf of the diagonal $\Delta : X \hookrightarrow X \times_k X$. Then $\Phi_K \simeq \mathrm{id}$. Indeed, we have

$$Lpr_1^*(N) \otimes K \simeq R\Delta_* N$$

by the projection formula for Δ, so the claim follows $pr_2 \circ \Delta = \mathrm{id}$.

olvePullback **Example 6.2.3** (Pushforward and pullback)**.** Let $f : X \to Y$ be a morphism of separated finite type k-schemes. Let $i : \Gamma \subset X \hookrightarrow Y$ by the graph, and $K = Ri_*\mathcal{O}_\Gamma \in D(X \times_k Y)$. Then $\Phi_K = Rf_*$ and $\Psi_K = Lf^*$. To see the former, note that for $N \in D(X)$, we have

$$Lpr_1^*(N) \otimes^L Ri_*\mathcal{O}_\Gamma \simeq Ri_*N$$

via the projection formula for i, so we get

$$\Phi_K(N) = Rpr_{2,*}(Lpr_1^*N \otimes^L K) \simeq Rpr_{2,*}(Ri_*N) \simeq Rf_*N.$$

Similarly, for $M \in D(Y)$, we have

$$Lpr_2^*(M) \otimes^L Ri_*\mathcal{O}_\Gamma \simeq Ri_*(Lf^*N)$$

via the projection formula for i, so we get

$$\Psi_K(M) = Rpr_{1,*}Ri_*(Lf^*N) \simeq Lf^*N,$$

as wanted.

Example 6.2.4 (Tensor product)**.** Fix a k-scheme X and some $M \in D(X)$. Set $K = R\Delta_*M$, where $\Delta : X \hookrightarrow X \times_k X$ is the diagonal. Then $\Phi_K(-) = (-) \otimes^L M$.

lveTransform

Proposition 6.2.5 (Convolution of integral transforms)**.** *Let X, Y, and Z be three k-schemes of finite type. Let $K \in D(X \times_k Y)$ and $L \in D(Y \times_k Z)$. Set*

$$K * L := Rpr_{13,*}\big(Lpr_{12}^* K \otimes^L Lpr_{23}^* L\big) \in D(X \times_k Z).$$

We call this the convolution *of K and L. Then we have*

$$\Phi_L \circ \Phi_K \simeq \Phi_{K*L}$$

as functors $D(X) \to D(Z)$. Similarly, we have

$$\Psi_K \circ \Psi_L \simeq \Psi_{K*L}$$

as functors $D(Z) \to D(X)$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & & X \times_k Y \times_k Z \arrow[dl, "p_{XY}"'] \arrow[dr, "p_{YZ}"] & & \\
 & X \times_k Y \arrow[dl, "pr_X"'] \arrow[dr, "pr_Y"] & & Y \times_k Z \arrow[dl, "pr_Y"'] \arrow[dr, "pr_Z"] & \\
X & & Y & & Z.
\end{tikzcd}$$

All maps above are flat, and the square is a fibre square. Here we have followed the convention that all projection maps coming out of $X \times_k Y \times_k Z$ are labelled with a suitable subscript on p, while all projection maps out of any product with only two factors are denoted pr with a suitable subscript. Thus, we also write $p_X : X \times_k Y \times_k Z \to X$ for the projection to X, etcetera.

By definition, we have

$$(\Phi_L \circ \Phi_K)(M) = \Phi_L(Rpr_{Y,*}(Lpr_X^*(M) \otimes^L K)) = Rpr_{Z,*}(Lpr_Y^*(Rpr_{Y,*}(Lpr_X^*(M) \otimes^L K) \otimes^L L)).$$

Applying flat base change to the cartesian square above shows

$$Lpr_Y^* Rpr_{Y,*} \simeq Rp_{YZ,*} Lp_{XY}^*,$$

so the above exrpression can be simplfied to give

$$(\Phi_L \circ \Phi_K)(M) = Rp_{Z,*}\big(Lp_X^* M \otimes Lp_{XY}^* K \otimes Lp_{YZ}^* L\big),$$

where we have also used the projection formula for p_{XY} and p_{YZ}. Applying the projection formula to p_{XZ} and the sheaf $Lpr_X^* M$, this simplifies further to

$$(\Phi_L \circ \Phi_K)(M) = Rpr_{Z,*}\big(Lpr_X^* M \otimes Rp_{XY,*}(Lp_{XY}^* K \otimes Lp_{YZ}^* L)\big) \simeq \Phi_{K*L}(M),$$

as wanted. The proof for Ψ is similar. $\square$

Remark 6.2.6. The operation of convolution gives a "composition" law

$$D(X \times_k Y) \times D(Y \times_k Z) \to D(X \times_k Z).$$

One can check that this is associative. Proposition 6.2.5 implies that this convolution law is compatible with composition of functors under the map

$$\Phi : D(X \times_k Y) \to \mathrm{Fun}(D(X), D(Y)).$$

In other words, the assignment $X \mapsto D(X)$ has more functoriality than just in morphisms of schemes: it is functorial in "correspondences" $X \to Y$ if one defines this to mean an object of $D(X \times Y)$. One can also formulate this in suitable 2-categorical terms, but we do not do that here.

Remark 6.2.7. Given finite type k-schemes X and Y and some $K \in D(X)$, we can view X as a space parametrizing certain objects of $D(Y)$ via the assignment $x \mapsto K|_{\{x\} \times Y}$. This perspective is quite useful in practice.

6.3 The Fourier-Mukai equivalence

Let A be an abelian variety over a field k. Write A^t for the dual and $\mathcal{P}_A \in \mathrm{Pic}(A \times A^t)$ for the Poincare bundle. Under the biduality isomorphism, we can also view $\mathcal{P}_A$ as a line bundle on $A^t \times A \simeq A^t \times (A^t)^t$ by switching factors; we have seen before that $\mathcal{P}_A = \mathcal{P}_{A^t}$ via this identification. For notational sanity, we write $\mathcal{P}_{A^t}$ for $\mathcal{P}_A$ viewed as a line bundle on $A^t \times A$.

Theorem 6.3.1. *The map $\Phi_{\mathcal{P}_A} : D(A) \to D(A^t)$ is an equivalence. More precisely, under the biduality isomorphism $A \simeq (A^t)^t$, we have the formulas*

$$\Phi_{\mathcal{P}_{A^t}} \circ \Phi_{\mathcal{P}_A} \simeq [-1]_A^*[-g] \quad \textit{and} \quad \Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{A^t}} \simeq [-1]_{A^t}^*[-g],$$

where $g = \dim(A)$. Moreover, $\Phi_{\mathcal{P}_A}$ induces an equivalence on D^b_{coh} with inverse determined by the same formulas.

The equivalence $\Phi_{\mathcal{P}_A}$ is called the *Fourier-Mukai equivalence* for A. The proof relies on the following observation:

CubeFM **Lemma 6.3.2.** *Let $\mu : A \times A^t \times A \to A \times A^t$ be the map $(a, b, c) \mapsto (m(a, c), b)$. Then the line bundle*

$$\mu^* \mathcal{P}_A^{-1} \otimes pr_{12}^* \mathcal{P}_A \otimes pr_{23}^* \mathcal{P}_{A^t}$$

on $A \times A^t \times A$ is trivial.

Proof. This follows from the theorem of the cube: to get triviality on $\{e\} \times A^t \times A$ and $A \times A^t \times \{e\}$, one uses that $\mathcal{P}_A|_{\{e\} \times A^t}$ is trivial, while triviality on $A \times \{e\} \times A$ uses that $\mathcal{P}_A|_{A \times \{e\}}$ is trivial. □

Proof. As the isomorphism $A \simeq (A^t)^t$ carries $\mathcal{P}_{A^t}$ on $A^t \times (A^t)^t$ to $\mathcal{P}_A$ on $A^t \times A$, it is enough to prove the first assertion. By Proposition 6.2.5 and Example 6.2.3, it is enough to show the following: if $pr_{13} : A \times A^t \times A \to A \times A$ is the projection, then the convolution

$$\mathcal{P}_A * \mathcal{P}_{A^t} := Rpr_{13,*}(Lpr_{12}^* \mathcal{P}_A \otimes^L Lpr_{23}^* \mathcal{P}_{A^t}) \in D(A \times A)$$

identifies with $\mathcal{O}_\Gamma[-g]$, where $\Gamma \subset A \times A$ is the graph of $[-1]$. By Lemma 6.3.2, this simplifies to give

$$\mathcal{P}_A * \mathcal{P}_{A^t} := Rpr_{13,*} L\mu^* \mathcal{P}_A,$$

where $\mu : A \times A^t \times A \to A \times A^t$ is the map $(a, b, c) \mapsto (m(a, c), b)$. Applying flat base change for the cartesian square

$$\begin{tikzcd} A \times A^t \times A \arrow[r, "\mu"] \arrow[d, "pr_{13}"] & A \times A^t \arrow[d, "pr_1"] \\ A \times A \arrow[r, "m"] & A, \end{tikzcd}$$

this simplifies to

$$\mathcal{P}_A * \mathcal{P}_{A^t} := Lm^* Rpr_{1,*} \mathcal{P}_A.$$

We have seen earlier that $Rpr_{1,*} \mathcal{P}_A \simeq \kappa(e)[-g]$ (see Corollary 5.2.5). We can then apply flat base change to the square

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & A \times A \arrow[d, "m"] \\ \{e\} \arrow[r] & A \end{tikzcd}$$

to conclude that

$$\mathcal{P}_A * \mathcal{P}_{A^t} := Lm^* Rpr_{1,*} \mathcal{P}_A \simeq Lm^* \kappa(e)[-g] \simeq \mathcal{O}_\Gamma[-g],$$

as wanted.

The last assertion is automatic as all functors involved preserve $D^b_{coh}(-)$ as all varieties in sight are proper and smooth. □

We briefly discuss some properties of the Fourier transform.

FM1 **Lemma 6.3.3** (Tensoring with degree 0 line bundles goes to translations)**.** *For any degree* 0 *line bundle* M_x *on* A *corresponding to a point* $x \in A^t(k)$*, we have a natural isomorphism*

$$\Phi_{\mathcal{P}_A}(F \otimes^L M_x) \simeq t_x^* \Phi_{\mathcal{P}_A}(F) \tag{6.1}$$ {FMTensoring

for all $F \in D(A)$*.*

Proof. Unwinding definitions, it is enough to check that the line bundles $pr_1^* M_x \otimes \mathcal{P}_A$ and $t_{(e,x)}^* \mathcal{P}_A$ on $A \times A^t$ agree. Both their restrictions to the fibres $A \times \{y\}$ of pr_2 are identified with $M_x \otimes M_y$. Hence, their difference is pulled back from a line bundle on A^t by the Seesaw theorem. On the other hand, both line bundles are also trivial along the section $\{e\} \times A^t \subset A \times A^t$: this is clear from $pr_1^* M_x \otimes \mathcal{P}_A$ from the corresponding statement for $\mathcal{P}_A$, and follows for the second as $t_{e,x}^* \mathcal{P}_A|_{\{e\} \times A^t} \simeq t_x^*(\mathcal{P}_A|_{\{e\} \times A^t})$, which is trivial by the trivialization in the definition of $\mathcal{P}_A$. ☐

kyScraperPic0 **Example 6.3.4.** Corollary 5.2.5 (and biduality) show that

$$\Phi_{\mathcal{P}_A}(\mathcal{O}_A) \simeq \kappa(e)[-g].$$

It follows from (6.1) that

$$\Phi_{\mathcal{P}_A}(M_x) \simeq \kappa(-x)[-g]$$

for any $M_x \in \mathrm{Pic}^0(A)$ corresponding to $x \in A^t(k)$. Hitting both sides with $\Phi_{\mathcal{P}_{A^t}}$ then gives

$$M_{-x} = [-1]^* M_x \simeq \Phi_{\mathcal{P}_{A^t}}(\kappa(-x))$$

for $x \in A^t(k)$ and thus (by switching the roles of A and A^t) we get

$$\Phi_{\mathcal{P}_A}(\kappa(y)) \simeq N_y$$

for $y \in A(k)$ corresponding to $N_y \in \mathrm{Pic}^0(A^t)$ via the biduality isomorphism. In other words, the Fourier-Mukai transform switches degree 0 line bundles and skyscraper sheaves (up to shifts and signs).

FM2 **Lemma 6.3.5** (Translations go to tensoring with degree 0 line bundles)**.** *If* $a \in A(k)$ *corresponds to a line bundle* N_a *on* A^t*, then we have a natural isomorphism*

$$\Phi_{\mathcal{P}_A}(t_x^* F) \simeq N_{-x} \otimes \Phi_{\mathcal{P}_A}(F)$$

for all $F \in D(A)$ *and* $x \in A(k)$*.*

Proof. Note that

$$Lpr_1^* t_x^* F \otimes \mathcal{P}_A \simeq t_{x,0}^* Lpr_1^* F \otimes \mathcal{P}_A \simeq t_{x,0}^*(Lpr_1^* F \otimes \mathcal{P}_A) \otimes pr_2^* N_{-x},$$

where we use the identification

$$t_{x,0}^* \mathcal{P}_A \simeq pr_2^* N_x \otimes \mathcal{P}_A$$

proven as in Lemma 6.3.3 (and the fact that $N_{-x} = N_x^{-1}$). Thus, we obtain

$$\Phi_{\mathcal{P}_A}(t_x^* F) \simeq Rpr_{2,*}\big(t_{x,0}^*(Lpr_1^* F \otimes \mathcal{P}_A) \otimes pr_2^* N_{-x}\big) \simeq N_{-x} \otimes Rpr_{2,*} t_{x,0}^*(Lpr_1^* F \otimes \mathcal{P}_A).$$

Now writing $t_{x,0}^* = (t_{-x,0})_*$, and observing that $pr_2 \circ t_{x,0} = pr_2$, the last expression above simplifies to give

$$\Phi_{\mathcal{P}_A}(t_x^* F) \simeq N_{-x} \otimes Rpr_{2,*} t_{x,0}^*(Lpr_1^* F \otimes \mathcal{P}_A) \simeq N_{-x} Rpr_*(Lpr_1^* F \otimes \mathcal{P}_A) \simeq N_{-x} \otimes \Phi_{\mathcal{P}_A}(F),$$

as wanted. ☐

FM3 **Lemma 6.3.6** (Tensor products go to convolutions)**.** *For any $M, N \in D(A)$, there is a natural isomorphism*

$$\Phi_{\mathcal{P}_A}(M) \otimes \Phi_{\mathcal{P}_A}(N) \simeq \Phi_{\mathcal{P}_A}(M *_A N),$$

where $M *_A N = Rm_*(Lpr_1^* M \otimes Lpr_2^* N) \in D(A)$.

In other words, the Fourier-Mukai transform carries tensor products in $D(A)$ to convolution on $D(A^t)$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & A \times A \times A^t \arrow[rr, "\mu=(m,id)"] \arrow[d, "pr_{12}"] \arrow[ddl, "pr_1"'] \arrow[ddr, "pr_2"] & & A \times A^t \arrow[r, "pr_2"] \arrow[d, "pr_1"] & A^t \\
 & A \times A \arrow[rr, "m"] \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & & A & \\
A & & A & &
\end{tikzcd}$$

with the square being cartesian. Now we have

$$\Phi_{\mathcal{P}_A}(M *_A N) = Rpr_{2,*}(\mathcal{P}_A \otimes Lpr_1^*(M *_A N)) \simeq Rpr_{2,*}(\mathcal{P}_A \otimes (Lpr_1^* Rm_*(Lp_1^* M \otimes^L Lp_2^* N)))$$

Simplifying the second largest parenthesized term via flat base change for the square above gives

$$\Phi_{\mathcal{P}_A}(M *_A N) \simeq Rpr_{2,*} R\mu_*(L\mu^* \mathcal{P}_A \otimes Lpr_1^* M \otimes Lpr_2^* N).$$

As the map μ above agrees with the one in Lemma 6.3.2 after reordering factors, that lemma impies

$$\mu^* \mathcal{P}_A \simeq pr_{13}^* \mathcal{P}_A \otimes pr_{23}^* \mathcal{P}_A.$$

As $pr_2 \circ \mu = pr_3$ as maps, this gives

$$\Phi_{\mathcal{P}_A}(M *_A N) \simeq Rpr_{3,*}\Big((Lpr_{13}^* \mathcal{P}_A \otimes Lpr_1^* M) \otimes (Lpr_{23}^* \mathcal{P}_A \otimes Lpr_2^* M)\Big).$$

The map $pr_3 \circ \mu : A \times A \times A^t \to A^t$ is the two fold fibre power of the flat map $p_2 : A \times A^t \to A^t$. The sheaves appearing on the right pulled back from each component of this fibre power. Thus, by Künneth, this simplifies to

$$\Phi_{\mathcal{P}_A}(M *_A N) \simeq Rpr_{2,*}(\mathcal{P}_A \otimes Lpr_1^* M) \otimes Rpr_{2,*}(\mathcal{P}_A \otimes Lpr_2^* N) =: \Phi_{\mathcal{P}_A}(M) \otimes \Phi_{\mathcal{P}_A}(N),$$

as wanted. □

FM4 **Lemma 6.3.7** (Exchange of $R\Gamma$ with fibers)**.** *For any $F \in D(A)$ and any $M_x \in \mathrm{Pic}^0(A)$ corresponding to $x \in A^t(k)$, there is a canonical isomorphism*

$$R\Gamma(A, F \otimes M_x) \simeq \Phi_A(F)|_{\{x\}}.$$

Similarly, for any $G \in D(A^t)$, there is a canonical isomorphism

$$R\Gamma(A, \Phi_{A^t}(G) \otimes M_x) \simeq G[-g]|_{\{-x\}}$$

for all $x \in A^t(k)$.

Proof. The second follows from the first by setting $F = \Phi_{A^t}(G)$ and using $\Phi_A \circ \Phi_{A^t} \simeq [-1]^*[-g]$. For the first, Applying base change to the cartesian square

$$\begin{tikzcd} A \times \{x\} \arrow[r] \arrow[d] & A \times A^t \arrow[d, "pr_2"] \\ \{x\} \arrow[r] & A^t \end{tikzcd}$$

shows that

$$\Phi_A(F)|_{\{x\}} \simeq Rpr_{2,*}(F \otimes \mathcal{P}_A)|_{\{x\}} \simeq R\Gamma(A, F \otimes \mathcal{P}_A|_{A \times \{x\}}) \simeq R\Gamma(A, F \otimes M_x),$$

as wanted. □

FMexRGamma **Remark 6.3.8.** In Lemma 6.3.7, taking $M_x = \mathcal{O}_A$ in the second isomorphism shows that $\chi(A, \Phi_{A^t}(G)) = (-1)^g \chi(G|_{\{e\}})$. Thus, if G is a line bundle, then $\chi(A, \Phi_{A^t}(G)) = (-1)^g$.

Remark 6.3.9. Combining Lemma 6.3.7 with 6.3.4, we learn that

$$R\Gamma(A, \mathcal{O}_A) \simeq \kappa(e)[-g] \otimes^L_{\mathcal{O}_{A^t,e}} \kappa(e).$$

We may replace $\mathcal{O}_{A^t,e}$ with its completion R without changing the right hand side. But then there is a non-canonical identification $R \simeq k[\![x_1, ..., x_g]\!]$ as A^t is smooth of dimension g. It follows from a standard calculation with the Koszul complex that that the homology of right hand side is an exterior algebra on H^1, and hence the same holds true for the left hand side, recovering Corollary 5.2.6; of course, this is essentially the same proof, repackaged using derived categories.

FM5 **Lemma 6.3.10** (Behaviour under isogenies)**.** *Say $f : A \to B$ is a homomorphism of abelian varieties with dual f^t. Then*

$$\Phi_B \circ f_* \simeq (f^t)^* \circ \Phi_A.$$

Proof. Consider the diagram

$$\begin{tikzcd} B & B \times B^t \arrow[l, "q_1"'] \arrow[r, "q_2"] & B^t \\ A \arrow[u, "f"] \arrow[d, equal] & A \times B^t \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[u, "\beta"] \arrow[d, "\alpha"] & B^t \arrow[u, equal] \arrow[d, "f^t"] \\ A & A \times A^t \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & A^t. \end{tikzcd}$$

The top left and bottom right squares are cartesian. By construction of the dual isogeny, we have

$$\alpha^* \mathcal{P}_A \simeq \beta^* \mathcal{P}_B. \tag{6.2}$$ {eq:FMIsog}

Thus, we get

$$\Phi_B \circ f_* M = q_{2,*}(q_1^* f_* M \otimes \mathcal{P}_B) \simeq q_{2,*}\beta_*(p_1^* M \otimes \beta^* \mathcal{P}_B) \simeq p_{2,*}(p_1^* M \otimes \alpha^* \mathcal{P}_A)$$

where the second equality uses flat base change for the top left square and the projection formula for β, and the last equality uses (6.2). As $p_1 = pr_1 \circ \alpha$, this simplifies to

$$\Phi_B \circ f_* M \simeq p_{2,*}\alpha^*(pr_1^* M \otimes \mathcal{P}_A).$$

The projection formula for the bottom right square then shows $p_{2,*}\alpha^* \simeq (f^t)^* pr_{2,*}$, which immediately gives the desired equality. □

Chapter 7

Applications of the Fourier-Mukai equivalence

Let A/k be an abelian variety of dimension g. Write $\Phi = \Phi_{\mathcal{P}_A}$ for notational simplicity. Also entirely for notational ease, we restrict to the case where k is algebraically closed.

7.1 Homogeneous bundles

An object $E \in D(A)$ is called *homogeneous* if $t_x^* E \simeq E$ for every geometric point x of A; this is the obvious derived analog of Definition 5.1.1, and will usually be applied when E is a vector bundle. Note that we do not demand any compatibility in x for the isomorphisms $t_x^* E \simeq E$.

Our first real application of the Fourier-Mukai equivalence is to classify homogeneous vector bundles:

HomogeneousFM **Theorem 7.1.1.** $\Phi[g]$ *identifies the category of homogeneous bundles on A with the category of coherent sheaves with finite support on A^t; this correspondence exchanges ranks with lengths. In particular, the collection of homogeneous vector bundles forms an abelian subcategory of* $\mathrm{Coh}(A)$ *that is closed under extensions.*

The proof relies on the following lemma:

PicHomFM **Lemma 7.1.2.** *. Say $G \in D^b_{coh}(A)$ is invariant under tensoring with degree 0 line bundles (i.e., there exist isomorphisms $G \otimes L \simeq G$ for all $L \in \mathrm{Pic}^0(A)$). Then G has finite support.*

Proof. As tensoring with line bundles preserves cohomology sheaves, we may assume G is actually a sheaf. Now if the support is not finite, then there is a reduced irreducible curve $C \subset A$ such that $G|_C$ has nonzero rank. Write $f : \widetilde{C} \to C \subset A$ for the normalization, and write $\overline{G}$ for the torsionfree quotient of f^*G. Then $\overline{G}$ is a *nonzero* vector bundle on C such that $\overline{G} \otimes f^*L \simeq \overline{G}$ for any $L \in \mathrm{Pic}^0(A)$. Passing to determinants (and this uses $\overline{G} \neq 0$), and using the divisibility of $\mathrm{Pic}^0(A)$, it follows that the same holds for $E = \det(G)$. But this simply means that f^*L is the trivial line bundle on C for every $L \in \mathrm{Pic}^0(A)$. Stated differently, the pullback $\pi^*\mathcal{P}_A$ along $\pi := (f, \mathrm{id}) : C \times A^t \to A \times A^t$ is trivial along all fibers of projection to C. By the Seesaw theorem, we must then have $\pi^*\mathcal{P}_A \simeq pr_1^* L$ for some line bundle L on C. Using our trivialization of $\mathcal{P}_A|_{A\times\{0\}}$, it follows that L is itself trivial, and thus $\pi^*\mathcal{P}_A$ is trivial. Viewing $\mathcal{P}_A$ now as a family of line bundles on A^t parametrized by A as in Theorem 5.3.3, it follows that the classifying map $C \to (A^t)^t$ is the constant map. But by unwinding definitions, this is simply the map $f : C \to A$ under the biduality isomorphism $A \simeq (A^t)^t$. Thus, we have shown that f is constant, but this is absurd: f was finite and C was a curve by construction. □

Proof of Theorem 7.1.1. If N is a torsion coherent sheaf with finite support on A^t, then write $N = \Phi(M)$ for some $M \in D(A)$. As N has finite support, it can be written as an iterated extension of skyscraper sheaves.

As Φ^{-1} of a skyscraper sheaf is a line bundle in homological degree g, it follows that $M[g]$ is coherent sheaf on A that can be expressed as an iterated extension of line bundles. In particular, $M[g]$ is a vector bundle. To see homogeneity, note that as N has finite support, N is invariant under tensoring with degree 0 line bundles. The claim now follows from Lemma 6.3.3.

Conversely, say $M \in D(A)$ is homogenous. Set $N = \Phi(M)[g]$. We must check that N is a coherent sheaf with finite support on A. In fact, it is enough to check that N has finite support: once this is known, it follows immediately from the previous paragraph that $\Phi^{-1}(N) = M[g]$ lives in exactly as many nonzero degrees as N, and thus N must live in a single degree since M does so. To check finite support, we simply invoke Lemma 7.1.2, noting that N is invariant under tensoring by degree 0 line bundles by Lemma 6.3.3 and the homogeneity of M. ☐

7.2 Unipotent vector bundles

A vector bundle E on A is called *unipotent* if it is an iterated extension of copies of the structure sheaf.

Theorem 7.2.1. *$\Phi_A[g]$ identifies the category of unipotent vector bundles on A with the category of coherent sheaves on A^t supported set-theoretically at 0 (i.e., the category of artinian $\mathcal{O}_{A^t,e}$-modules); this correspondence exchanges ranks and lengths. In particular, the collection of unipotent vector bundles form an abelian subcategory of* $\mathrm{Coh}(A)$ *closed under extensions.*

Proof. Recall the formulae

$$\Phi_A(\mathcal{O}_A) \simeq \kappa(e)[-g] \quad \text{and} \quad \Phi_{A^t}(\kappa(e)) = \mathcal{O}_A$$

coming from Example 6.3.4. By induction on the number of extensions needed to express a unipotent vector bundle in terms of $\mathcal{O}_A$, it follows from the first formula that $\Phi_A[g]$ carries unipotent vector bundles to sheaves on A^t supported at the origin. The claim now follows by applying the same reasoning to Φ_{A^t} using the second formula above. ☐

7.3 Non-degenerate line bundles and cohomology of ample line bundles

sec:ndg

We say that a line bundle L on A is *non-degenerate* if the group scheme $K(L)$ is finite; equivalently, the homomorphism $\phi_L : A \to A^t$ is an isogeny (i.e., finite surjective). An ample line bundle and its inverse are non-degenerate. Our goal is to show that the cohomology of such line bundles is particularly constrained.

thm:nondeg **Theorem 7.3.1.** *Let L be a nondegenerate line bundle.*

1. *There exists an integer $0 \leq i(L) \leq g$ and a vector bundle E on A^t such that $\Phi_A(L) = E[-i(L)]$. In particular, $H^i(A, L) = 0$ for all $i \neq i(L)$.*

2. *We have $\left(\dim H^{i(L)}(A, L)\right)^2 = rank(K(L))$.*

In particular, the rank of $K(L)$ is a square and $H^{i(L)}(A, L) \neq 0$.

The integer $i(L)$ is called the *index* of L and will be analyzed later. To prove the theorem, we proceed in steps. First, we explain why the structure of $\Phi_A(L)$ is quite simple after pullback along the isogeny of ϕ_L.

Nondeg1 **Lemma 7.3.2.** *Let L be a non-degenerate line bundle on A. Then there is an isomorphism*

$$\phi_L^* \Phi_A(L) \simeq R\Gamma(A, L) \otimes_k L^{-1}.$$

By the faithful flatness of ϕ_L, it follows that each cohomology group of $\Phi_A(L)$ is a vector bundle on A. (We shall subsequently show that there is only one nonzero cohomology group.)

Proof. The map $\phi_L : A \to A^t$ classifies a line bundle M on $A \times A$ that is trivial on each fibre of pr_1 (and trivialized at the origin on the second factor). By definition, if we write $\alpha = (\mathrm{id}, \phi_L) : A \times A \to A \times A^t$, then $\alpha^* \mathcal{P}_A = M$. It is an exercise in unwinding definitions that $M \simeq \Lambda(L)$; more precisely, such an isomorphism is determined by the choice of a trivialization of $L|_{\{e\}}$. (In fact, the entire construction in §5.2 could have been carried out using any non-degenerate line bundle L, not just an ample one.)

Now consider

$$\Phi_A(L) = pr_{2,*}(pr_1^* \times \mathcal{P}_A).$$

Using the diagram

$$\begin{tikzcd}
 & A \times A \arrow[dl, "p_1"'] \arrow[dd, "\alpha"] \arrow[dr, "p_2"] & \\
A \arrow[dd, equal] & & A \arrow[dd, "\phi_L"] \\
 & A \times A \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
A & & A^t
\end{tikzcd}$$

and the cartesianness of the second square, we get

$$\phi_L^* \Phi_A(L) \simeq p_{2,*}(p_1^* L \otimes \alpha^* \mathcal{P}_A).$$

As explained above, we have $\alpha^* \mathcal{P}_A \simeq \Lambda(L) \simeq m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$. Plugging this in, we get

$$\phi_L^* \Phi_A(L) \simeq p_{2,*}(m^* L \otimes p_2^* L^{-1}) \simeq p_{2,*}(m^* L) \otimes L^{-1},$$

where the second isomorphism uses the projection formula. Now the cartesian square

$$\begin{tikzcd}
A \times A \arrow[r, "m"] \arrow[d, "p_2"] & A \arrow[d] \\
A \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

then implies that

$$\phi_L^* \Phi_A(L) \simeq R\Gamma(A, L) \otimes L^{-1},$$

as wanted. $\square$

Next, we compute $\chi(A, L)$.

Nondeg2

Corollary 7.3.3. *We have* $\chi(A, L)^2 = \mathrm{rank}(K(L))$.

Proof. We shall use a fancier version of Theorem 4.2.4 (7) applied to the finite quotient $A \to A/K(L) \simeq A^t$: for any $M \in D^b_{coh}(A^t)$, we have

$$\chi(A, \phi_L^* M) = \deg(\phi_L) \cdot \chi(A^t, M).$$

Applying this to $M = \Phi_A(L)$ and using Lemma 7.3.2 gives

$$\chi(A, R\Gamma(A, L) \otimes L^{-1}) = \mathrm{rank}(K(L)) \cdot \chi(A^t, \Phi_A L).$$

Now the projection formula simplifies the left side as

$$\chi(A, R\Gamma(A, L) \otimes L^{-1}) = \chi(A, L) \cdot \chi(A, L^{-1}).$$

By Serre duality (as $K_A \simeq \mathcal{O}_A$), we have $\chi(A, L^{-1}) = (-1)^g \chi(A, L)$. So the left side above simplifies to

$$\chi(A, R\Gamma(A, L) \otimes L^{-1}) = \chi(A, L)^2 \cdot (-1)^g.$$

On the other hand, by Exercise 6.3.8, we have

$$\chi(A^t, \Phi_A L) = (-1)^g \chi(L|_{\{e\}}) = (-1)^g.$$

Putting everything together now gives the claim. □

The following general fact about convolutions of line bundles will help in the proof.

Nondeg3 **Lemma 7.3.4.** *Let L be any line bundle on A, and let $i : K(L) \to A$ be the defining inclusion. Then*

$$L *_A [-1]^* L^{-1} \simeq i_*(L|_{K(L)})[-g].$$

In particular, if L is non-degenerate, then this complex has finite support.

Proof. We must show

$$m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) \simeq i_*(L|_{K(L)})[-g].$$

To understand the LHS, consider the factorization the automorphism η of $A \times A$ given by

$$\eta(a, b) = (m(a, b), -b).$$

Then $p_1 \circ \eta = m$ and $\eta^2 = \mathrm{id}$. In particular, the latter implies $\eta_* = \eta^*$, so we may compute

$$m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) \simeq p_{1,*}\eta^*(p_1^* L \otimes p_2^*[-1]^* L^{-1}).$$

Now $p_1 \circ \eta = m$ and $[-1] \circ p_2 \circ \eta = p_2$, so this simplifies to

$$m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) \simeq p_{1,*}(m^* L \otimes p_2^* L^{-1}).$$

Using the formula $\Lambda(L) \otimes p_1^* L \simeq m^* L \otimes p_2^* L^{-1}$, this can be rewritten as

$$m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) \simeq p_{1,*}(\Lambda(L) \otimes p_1^* L) \simeq p_{1,*}\Lambda(L) \otimes L,$$

where the last isomorphism uses the projection formula. Now using flat base change for the cartesian square

$$\begin{tikzcd} A \times A \arrow[r, "\alpha=(\phi_L,\mathrm{id})"] \arrow[d, "p_1"] & A^t \times A \arrow[d, "pr_1"] \\ A \arrow[r, "\phi_L"] & A^t, \end{tikzcd}$$

as well as the formula $\alpha^* \mathcal{P}_A \simeq \Lambda(L)$, we learn that

$$m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) \simeq L \otimes \phi_L^* pr_{1,*}\mathcal{P}_A.$$

By Corollary 5.2.5, this simplies as

$$m_*(p_1^* L \otimes p_2^*[-1]^* L^{-1}) \simeq L \otimes \phi_L^* \kappa(e)[-g].$$

As the kernel of ϕ_L is exactly $K(L)$, the claim follows. □

Proof of Theorem 7.3.1. Thanks to Corollary 7.3.3 and Lemma 7.3.2, it is enough to show the following: if L is a nondegenerate line bundle, then $\Phi_A(L)$ has the form $E[-i(L)]$ for some vector bundle E and some integer $0 \leq i(L) \leq g$. Note that this assertion can be checked after pullback along ϕ_L. By Lemma 7.3.2, we have $\phi_L^*\Phi_A(L) \simeq R\Gamma(A, L) \otimes L^{-1}$. It is thus enough to show that $R\Gamma(A, L)$ is concentrated in a single degree; this degree is necessarily in $[0, g]$ for general reasons as A has dimension g.

As tensor products and convolutions are interchanged (Lemma 6.3.6), applying Φ_A to Lemma 7.3.4 gives

$$\Phi_A(L) \otimes \Phi_A([-1]^*L^{-1}) \simeq \Phi_A(i_*(L|_{K(L)}))[-g].$$

Now $K(L)$ is finite as L is non-degenerate, so $L|_{K(L)}$ is non-canonically identified with $\mathcal{O}_{K(L)}$. By the finiteness of the support and Theorem 7.1.1, it follows that the RHS above has the form $E'[-g]$ for a homogeneous vector bundle E' on A^t of rank $\mathrm{rank}(K(L))$. Pulling back along ϕ_L and using Lemma 7.3.2 then shows that

$$(R\Gamma(A, L) \otimes L^{-1}) \otimes (R\Gamma(A, [-1]^*L^{-1}) \otimes [-1]^*L) \simeq \phi_L^*E'[-g].$$

Reorganizing the LHS, we get

$$R\Gamma(A, L) \otimes R\Gamma(A, [-1]^*L^{-1}) \otimes \mathcal{O}_A \simeq E[-g]$$

for some vector bundle E on A of rank $\mathrm{rank}(K(L))$. As the LHS is pulled back from a point, it is easy to see that this forces both $R\Gamma(A, L)$ and $R\Gamma(A, [-1]^*L^{-1})$ to have exactly one non-zero cohomology group, as wanted. □

Next, we analyze the index $i(L)$ of a nondegenerate line bundle. We begin by observing that it is invariant in families:

Nondeg4 **Lemma 7.3.5.** *Say L and L' are algebraically equivalent nondegenerate line bundles on A. Then $i(L) = i(L')$.*

Proof. It is enough to show that $L' = t_x^*L$ for some $x \in A^t(k)$: granting this, pullback along t_x induces an isomorphism $H^*(A, L) \simeq H^*(A, L')$, so the indices must be the same. By hypothesis, there is a smooth proper connected curve and a line bundle $\mathcal{L}$ on $C \times A$ such that $\mathcal{L}|_{\{c_0\}\times A} \simeq L$ and $\mathcal{L}|_{\{c_1\}\times A} \simeq L'$ for some $c_0, c_1 \in C(k)$. The twist $M := \mathcal{L} \otimes pr_2^*L^{-1}$ is thus a line bundle on $C \times A$ whose fibre over c_0 is trivial. By Lemma 5.1.7, the fibre over c_1 lies in $\mathrm{Pic}^0(A)$, so $L' \otimes L^{-1} \in \mathrm{Pic}^0(A)$. But the map $\phi_L : A \to A^t$ is surjective by non-degeneracy of L, so we must have

$$L' \otimes L^{-1} = \phi_L(x) = t_x^*L \otimes L^{-1},$$

and thus $L' \simeq t_x^*L$, as wanted. □

The index is also invariant under isogeny:

Nondeg5 **Lemma 7.3.6.** *If $f : A \to B$ is an isogeny and L is a nondegenerate line bundle on B, then f^*L is nondegenerate and $i(L) = i(f^*L)$.*

Proof. By construction, we have

$$\phi_{f^*L} = f^t \circ \phi_L \circ f.$$

Indeed, we have $\phi_{f^*L}(x) = t_x^*f^*L \otimes f^*L^{-1} \simeq f^*(t_{f(x)}^*L \otimes L^{-1}) = f^*\phi_L(x)$, so the claim follows that $f^t = f^*$ under the identifications $A^t(k) \simeq \mathrm{Pic}^0(A)$ and $B^t(k) \simeq \mathrm{Pic}^0(B)$. As the RHS is a composition of 3 finite surjective maps, the same is true for the LHS, so f^*L is non-degenerate.

For equality of indices, we use Lemma 6.3.10 applied to f^t to get

$$\Phi_A f^*L \simeq f_*^t\Phi_B(L).$$

As L is nondegenerate, $\Phi_B(L)$ has the form $E[-i(L)]$ for a vector bundle E on B^t. As f^t is finite flat, the pushforward f_*^t preserves vector bundles, so $\Phi_A f^*L = F[-i(L)]$ for a vector bundle F on A^t. The claim now follows from the definition of the index in Theorem 7.3.1. □

Nondeg6 **Lemma 7.3.7.** *For any $n > 0$ and any nondegenerate line bundle L on A, we have $i(L) = i(L^n)$.*

Proof. We only give a proof when $n = m^2$ is a square; the general case can be deduced by a trick using Lagrange's 4 square theorem. Lemma 7.3.6 applied to $f = [m]$ shows that $i([m]^*L) = i(L)$. Now Lemma 5.1.5 shows that $L^n = L^{m^2}$ and $[m]^*L$ lie in the same algebraic family: their difference is degree 0, and hence lies in the same family as the trivial bundle as A^t is connected. Lemma 7.3.5 then shows that $i(L) = i(L^n)$, proving the claim. □

Putting everything together, we learn a vanishing (and non-vanishing) theorem for ample line bundles:

AmpleH0 **Corollary 7.3.8.** *Say L is an ample line bundle on A. Then $H^j(A, L) = 0$ for $j > 0$, and $\dim H^0(A, L) = \sqrt{\mathrm{rank}(K(L))} > 0$.*

If $\dim(H^0(A, L)) = 1$, then we say that L gives a *principal polarization*.

Proof. Lemma 7.3.7 ensures that $i(L) = i(L^n)$ for all $n > 0$. On the other hand, by Serre vanishing, we have $H^j(A, L^n) = 0$ for $j > 0$ and $n \gg 0$. It follows that $i(L^n) = 0$ for $n \gg 0$, and hence $i(L) = 0$. The rest follows from Theorem 7.3.1. □

7.4 Classification of stable vector bundles on an elliptic curve

Let E be an elliptic curve over a field k. We shall review the notion of semistable and stable bundles, and then to classify them (following a result a Atiyah).

7.4.1 Semistable and stable bundles

Fix a smooth projective geometrically connected curve C over a field k. Recall that each vector bundle E on C has a degree $d(E) := d(\det(E)) \in \mathbf{Z}$ and a rank $r(E) \in \mathbf{Z}_{\geq 0}$. Both these invariants are additive in short exact sequences, and thus give homomorphism $K_0(C) \to \mathbf{Z}$. It follows that we can define $d(E)$ and $r(E)$ for any $E \in D^b_{coh}(C)$, and these invariants are also additive in E.

Definition 7.4.1. For a nonzero vector bundle E on C, define the slow $\mu(E) = \deg(E)/\mathrm{rank}(E)$. We say that E is *semistable* (resp. *stable*) if for every quotient bundle $E \twoheadrightarrow F$, we have $\mu(E) \leq \mu(F)$ (resp. $\mu(E) < \mu(F)$).

Stable0 **Lemma 7.4.2.** *Let*

$$0 \to E_1 \to E \to E_2 \to 0$$

be a short exact sequence of nonzero vector bundles. Then

$$\min(\mu(E_1), \mu(E_2)) \leq \mu(E) \leq \max(\mu(E_1), \mu(E_2)).$$

Moreover, if $\mu(E_1) = \mu(E_2)$, then $\mu(E) = \mu(E_1) = \mu(E_2)$.

Proof. This is a complete elementary statement. Say $d_1, r_1, d_2, r_2 \in \mathbf{Z}$ with r_1, r_2 nonzero and $\frac{d_1}{r_1} \leq \frac{d_2}{r_2}$. Then we have

$$\frac{d_1 + d_2}{r_1 + r_2} \leq \frac{d_2 \cdot \frac{r_1}{r_2} + d_2}{r_1 + r_2} = \frac{d_2 \cdot \frac{r_1 + r_2}{r_2}}{r_1 + r_2} = \frac{d_2}{r_2},$$

which gives the the assertion $\mu(E) \leq \max(\mu(E_1), \mu(E_2))$; the assertion $\min(\mu(E_1), \mu(E_2)) \leq \mu(E)$ is proven in exactly the same way. The last part is clear. □

Example 7.4.3 (Stable vector bundles on $\mathbf{P}^1$)**.** For any curve C, every line bundle is trivially stable and thus also semistable. If L, M are two line bundles, then $E = L \oplus M$ is semistable if and only if $d(L) = d(M)$: the if direction follows from Lemma 7.4.6 below. Conversely, if $d(L) \neq d(M)$, then say $d(L) < d(M)$. We have $\mu(E) = \frac{d(L)+d(M)}{2} > d(L) = \mu(L)$. Realizing L as a quotient of E then shows that E is not semistable.

By the same reasoning, a finite direct sum $\oplus_i L_i$ of line bundles is semistable exactly when all the L_i's have the same degree.

Specializing now to $\mathbf{P}^1$, by Grothendieck's theorem, there exists semistable vector bundles on $\mathbf{P}^1$ of slope μ exactly when μ is an integer. In this case, the category of semistable vector bundles of slope μ (for fixed μ) is identified with the category of vector spaces V via the functor $V \mapsto V \otimes \mathcal{O}(\mu)$. In particular, stable vector bundles on $\mathbf{P}^1$ coincide with line bundles.

Stable1

Lemma 7.4.4. *Fix a nonzero vector bundle E. Then E is semistable (resp. stable) if and only if for every nonzero subsheaf $F \subset E$, we have $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$.*

Proof. We explain the semistable case. The stable case is exactly the same with strict inequalities instead.

Assume that E is semistable. For a nonzero subsheaf $F \subset E$, write $\overline{F} \subset E$ for the saturation of F, so $\overline{F}/F$ is torsion. Then the exact sequence

$$0 \to F \to \overline{F} \to \overline{F}/F \to 0$$

shows that $\deg(F) \leq \deg(\overline{F})$. As F and $\overline{F}$ have the same rank, it follows that $\mu(F) \leq \mu(\overline{F})$. Thus, after replacing F with $\overline{F}$, we may assume that F is saturated in E, and thus $Q := E/F$ is a vector bundle. We must check that $\mu(F) \leq \mu(E)$, i.e., that

$$d(F)r(E) \leq d(E)r(F).$$

By semistability, we know that $\mu(E) \leq \mu(Q)$, i.e., we have

$$d(E)r(Q) \leq d(Q)r(E).$$

Now we know that $d(Q) = d(E) - d(F)$ and $r(Q) = r(E) - r(F)$. Plugging these in to last inequality, we get

$$d(E)(r(E) - r(F)) \leq (d(E) - d(F))r(E).$$

Canceling the $d(E)r(E)$ term on both sides and switching signs gives

$$d(E)r(F) \geq d(F)r(E),$$

as wanted. The exact same proof also works when E is stable.

Conversely, assume that E satisfies the conclusion of the lemma in the semistable case. Fix a bundle quotient $E \twoheadrightarrow F$ with kernel K. We must show $\mu(E) \leq \mu(F)$, and we know that $\mu(K) \leq \mu(E)$. The same reasoning used above in the opposite direction proves the desired implication. ☐

Stable2

Lemma 7.4.5. *Let E and F be semistable bundles with $\mu(E) > \mu(F)$. Then* $\mathrm{Hom}(E, F) = 0$.

Proof. Assume there exists a nonzero map $E \to F$ with image $Q \subset F$. Then $\mu(Q) \geq \mu(E)$ by semistability of E, and $\mu(Q) \leq \mu(F)$ by semistability of F (and Lemma 7.4.4). It follows that $\mu(E) \leq \mu(F)$, which is a contradiction. ☐

Stable3

Lemma 7.4.6. *Let $0 \to E_1 \to E \to E_3 \to 0$ be a short exact sequence of vector bundles with equal slopes. Then E is semistable if and only if E_1 and E_2 are so.*

In particular, direct sums of semistable vector bundles of the same slope are semistable.

Proof. Assume E is semistable. As $\mu(E_1) = \mu(E)$, it is immediate the characterisation of semistability via subsheaves that E_1 is semistable. Dually, as $\mu(E) = \mu(E_2)$, it follows from semistability of E that E_2 is semistable.

Conversely, assume both E_1 and E_2 are semistable, and write μ for the common slope of all bundles. Assume towards contradiction that E admits a subsheaf G with $\mu(G) > \mu(E)$. Then $G \cap E_1 \subset E_1$ so $\mu(G \cap E_1) \leq \mu$ if $G \cap E_1 \neq 0$; likewise, $G/G \cap E_1 \subset E_2$ so $\mu(G/G \cap E_1) \leq \mu$ if $G/(G \cap E_1) \neq 0$. Thus, if $0 \not\subset G \cap E_1 \not\subset G$, then Lemma 7.4.2 gives $\mu(G) \leq \max(\mu(G \cap E_1), \mu(G/(G \cap E_1))) \leq \mu$, which is a contradiction. Thus, we must have either $G \cap E_1 = 0$ or $G \subset E_1$. Now if $G \cap E_1 = 0$, then $G \to E_2$ is injective, so $\mu(G) \leq \mu$, which is a contradiction; a similar contradiction is reached if $G \subset E_1$, so we are done. ☐

Stable4 **Lemma 7.4.7.** *A stable vector bundle E is simple, i.e.,* $\mathrm{Hom}(E,E)$ *is a division ring. In particular, if k is algebraically closed, then* $\mathrm{Hom}(E,E) \simeq k$.

Proof. Say $f : E \to E$ is a nonzero map with image Q. As f is nonzero, Q is also nonzero, so Q is a bundle. If f is not surjective, then $\mu(Q) < \mu(E)$ as Q is a proper submodule of E. On the other hand, $\mu(Q) \geq \mu(E)$ as Q is a quotient of E; if Q is a non-trivial quotient one gets a strict inequality, and the only other option is $E \simeq Q$, which gives equality. Combining the two, this leads to the absurd statement that $\mu(E) < \mu(E)$, which is a contradiction, so f must be surjective. By rank considerations, f is also injective, and hence f is an isomorphism. □

Remark 7.4.8. In Lemma 7.4.7, if k is not algebraically closed, then it is indeed possible that $\mathrm{Hom}(E,E)$ is a division ring larger than k. For example, say C is a genus 0 curve over k without points. Fix an identification $C \otimes \overline{k} \simeq \mathbf{P}^1$. Then there exists a rank 2 vector bundle E on C such that $E \otimes \overline{k} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$ via the previous identification; for instance, the group $H^1(C, K_C) \simeq \mathrm{Ext}^1_C(\mathcal{O}_C, K_C)$ is a 1-dimensional k-vector space (as this can be checked after passage to $\overline{k}$), and we can choose E to be any non-split extension of $\mathcal{O}_C$ by K_C. It is easy to see that semistability can be detected after passage to $\overline{k}$, so E is semistable by Lemma 7.4.6. In fact, we claim that E is also stable. To see this, fix a line subbundle $L \subset E$. We must show that $d(L) < \mu(E) = \mu(E \otimes \overline{k}) = -1$. By base change, we get a line subbundle $M := L \otimes \overline{k} \subset \mathcal{O}(-1)^{\oplus 2}$. Then $M \simeq \mathcal{O}(-i)$ for some $i > 0$, and we must check that $i > 1$. If not, then $i = 1$, so L provides a k-model for $\mathcal{O}(-1)$. But then L^{-1} provides a k-model for $\mathcal{O}(1)$. But then $H^0(C, L^{-1})$ is a 1-dimensional k-vector space, and the zero locus of any nonzero section of L^{-1} gives a k-rational point of C (as the same holds true over $\overline{k}$), which contradicts our assumption that C does not have points. It follows that E is stable, and thus $A := \mathrm{Hom}(E,E)$ is a division ring over k. By base change, we also know that $A \otimes \overline{k} \simeq \mathrm{End}(\mathcal{O}(-1)^{\oplus 2}) \simeq M_2(\overline{k})$. In particular, A is a division algebra over k that is strictly larger than k. In fact, a more careful analysis shows that A is the quaternion algebra over k corresponding to C under the identification between twisted forms of $\mathbf{P}^1$ and twisted forms of $M_2(-)$ (as both sides identify with $H^1(k, \mathrm{PGL}_2)$).

Theorem 7.4.9. *Let E be a vector bundle. Then there is a unique filtration*

$$0 = E_0 \subset E_1 \subset \dots \subset E_r = E$$

such that $Q_i := E_i/E_{i-1}$ is semistable, and $\mu(Q_i) > \mu(Q_{i+1})$ for $i \geq 1$.

This filtration is called the *Harder-Narasimhan* (or just HN) filtration. The subsheaf $E_0 \subset E$ is called the *maximal destabilizing subsheaf* of E.

Proof. We first observe that set of all possible slopes of all nonzero subsheaves of E is bounded from above. To see this, note that E can be realized as a subsheaf of $L^{\oplus n}$ where L is sufficiently ample and $n \gg 0$ (by writing $E^\vee$ as a quotient of $L^{\oplus -n}$ for $n \gg 0$). Lemma 7.4.6 implies that $L^{\oplus -n}$ is semistable, so slopes of its subsheaves are bounded from the above, and thus the same holds true for E. Let μ be the largest possible slope that occurs amongst all nonzero subsheaves of E.

Next, we observe that $F, G \subset E$ are two subsheaves with $\mu(F) = \mu(G) = \mu$, then $\mu(F+G) = \mu$ as well. Indeed, $F + G$ is a quotient of $F \oplus G$. Now Lemma 7.4.2 implies that $F \oplus G$ is semistable of slope μ, and hence $\mu(F+G) \geq \mu$. But μ was chosen to be the maximal slope amongst all subsheaves of E, so $\mu(F+G) = \mu$ as well. It follows that there is a maximal nonzero subsheaf $E_1 \subset E$ of maximal rank with $\mu(E_1) = \mu$. It is then also clear from maximality of μ that E_1 is semistable: any subsheaf of E_1 is also a subsheaf of E, and hence must have slope $\leq \mu$. This gives the first step of the filtration.

By induction, E/E_1 has a filtration as prescribed in the theorem. Taking preimages in E then gives a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_r = E$ of E with $\mu(Q_i) \geq \mu(Q_{i+1})$ for $i \geq 2$. It remains to check that $\mu(E_1) > \mu(E_2/E_1)$. If not, then $\mu(E_1) \leq \mu(E_2/E_1)$. But then $\mu(E_1) = \min(\mu(E_1), \mu(E_2/E_1)) \leq \mu(E_2)$, so we have found a subsheaf $E_2 \subset E$ that is strictly larger than E_1 with slope $\geq \mu$, which is impossible by choice of E_1. Thus, we have constructed the filtration.

For uniqueness, say $0 = G_0 \subset G_1 \subset ... \subset G_k = G$ is another filtration with G_i/G_{i+1} being semistable and $\mu(G_i/G_{i-1}) > \mu(G_{i+1}/G_i)$ for $i \geq 1$. Let k be minimal such that $E_1 \subset G_k$, so the map $E_1 \to G_k/G_{k-1}$ is nonzero. We then have

$$\mu(E_1) \leq \mu(G_k/G_{k-1}) \leq \mu(G_1) \leq \mu(E_1),$$

where the first inequality uses Lemma 7.4.5, the second is true by assumption on the filtration $\{G_i\}$, and the last holds true as $\mu(E_1) = \mu$ is maximal amongst the slope of all subsheaves of E. It follows that all these numbers must be the same. As $\mu(G_k/G_{k-1})$ is a strictly decreasing function of k, it follows that $k = 1$, and thus $E_1 \subset G_1$ with $\mu(E_1) = \mu(G_1)$. But then the maximality of E_1 amongst all subsheaves of E with slope μ ensures that $E_1 = G_1$. Quotienting by this subsheaf, one proceeds by induction. ☐

leSSEElliptic **Corollary 7.4.10.** *The HN filtration of any vector bundle E on an elliptic curve C is split. In particular, indecomposable vector bundles are semistable.*

Proof. Let $0 = E_0 \subset E_1 \subset ... \subset E_r = E$ be the HN filtration for E, and let $Q_i = E_i/E_{i+1}$. We have $\mu(Q_i) > \mu(Q_{i+1})$. By induction, the HN filtration for E_{r-1} is split, i.e., we have an isomorphism $E_{r-1} \cong \oplus_{i=1}^{r-1} Q_i$. The obstruction to splitting the HN filtration from E is thus an element of $\oplus_{i=1}^{r-1} \mathrm{Ext}^1(Q_r, Q_i)$. By Serre duality (as C has genus 1), it is enough to show that $\mathrm{Hom}(Q_i, Q_r) = 0$ for $i \in \{1, ..., r-1\}$. But $\mu(Q_i) > \mu(Q_r)$ for such i, so the claim follows from Lemma 7.4.5. ☐

We give one source of examples of semistable bundles.

Example 7.4.11. Say $f : D \to C$ is a finite étale map of curves. Then $A = f_*\mathcal{O}_D$ is semistable of slope (or, equivalently, degree 0). To see this, we are allowed to pass to finite flat covers of C by smooth connected curves: this passage scales the slope by the degree of the cover, and thus semistability can be descended through the pullback. But, as f is finite étale, there is some finite flat cover $g : C' \to C$ of smooth connected curves such that $D' := g^*D \to C'$ is simply $\sqcup_{i=1}^n C' \to C'$. The corresponding structure sheaf is $\oplus_{i=1}^n \mathcal{O}_{C'}$, which is semistable by Lemma 7.4.6.

7.4.2 Atiyah's theorem

Let E be an elliptic curve over k. The base point $e \in E(k)$ determines a line bundle $L = \mathcal{O}_E(e)$ of degree 1, and the map $\phi_L : E \to E^t$ is an isomorphism: its degree is the square of $H^0(E, L)$ by Corollary 7.3.8, and is thus 1-dimensional by Riemann-Roch. From now on, identify E with E^t via this map.

Our first goal is to prove the following classification of semistable bundles on E.

SSElliptic **Theorem 7.4.12.** *Fix a rational number μ. The category $\mathrm{Vect}(E)_\mu$ of semistable bundles with slope μ on E is identified with the category $\mathrm{Coh}_{tors}(E)$ of coherent sheaves on E with finite support. If we denote this association by $F \mapsto T(F)$, then $\ell(T(F)) = \gcd(\deg(F), \mathrm{rank}(F))$.*

In particular, there exist semistable bundles with all slopes. The first step of the proof is to understand the behaviour of semistable bundles under the Fourier-Mukai transform.

Atiyah1 **Lemma 7.4.13.** *For any $F \in D^b_{coh}(E)$, we have*

$$\deg(\Phi_E(F)) = -\mathrm{rank}(F) \quad \textit{and} \quad \mathrm{rank}(\Phi_E(F)) = \deg(F).$$

In particular, $\mu(\Phi_E(F)) = -\mu(F)^{-1}$ provided $\mu(F) \neq 0$.

Proof. As E is an elliptic curve, we have $\chi(E, F) = \deg(F)$: this is true for vector bundles by Riemann-Roch, and thus follows in general by additivity. Using this observation, the claims follow from the canonical isomorphisms

$$R\Gamma(E, \Phi_E(F)) = F[-1]|_{\{e\}} \quad \text{and} \quad R\Gamma(E, F) = \Phi_E(F)|_{\{e\}}$$

from Lemma 6.3.7. ☐

Atiyah2 **Lemma 7.4.14.** *Say F is a vector bundle on E with $\mu := \mu(F) < 0$. Then $\Phi_E(F)[1]$ is a semistable bundle with slope $-\mu^{-1}$.*

Proof. It is enough to prove this after base change to the algebraic closure. By Lemma 7.4.6, we may also assume that F is indecomposable. Then $\Phi_E(F)$ is also indecomposable. As F has negative slope, we have $H^0(E, F \otimes L) = \mathrm{Hom}(L^{-1}, E) = 0$ for all $L \in \mathrm{Pic}^0(E)$ by Lemma 7.4.5 as L^{-1} is semistable with slope 0. It follows that all fibers of $\Phi_E(F)$ lie in cohomological degree 1. By semicontinuity, it follows that $\Phi_E(F)[1]$ is a vector bundle whose slope is $-\mu^{-1}$ by Lemma 7.4.13. As we already know this bundle is indecomposable, it follows from Corollary 7.4.10 that $\Phi_E(F)[1]$ is indeed semistable. ☐

Atiyah3 **Lemma 7.4.15.** *For any slope μ, there is an equivalence $\mathrm{Vect}(E)_\mu \simeq \mathrm{Vect}(E)_0$ obtained by repeatedly applying $\Phi_E[1]$ and $- \otimes \mathcal{O}_E(e)$ (and their inverses) in some order. Any such equivalence preserves the quantity $\gcd(\mathrm{rank}(F), \deg(F))$.*

Proof. If $\mu = 0$, there is nothing to do. If not, then using Lemma 7.4.14, we are allowed to replace μ with $-\mu^{-1}$ using the Fourier-Mukai equivalence. Moreover, tensoring with $\mathcal{O}(e)$ certainly gives an equivalence $\mathrm{Vect}(E)_\mu \simeq \mathrm{Vect}(E)_{\mu+1}$. To see why this allows us to eventually reach $\mu = 0$, we use some group theory. Recall that $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on $\mathbf{Q} \cup \{\infty\} = \mathbf{P}^1(\mathbf{Q})$ via fractonal linear transformation. Moreover, $\mathrm{SL}_2(\mathbf{Z})$ is generated by the matrices

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \textit{ and } T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrix S acts on $\mathbf{Q} \cup \{\infty\}$ via $\mu \mapsto -\mu^{-1}$, while T acts via $\mu \mapsto \mu + 1$. These coincide by the effect on slopes of the equivalences induces by $\Phi_E[1]$ and $- \otimes \mathcal{O}(e)$ of semistable bundles. As S and T generate $\mathbf{SL}_2(\mathbf{Z})$ and the latter acts transitively on $\mathbf{Q} \cup \{\infty\}$, it follows that some combination of iterations of $\Phi_E[1]$ and $- \otimes \mathcal{O}(e)$ (and their inverses) gives an equivalence $\mathrm{Vect}(E)_\mu \simeq \mathrm{Vect}(E)_0$.

The statement about preservation of $\gcd(\mathrm{rank}(F), \deg(F))$ under the above equivalence follows from Lemma 7.4.14 for the Fourier-Mukai functor, and the following observation for the $- \otimes \mathcal{O}_E(e)$ functor: if $\frac{a}{b}$ is a rational number, then $\gcd(a, b) = \gcd(a + b, b)$. ☐

Proof of Theorem 7.4.12. By Lemma 7.4.15, it is enough to show

1. $\Phi_E[1]$ gives an equivalence $\mathrm{Vect}(E)_0 \simeq \mathrm{Coh}_{tors}(E)$.
2. For any $F \in \mathrm{Vect}(E)_0$, we have $\ell(\Phi_E(F)[1]) = \gcd(\deg(F), \mathrm{rank}(F)) = \mathrm{rank}(F)$ (as $\deg(F) = 0$).

For both of these, we may assume $k = \overline{k}$.

For (1), say F is a semistable bundle of degree 0. We shall prove by induction on the rank of F that $\Phi_E(F[1])$ is a torsion coherent sheaf. This gives a functor $\mathrm{Vect}(E)_0 \to \mathrm{Coh}_{tors}(E)$. The case of rank 1 follows from Lemma 6.3.3. In larger rank, assume first that there exists a degree 0 line bundle L and a nonzero map $L \to F$. Then $L \subset F$ must be saturated: if it was not saturated, its saturation $\overline{L} \subset F$ would give a line subbbundle in F with $\mu(\overline{L}) = \deg(\overline{L}) > \deg(L) = 0$, which is not possible by semistability of F. But then F/L is a semistable vector bundle of degree 0, and hence $F/L \in \mathrm{Vect}(E)_0$ as well. By induction, we know L and F/L carried to torsion coherent sheaves by $\Phi_E[1]$, and thus the same holds true for E. It remains to check the claim for those $E \in \mathrm{Vect}(E)_0$ that admit no non-trivial maps from degree 0 bundles. But then $H^0(E, F \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(E)$, so all fibers $\Phi_E(F)|_{\{x\}}$ of $\Phi_E(F)$ are concentrated in cohomological degree -1. By semicontinuity, it follows that $\Phi_E(F[1])$ is a vector bundle. But the rank of this vector bundle is given by $\chi(X, F) = \deg(F)$ by Lemma 7.4.13, and is thus 0 as F has slope 0. But this means that $\Phi_E(F) = 0$, and hence $F = 0$, as Φ_E was an equivalence. This gives the functor required in (1).

Conversely, as any torsion coherent sheaf on E is an iterated extension of copies of the structure sheaves of closed points, it is easy to see that if $\Phi_E(F[1])$ is a torsion coherent sheaf, then F must be an iterated extension of degree 0 line bundles on E, and hence $E \in \mathrm{Vect}(E)_0$ by Lemma 7.4.6. This finishes the proof of (1).

It remains to check that $\ell(\Phi_E(F[1])) = \mathrm{rank}(F)$. As $\Phi_E(\Phi_E(F[1])) = [-1]^*F$, it follows that

$$\mathrm{rank}(F) = \mathrm{rank}([-1]^*F) = \chi(E, \Phi_E(F[1])) = \ell(\Phi_E(F[1])),$$

where the first equality is obvious, the second uses Lemma 6.3.7, and third uses that $\Phi_E(F[1])$ is a coherent sheaf with finite support. $\square$

StableEll1

Corollary 7.4.16. *Assume $k = \overline{k}$. Let F be an indecomposable vector bundle of rank r and degree d on E. Set $\mu = \frac{d}{r}$ to be the slope. The following are equivalent:*

1. *F is stable.*
2. *$\gcd(d, r) = 1$.*
3. *F is simple as an object of $\mathrm{Vect}(E)_\mu$ (i.e., has no nonzero nontrivial subobjects).*
4. *F is simple as a sheaf (i.e., $\mathrm{Hom}(E, E) = k$).*

In particular, there exist stable bundles of every slope μ.

Proof. Lemma 7.4.5 gives (1) $\Rightarrow$ (2).

The equivalence of (2) and (3) follows from Theorem 7.4.12 as a torsion coherent sheaf on E is simple exactly when it has length 1; this requires $k = \overline{k}$.

The equivalence of (2) and (4) also follows from Theorem 7.4.12 as a torsion coherent sheaf on E has no non-scalar endomorphisms exactly when it is the skyscraper sheaf for a closed point.

For (3) $\Rightarrow$ (1), assume F is simple as an object of $\mathrm{Vect}(E)_\mu$. As F is indecomposable, we know that F is semistable by Corollary 7.4.10. We wish to show that F is stable. Say $G \subset F$ is a nonzero nontrivial subsheaf. By semistability, $\mu(G) \leq \mu(F) = \mu$. If $\mu(G) = \mu(F) = \mu$, then the semistability of F would pass down to G, so $G \subset F$ would be a nonzero nontrivial subobject of F in $\mathrm{Vect}(E)_\mu$, which violates the simplicity of F. Thus, $\mu(G) < \mu(F)$, which proves stability.

To see the existence of stable bundles, it is enough to produce a simple indecomposable object of $\mathrm{Vect}(E)_\mu$. But any F with $T(F)$ being a skyscraper sheaf of length 1 provides such a sheaf. $\square$

Chapter 8

Generic vanishing theorems

In this chapter, we work exclusively over $\mathbf{C}$. For any smooth proper variety X with a base point $x \in X(\mathbf{C})$, write $a : X \to \mathrm{Alb}(X)$ for the Albanese morphism: this is the universal map from X into an abelian variety that carries the base point to the identity; other properties shall be explained later. The goal is to prove the following theorem:

GVprototype **Theorem 8.0.1** (Green-Lazarsfeld)**.** *Let X be a smooth projective variety. For a general $L \in \mathrm{Pic}^0(X)$ and $i < \dim(a(X))$, we have $H^i(X, L) = 0$. In particular, if a is generically finite, then $(-1)^{\dim(X)}\chi(X, \mathcal{O}_X) \geq 0$.*

This theorem is a prototype of a "generic vanishing theorem" as it asserts that some cohomology group is 0 is a generic value of the parameter. We shall give an exposition of this theorem following a paper of Hacon that heavily uses the Fourier-Mukai transform; the non-algebraic input comes from a deep vanishing theorem of Kollár in Hodge theory, that we use as a blackbox.

Example 8.0.2. When X is a smooth projective curve of genus $g \geq 1$, the Albanese $\mathrm{Alb}(X)$ coincides with the Jacobian of C. The map $a : X \to \mathrm{Alb}(X)$ is non-constant, and thus generically finite. Thus, in this case, Theorem 8.0.1 states that $H^0(C, L) = 0$ for a general degree 0 line bundle L on C, which is easy to see directly: any non-constant map $\mathcal{O}_C \to L$ must be an isomorphism for degree reasons. Note that if $g \geq 2$, then $\chi(C, L) = 1 - g < 0$, so $H^1(C, L)$ is typically nonzero. In particular, the bound on i in Theorem 8.0.1 is sharp.

8.1 The Picard scheme and the Albanese variety

Let X be a smooth proper geometrically connected variety over a field k of characteristic 0. Fix a base point $x \in X(k)$. We shall use the following facts about Picard and Albanese varieties attached to X.

1. Consider the functor $\underline{\mathrm{Pic}}(X)$ on k-schemes defined by attaching to each k-scheme T the set of isomorphism classes of pairs $(L \in \mathrm{Pic}(T \times X), \iota : L|_{T \times \{x\}} \simeq \mathcal{O}_T)$. This functor is naturally valued in abelian groups via tensor products of line bundles. Moreover, it is representable by a locally finitely presented group scheme also denoted $\underline{\mathrm{Pic}}(X)$ and called the *Picard scheme*; write $\mathcal{P}_X$ for the universal or *Poincare* line bundle on $\underline{\mathrm{Pic}}(X)) \times X$ and ι^{univ} for the universal trivialization over x. The association $X \mapsto \underline{\mathrm{Pic}}(X)$ is contravariantly functorial in X (for base point preserving maps).

   Proof. The existence of the Picard scheme is a non-trivial theorem, and we do not prove it here. □

2. The identity component $\underline{\mathrm{Pic}}^0(X)$ of $\underline{\mathrm{Pic}}(X)$ is an abelian variety. We shall write $\mathrm{Pic}^0(X) = \underline{\mathrm{Pic}}^0(X)(k)$ for the "degree 0" line bundles on X; the notion of "degree 0" can also be defined in other more direct

ways. The tangent space $T_e\underline{\mathrm{Pic}}^0(X)$ is canonically identified with $H^1(X, \mathcal{O}_X)$. In particular, $\underline{\mathrm{Pic}}^0(X)$ has dimension

$$\dim H^1_k(X, \mathcal{O}_X) = \frac{1}{2}\dim_{\mathbf{C}} H^1(X^{an}, \mathbf{C}) =: \frac{\beta_1(X)}{2},$$

where the second equality makes sense when $k = \mathbf{C}$ and uses Hodge theory.

Proof. To calculate the tangent space, we must identify the kernel of $\underline{\mathrm{Pic}}(X)(k[\epsilon]) \to \underline{\mathrm{Pic}}(X)(k)$ with $H^1(X, \mathcal{O}_X)$. Unwinding definitions, we must identify the kernel of $\mathrm{Pic}(X[\epsilon]) \to \mathrm{Pic}(X)$ with $H^1(X, \mathcal{O}_X)$, where $X[\epsilon] := X \otimes_k k[\epsilon]$. Consider the exact sequence of sheaves

$$0 \to 1 + \epsilon\mathcal{O}_X \to \mathcal{O}_X[\epsilon]^* \to \mathcal{O}_X^* \to 0$$

of Zariski sheaves on X. The term on the left is identified with $\mathcal{O}_X$ via the map $1 + \alpha \mapsto \alpha$ on local sections. Passing to the long exact sequence, we get an exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X[\epsilon]^*) \to H^1(X, \mathcal{O}_X^*),$$

where injectivity on the left comes from the surjectivity of $H^0(X, \mathcal{O}_X[\epsilon]^*) \to H^0(X, \mathcal{O}_X^*) \simeq k^*$. As the ringed space $(X, \mathcal{O}_X[\epsilon])$ identifies with the scheme $X[\epsilon] := X \otimes_k k[\epsilon]$, the above sequence can be viewed as a short exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X[\epsilon]) \to \mathrm{Pic}(X),$$

which gives the promised identification. □

3. The pair $(\mathcal{P}_X, \iota^{univ})$ defines, by the universal property of the dual abelian variety, a morphism

$$a : X \to \underline{\mathrm{Pic}}^0(X)^t.$$

that carries $x \in X(k)$ to the origin via our trivialization ι^{univ}. Informally, this map sends $y \in X(k)$ to the line bundle $\mathcal{P}_X|_{\underline{\mathrm{Pic}}^0(X)\times\{y\}}$. This is the universal map from X to an abelian variety that carries $x \in X(k)$ to the origin. We often write this map as $a : X \to \mathrm{Alb}(X)$ and call it the *Albanese map* for X. This map is proper, and the $\dim(a(X))$ is often called the *Albanese dimension* of X.

Proof of universality of $\mathrm{Alb}(X)$. Fix a map $b : X \to B$ into an abelian variety B that carries $x \in X(k)$ to the origin. Viewing $B = A^t$ for $A = B^t$, this corresponds to a line bundle $M \in \mathrm{Pic}(X \times A)$ that is trivialized along $X \times \{e\}$. By the geometric connectedness of X, there is a unique trivialization ι of $M|_{\{x\}\times A}$ compatible with the given trivialization over $M|_{\{x\}\times\{e\}}$. The resulting datum (M, ι) defines a map $A \to \underline{\mathrm{Pic}}^0(X)$ which induces a map $\mathrm{Alb}(X) \to B$ on passage to duals. One can check using universal properties that the map $X \xrightarrow{a} \mathrm{Alb}(X) \to B$ obtained this way agrees with b. □

4. The Albanese map $a : X \to \mathrm{Alb}(X)$ induces a map $\underline{\mathrm{Pic}}(\mathrm{Alb}(X)) \to \underline{\mathrm{Pic}}(X)$ by the functoriality of the Picard scheme. This map induces an isomorphism on identity components, i.e., $\underline{\mathrm{Pic}}^0(\mathrm{Alb}(X)) \simeq \underline{\mathrm{Pic}}^0(X)$. In particular, any $L \in \mathrm{Pic}^0(X)$ is obtained via pullback along a.

Proof. This is simply a manifestation of biduality for abelian varieties. □

8.2 Hacon's theorem: abstract version

Let A be an abelian variety over a field k of dimension g. We adopt the following notation:

1. Write $\Phi_A : D(A) \to D(A^t)$ for the Fourier transform attached to the Poincare bundle $\mathcal{P}_A$ on $A \times A^t$.

2. For any smooth proper k-scheme X of dimension n, write $\mathbb{D}_X(-) = \underline{\mathrm{RHom}}(-, \omega_X[n])$ for the Verdier duality functor. This functor satisfies $\mathbb{D}_X(F \otimes E^\vee) = \mathbb{D}_X(F) \otimes E$ for any vector bundle E. The crucial property we need is the following: if $f : X \to Y$ be a map between two such schemes, then $Rf_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ Rf_*$.

3. For any ample line bundle L on A^t, write $E(L) = \Phi_{A^t}(L)$. By §7.3, we know the following:
   - $E(L)$ is a nonzero vector bundle placed in degree 0.
   - The pullback $\phi_L^* E(L)$ identifies with $L^{-1} \otimes H^0(A, L)$ on A^t.

Hacon's basic theorem is the following:

Hacon1 **Theorem 8.2.1.** *Let $F \in D^b_{coh}(A)$. The following are equivalent:*

1. *For L sufficiently ample, we have $H^i(A, F \otimes E(L)^\vee) = 0$ for $i \neq 0$.*
2. *For L sufficiently ample, we have $H^i(A^t, \Phi_A(\mathbb{D}_A(F)) \otimes L) = 0$ for $i \neq 0$.*
3. *The complex $\Phi_A(\mathbb{D}(F))$ is concentrated in degree 0.*

Proof. (1) is equivalent to $\mathbb{D}_k(R\Gamma(A, F \otimes E(L)^\vee))$ is concentrated in degree 0. By Grothendieck duality, we have

$$\mathbb{D}_k R\Gamma(A, F \otimes E(L)^\vee) \simeq R\Gamma(A, \mathbb{D}_A(F \otimes E(L)^\vee)) \simeq R\Gamma(A, \mathbb{D}_A(F) \otimes E(L)),$$

where the last isomorphism uses that $E(L)$ is a vector bundle. Now if $pr_1 : A \times A^t \to A$ and $pr_2 : A \times A^t \to A^t$ are the two projection maps, then we have

$$R\Gamma(A, \mathbb{D}_A(F) \otimes E(L)) = R\Gamma(A, \mathbb{D}_A(F) \otimes pr_{1,*}(\mathcal{P}_A \otimes pr_2^* L)) \simeq R\Gamma(A \times A^t, pr_1^* \mathbb{D}_A(F) \otimes \mathcal{P}_A \otimes pr_2^* L) \simeq R\Gamma(A^t, \Phi_A(\mathbb{D}_A(F)) \otimes L),$$

where the second and third isomorphisms use the projection formulas for pr_1 and pr_2 respectively. This shows (1) and (2) are equivalent.

The equivalence of (2) and (3) is a general fact:

Lemma 8.2.2. *Fix a projective k-scheme X and $M \in D^b_{coh}(X)$. Then M is concentrated in degree 0 if and only if $R\Gamma(A^t, M \otimes L)$ is concentrated in degree 0 for L sufficiently ample.*

Proof. The "only if" direction is clear from Serre vanishing. Conversely, given $M \in D^b_{coh}(X)$, we have a spectral sequence

$$E_2^{p,q} : H^p(X, \mathcal{H}^q(M) \otimes L) \Rightarrow H^{p+q}(X, M \otimes L).$$

Assume that $R\Gamma(A^t, M \otimes L)$ is concentrated in degree 0 for all sufficiently ample L. For sufficiently ample L, the spectral sequence above then degenerates to give

$$H^0(X, \mathcal{H}^j(M) \otimes L) = H^j(X, M \otimes L)$$

for all j. Our hypothesis then ensures that $H^0(X, \mathcal{H}^j(M) \otimes L) = 0$ for all $j \neq 0$ and sufficiently ample L. But this cannot be true if $\mathcal{H}^j(M) \neq 0$ as a sufficiently positive twist of any nonzero coherent sheaf has non-trivial global sections on a projective scheme. It follows that $\mathcal{H}^j(M) = 0$ for $j \neq 0$. □

□

Next, we analyze some basic structural features of sheaves satisfying the conclusion of the theorem above. For convenicen, let us make the following *ad hoc* definition.

Definition 8.2.3. Any $F \in D^b_{coh}(A)$ satisfying the conclusion of Theorem 8.2.1 is called a *Hacon complex* on A; if $F \in \mathrm{Coh}(A)$ is a coherent sheaf, then we say that F is a *Hacon sheaf*.

Our main goal is to prove the following property of Hacon sheaves:

Hacon3 **Theorem 8.2.4.** *Let F be a Hacon complex on A.*

1. *The locus $S^i(A^t, F) = \{x \in A^t(k) \mid H^i(A, F \otimes L_x) \neq 0\}$ is a closed subset of A^t. More each irreducible component of this set has codimension $\geq i$.*
2. *We have $S^i(A^t, F) \subset S^{i-1}(A^t, F)$ for $i > 0$.*
3. *If F is additionally a coherent sheaf, then have $\chi(A, F) \geq 0$.*

To prove this, we shall need the following compatibility of the Fourier transform with duality.

FMDuality **Lemma 8.2.5.** *For any $F \in D^b_{coh}(A)$, there is an isomorphism*

$$\mathbb{D}_{A^t}(\Phi_A(F)) \simeq [-1]^*\Phi_A(\mathbb{D}_A(F))[g].$$

Proof. By Grothendieck duality for $pr_2 : A \times A^t \to A^t$, we have

$$D_{A^t}\Phi_A(F) \simeq pr_{2,*}\mathbb{D}_{A\times A^t}(pr_1^*F \otimes \mathcal{P}_A).$$

Note that $\mathbb{D}_{A\times A^t} = \underline{\mathrm{RHom}}(-, \mathcal{O}_{A\times A^t}[2g])$. Using the modular interpretation of the Poincare bundle, one shows that $([1] \times [-1])^*\mathcal{P}_A \simeq \mathcal{P}_A^{-1}$. Using these, the above becomes

$$pr_{2,*}\mathbb{D}_{A\times A^t}(pr_1^*F \otimes \mathcal{P}_A) \simeq pr_{2,*}\Big(pr_1^*\mathbb{D}_A(F)[g] \otimes ([1] \times [-1])^*\mathcal{P}_A\Big) \simeq pr_{2,*}([1] \times [-1])^*\Big(pr_1^*\mathbb{D}_A(F)[g] \otimes \mathcal{P}_A\Big),$$

where we use $([1] \times [-1]) \circ pr_1 = pr_1$ for the second isomorphism. Using the projection formula for the square

$$\begin{tikzcd} A \times A^t \arrow[r, "pr_2"] \arrow[d, "{[1]\times[-1]}"] & A^t \arrow[d, "{[-1]}"] \\ A \times A^t \arrow[r, "pr_2"] & A^t \end{tikzcd},$$

the above formula simplifies to

$$pr_{2,*}([1] \times [-1])^*\Big(pr_1^*\mathbb{D}_A(F)[g] \otimes \mathcal{P}_A\Big) \simeq [-1]^*pr_{2,*}(pr_1^*\mathbb{D}_A(F) \otimes \mathcal{P}_A)[g] \simeq [-1]^*\Phi_A\mathbb{D}_A(F)[g],$$

as wanted. ∎

Using this property, we obtain an alternative and more useful characterization of a Hacon complex:

HaconGV **Proposition 8.2.6** (Characterization of Hacon complexes)**.** *Let $F \in D^b_{coh}(A)$. The following are equivalent:*

1. *F is a Hacon complex.*
2. *$\underline{\mathrm{RHom}}(\Phi_A(F), \mathcal{O}_{A^t})$ is concentrated in degree 0.*
3. *$\Phi_A(F) = \underline{\mathrm{RHom}}(E, \mathcal{O}_{A^t})$ for a coherent sheaf E on A^t.*

Proof. For (1) ⇔ (2): note that F is a Hacon complex exactly when $\Phi_A(\mathbb{D}_A(F))$ is concentrated in degree 0. By Lemma 8.2.5, this happens exactly when $\mathbb{D}_{A^t}\Phi_A(F)$ is concentrated in homological degree g. But $\mathbb{D}_{A^t} = \underline{\mathrm{RHom}}(-, \mathcal{O}_{A^t}[g])$ as ω_{A^t} is the trivial bundle. Thus, F is a Hacon complex exactly when $\underline{\mathrm{RHom}}(\Phi_A(F), \mathcal{O}_{A^t})$ is concentrated in degree 0.

For (2) ⇔ (3): Write $(-)^\vee = \underline{\mathrm{RHom}}(-, \mathcal{O}_{A^t}) \simeq \mathbb{D}_{A^t}[-g]$. Then $(-)^\vee$ is a duality: it is a contravariant equivalence, and there is a canonical identification $(G^\vee)^\vee \simeq G$ for any $G \in D^b_{coh}(A)$. The equivalence of (2) and (3) is now immediate. ∎

Thus, to understand the supports of the Fourier transform of a Hacon complex, we must understand locally the structure of $\mathrm{Ext}^*(M, R)$. The following general commutative algebra lemma will be quite useful in this regard:

CASupport **Lemma 8.2.7.** *Let R be a regular ring, and let M be a finitely generated R-module. Then $\mathrm{Ext}^i_R(M, R)$ is supported in codimension $\geq i$, i.e., it vanishes when localized at any prime of R with height $< i$.*

Proof. Let $\mathfrak{p}$ be a height j prime of R. We must show that $\mathrm{Ext}^i_R(M, R) \otimes_R \kappa(\mathfrak{p}) = 0$ for $i > j$. But we have

$$\mathrm{Ext}^i_R(M, R) \otimes_R R_{\mathfrak{p}} \simeq \mathrm{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}})$$

as the formation of Ext-groups commutes with localization for finitely generated R-modules. It is enough to show the groups displayed above vanish for $i > j$. Now $R_{\mathfrak{p}}$ is a regular local ring of dimension j, so the category of $R_{\mathfrak{p}}$-modules has global dimension j, and hence the above Ext-groups must vanish for $i > j$, as wanted. $\square$

Next, we explain why the Fourier transform of a Hacon complex is constrained:

Hacon2 **Corollary 8.2.8** (Supports of the Fourier transform)**.** *Let F be a Hacon complex. Then $\mathrm{Supp}(\mathcal{H}^i\Phi_A(F)) \subset A^t$ has codimension $\geq i$ for all i.*

If F is additionally assumed to be a coherent sheaf (or merely that $F \in D^{\geq 0}$), then $\Phi_A(F)|_U$ is concentrated in degree 0 for a sufficiently small non-empty open subset $U \subset A^t$.

Proof. As F is a Hacon complex, we know from Proposition 8.2.6 that $\Phi_A(F)$ has the form $\underline{\mathrm{RHom}}(E, \mathcal{O}_{A^t})$ for a coherent sheaf E on A^t. The first part of the claim then follows from Lemma 8.2.7.

For the second part, note that $M := \Phi_A(F)|_U \in D^{\leq 0}$ by the first part for $U \subset A^t$ sufficiently small and non-empty. On the other hand, if $F \in D^{\geq 0}$, then $\Phi_A(F) \in D^{\geq 0}$ as well, as one immediately checks from that flatness of pr_1 and $\mathcal{P}_A$. Combining these assertions shows that $\Phi_A(F)|_U$ is concentrated in degree 0 for such F, as wanted. $\square$

We can now prove Theorem 8.2.4.

Proof of Theorem 8.2.4. Let F be a Hacon complex on A. For (1), we must show that

$$S^i(A^t, F) = \{x \in A^t(k) \mid H^i(A, F \otimes L_x) \neq 0\}$$

is a closed subset of A^t of codimension $\geq i$. This follows Corollary 8.2.8 and the following general lemma applied to $M = \Phi_A(F)$:

Lemma 8.2.9. *Let R be a commutative ring, and let M be a perfect complex. Then, for each integer i, we have*

$$S^i(R, M) := \{x \in \mathrm{Spec}(R) \mid H^i(M \otimes_R \kappa(x)) \neq 0\} \subset \cup_{j \geq i} \mathrm{Supp}(H^j(M)).$$

Moreover, each $S^i(R, M)$ is a closed subset of $\mathrm{Spec}(R)$.

Proof. Assume that containment in the statement of the lemma is false. Then there exists some prime $x \in \mathrm{Spec}(R)$ such that $H^i(M \otimes_R \kappa(x)) \neq 0$ but $H^j(M) \otimes_R R_x = 0$ for $j \geq i$. The second condition implies that $M_x := M \otimes_R R_x \in D^{<i}$. But then $M_x \otimes_{R_x} \kappa(x) \in D^{<i}$, which is a contradiction because $M \otimes_R \kappa(x) \simeq M_x \otimes_{R_x} \kappa(x)$ is assumed to have a non-trivial H^i.

The closedness is a general fact, proven as in Corollary 2.2.6. $\square$

For (2), we must show that $S^i(A, F) \subset S^{i-1}(A, F)$ for $i > 0$. As F is a Hacon complex, we have $F \simeq \underline{\mathrm{RHom}}(E, \mathcal{O}_{A^t})$ for a coherent sheaf E on A^t from Proposition 8.2.6. For any $x \in A^t(k)$, we have the base change isomorphism

$$\underline{\mathrm{RHom}}(E, \mathcal{O}_{A^t}) \otimes \kappa(x) \simeq \underline{\mathrm{RHom}}(E, \kappa(x)).$$

The claim then follows from the following general commutative algebra lemma:

Lemma 8.2.10. *Let R be a noetherian local ring, and let M be a finitely generated R-module. If $\mathrm{Ext}^i_R(M,k) \neq 0$ and some $i > 0$, then $\mathrm{Ext}^{i-1}_R(M,k) \neq 0$.*

Proof. Let $K^\bullet$ be a minimal free resolution of M; this means that each K^i is finite free, and the differential is 0 modulo the maximal ideal of R. Any such complex is unique up to non-unique isomorphism. Then $\dim_k \mathrm{Ext}^i_R(M,k) = \mathrm{rank}(K^i)$. We must thus show that if $K^i \neq 0$ for some $i > 0$, then $K^{i-1} \neq 0$ as well. But this is just a general fact about minimal free resolutions: if some $K^j = 0$ for $j \geq 0$, then the complex $0 \to K^{j-1} \to .. \to K^0$ would give a necessarily minimal resolution for M, and hence minimality ensures that $K^i = 0$ for all $i > j$. □

For (3): we must show that if F is a Hacon sheaf, then $\chi(A,F) \geq 0$. Note that (1) implies that $H^i(A, F \otimes L) = 0$ for $i > 0$ and L lying in the complement of the proper closed subset $\cup_{i>0} S^i(A,F) \subset A^t$. As F is a sheaf, this implies $\chi(A, F \otimes L) = h^0(A, F \otimes L) \geq 0$ for any such L. On the other hand, As A^t is connected, we also have $\chi(A, F \otimes L) = \chi(A,F)$ for any $L \in A^t$, so we obtain the claim. □

8.3 Hacon's theorem: geometric version

Let X be a smooth projective variety over an algebraically closed field k of characteristic 0. Let $a : X \to A := \mathrm{Alb}(X)$ be the Albanese map.

Theorem 8.3.1. *For each $k \geq 0$, the sheaf $F := R^k a_* \omega_X$ is a Hacon sheaf.*

Proof. Let L be an ample line bundle on A^t, and let $E(L) = \Phi_{A^t}(L)$ be the attached vector bundle on A^t; we know that $\phi_L^* E(L) \simeq H^0(A^t, L) \otimes L^{-1}$, where $\phi_L : A^t \to A$ is the isogeny attached to L. By Theorem 8.2.1, we must show that $H^i(A, F \otimes E(L)^\vee) \simeq 0$ for $i \neq 0$.

First, we check that it suffices to show that $H^i(A, F \otimes E(L)^\vee \otimes \phi_{L,*}\mathcal{O}_{A^t}) = 0$ for $i \neq 0$. Indeed, the map $\mathcal{O}_A \to \phi_{L,*}\mathcal{O}_A$ is a direct summand of sheaves: a retraction is provided by the trace map divided by the degree of ϕ_L.

By the projection formula and the preceding formula for $\phi_L^* E(L)$, we are reduced to showing that $H^i(A^t, \phi_L^* F \otimes V \otimes L) = 0$ for $i \neq 0$, where $V = H^0(A,L)^\vee$ is a finite dimensional k-vector space. This is equivalent to checking that $H^i(A^t, \phi_L^* F \otimes L) = 0$ for $i \neq 0$. Applying base change to the cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "b"] & X \arrow[d, "a"] \\ A^t \arrow[r, "\phi_L"] & A \end{tikzcd}$$

and using that ϕ_L is étale, we learn that $\phi_L^* R^i a_* \omega_X \simeq R^i b_* \omega_{X'}$. We are thus reduced to showing that $H^i(A^t, R^i b_* \omega_{X'} \otimes L) = 0$ for $i \neq 0$. This is precisely the statement of Kollár vanishing Theorem 8.3.2 applied to the map $X' \to b(X')$. □

The following theorem of Kollár was used above.

Kollar

Theorem 8.3.2 (Kollár)**.** *Let $f : X \to Y$ be a proper surjective morphism of projective algebraic varieties over k. Assume X is smooth.*

1. *For any ample line bundle L on Y, we have $H^i(Y, R^j f_* \omega_X \otimes L) = 0$ for $i > 0$ and any j.*

2. *Each $R^j f_* \omega_X$ is torsionfree. In particular, we have $R^j f_* \omega_X = 0$ of $j > \dim(X) - \dim(Y)$.*

We now obtain consequences.

Corollary 8.3.3 (Green-Lazarsfeld)**.** *Let $S^i(\omega_X) := \{L \in \mathrm{Pic}^0(X) \mid H^i(X, \omega_X \otimes a^* L) \neq 0\} \subset A^t$. Then each $S^i(X) \subset A^t$ is closed, and each component has codimension $\geq i - (\dim(X) - \dim(a(X)))$.*

In particular, if the Albanese map is generically finite, then each component of $S^i(\omega_X)$ has codimension $\geq i$. Consequently, $\chi(X, \omega_X) \geq 0$.

Proof. As the Poincare bundle on $X \times \underline{\mathrm{Pic}}^0(X)$ is pulled back from the Poincare bundle on $\mathrm{Alb}(X) \times \underline{\mathrm{Pic}}^0(X)$ under the Albanese map, one checks by the projection formula that $S^i(\omega_X) = S^i(A^t, Ra_*\omega_X)$. An elementary spectral sequence argument shows that

$$S^i(A^t, Ra_*\omega_X) \subset \cup_j S^i(A^t, R^j a_*\omega_X[-j]) = \cup_i S^{i-j}(A^t, R^j a_*\omega_X).$$

For the first part, it is thus enough to show that the maximal j that can occur on the right above is $\dim(X) - \dim(a(X))$. It is thus enough to show that $R^j a_*\omega_X = 0$ for $j > \dim(X) - \dim(a(X))$. But this follows from Theorem 8.3.2 (2).

For the last part, we simply observe that if $\dim(X) = \dim(a(X))$, then $\cup_{i>0} S^i(\omega_X) \subset A^t$ is a proper closed subset by the first part. Thus, there exists some $L \in A^t(k)$ outside this locus. But then $R\Gamma(X, \omega_X \otimes a^*L) = H^0(X, \omega_X \otimes a^*L)$, so $\chi(X, \omega_X) = \chi(X, \omega_X \otimes L) \geq 0$, where the equality follows from the invariance of Euler characteristics in flat families. $\square$

Chapter 9

Jacobians and the Torelli theorem

We fix an algebraically closed field k of characteristic 0[1]. Let C be a smooth projective connected curve over k of genus $g \geq 1$. Write $\mathrm{Jac}(C) := \underline{\mathrm{Pic}}^0(C)$ for the identity component of the Picard scheme. Our goal is to prove the following:

Theorem 9.0.1 (Torelli). *The Jacobian* $\mathrm{Jac}(C)$ *comes equipped with a canonical principal polarization* Θ. *The pair* $(\mathrm{Jac}(C), \Theta)$ *determines* C *uniquely up to isomorphism.*

9.1 Constructing the principal polarization

9.1.1 Symmetric powers of C

For any integer $r \geq 0$, there is an obvious action of the symmetric group S_r on C^r. We shall use the following theorem:

Theorem 9.1.1. *The quotient* $\mathrm{Sym}^r(C) = C^r/S_r$ *exists in the category of schemes. Moreover, it is smooth, and* $\mathrm{Sym}^r(C)(k)$ *identifies with the set* $\mathrm{Div}_{eff}(C)_r$ *of effective divisors of degree* r *on* C.

Proof. The existence of a quotient of a quasi-projective variety by a finite group action is a standard construction, and we do not give it here. Instead, we simply sketch how it works: one chooses an S_r-equivariant cover of C^r by affine open subsets U_i, forms the affine schemes $V_i = \mathrm{Spec}(\mathcal{O}(U_i)^{S_r})$ so that $V_i = U_i/S_r$, and glues the V_i's together using the compatibility of forming invariants with localization.

Granting the existence of the quotient, let us explain smoothness at the point most likely to cause trouble, i.e., a point where all co-ordinates coinice. Fix a point $x \in C(k)$, which gives a point $y := (x, ..., x) \in C^r(k)$ and its image $z \in \mathrm{Sym}^r(C)(k)$. By smoothness of C, we have $\widehat{\mathcal{O}_{C,x}} \simeq k[\![t]\!]$ and thus $\widehat{\mathcal{O}_{C^r,y}} = k[\![t_1, ..., t_r]\!]$, where each t_i comes via a projection map to C. As y is a fixed point for the S_r-action, there is an induced S_r-action on $\widehat{\mathcal{O}_{C^r,y}}$ which, under our choice of co-ordinates, is the standard S_r-action on $k[\![t_1, .., t_r]\!]$. As y is a fixed point, we have $\widehat{\mathcal{O}_{\mathrm{Sym}^r(C),z}} \simeq k[\![t_1, ..., t_r]\!]^{S_r}$, which is a formal power series ring by the fundamental theorem on symmetric functions. Thus, $\mathrm{Sym}^r(C)$ is smooth at z.

To identify the points of $\mathrm{Sym}^r(C)(k)$, we first note that $\mathrm{Sym}^r(C)(k) = C^r(k)/S_r$ by a general property of finite quotients. Now the map that carries $(x_1, ..., x_r) \in C^r(k)$ to the divisor $\sum_i x_i \in \mathrm{Div}(C)_r$ is S_r-equivariant, and thus gives a map $\tau : C^r(k)/S_r \to \mathrm{Div}_{eff}(C)_r$. The injectivity of τ is the assertion that $(x_1, ..., x_r) \in C^r(k)$ is determined by subscheme attached to the divisor $\sum_i x_i$, which is clear. The surjectivity of τ is precisely what the effectivity of a divisor guarantees. $\square$

SymFunctor

Remark 9.1.2. More generally, one can identify the functor of points of $\mathrm{Sym}^r(C)$ in a similar way: for any k-scheme T, specifying a map $T \to \mathrm{Sym}^r(C)$ is the same as specifying an effective Cartier divisor $Z \subset C \times T$ that is finite and flat over T of degree r. We do not prove this assertion here.

[1] This assumption is strictly for simplicity of some arguments, and is not necessary for essentially any theorem.

9.1.2 Relating $\mathrm{Sym}^n(C)$ to the Picard scheme

To proceed further, we use the following two maps:

1. For any $r \geq 0$, there is natural map $C^r \to \underline{\mathrm{Pic}}(C)$ given by $(x_1, ..., x_r) \mapsto \mathcal{O}_C(\sum_i [x_i])$. This map is S_r-equivariant, and thus factors as a map $\sigma_r : \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)$. Under the interpretation of $\mathrm{Sym}^r(C)(k)$ with effective divisors of degree r, this map simply sends an effective divisor D to the corresponding line bundle $\mathcal{O}_C(D)$.

2. There is a degree map $\deg : \underline{\mathrm{Pic}}(C) \to \mathbf{Z}$. Write $\underline{\mathrm{Pic}}(C)_r = \deg^{-1}(r)$ for the clopen subset parametrizing line bundles of degree r. The map σ_r factors as $C^r \to \underline{\mathrm{Pic}}(C)_r \to \underline{\mathrm{Pic}}(C)$, we often refer to the first map as σ_r as well.

SymPicFibers **Theorem 9.1.3.** *The maps $\sigma_r : \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)_r$ satisfy the following properties.*

1. *Every fiber of σ_r is set-theoretically a projective space (but possibly empty).*

2. *For $r > g-1$, the map σ_r is surjective with geometrically connected fibers.*

3. *For $r > 2g-2$, the map σ_r is smooth. In fact, it is a projective bundle.*

4. *For $r \leq g$, the map σ_r is birational onto its image $W^r \subset \underline{\mathrm{Pic}}(C)_r$. In particular, $W^g = \underline{\mathrm{Pic}}(C)_g$ and $W^{g-1} \subset \underline{\mathrm{Pic}}(C)_{g-1}$ is an irreducible divisor.*

5. *The Jacobian $\mathrm{Jac}(C)$ coincides with $\underline{\mathrm{Pic}}(C)_0$. Thus, each $\underline{\mathrm{Pic}}(C)_r$ is a transate of $\mathrm{Jac}(C)$, and we write $\underline{\mathrm{Pic}}^r(C) = \underline{\mathrm{Pic}}(C)_r$ from henceforth.*

Proof. It is enough to check all assertions on k-points.

1. The fiber of σ_r over a degree r line bundle L is the set of effective divisors linearly equivalent to L, i.e., it identifies with $\mathbf{P}(H^0(C, L))$. In particular, it is a projective space.

2. If a line bundle L has degree $r > g-1$, then $h^0(L) \geq \chi(L) = r + 1 - g > 0$, so the fibre of σ_r over L is non-empty by (1).

3. If a line bundle L has degree $r > 2g-2$, then $h^1(L) = h^0(K_C \otimes L^{-1}) = 0$ as $K_C \otimes L^{-1}$ has negative degree, and thus $h^0(L) = \chi(L) = r + 1 - g$. In particular, it follows that if $r > 2g-2$, then all fibers of σ_r are at least set-theoretically projectivizations of vector spaces of the same dimension $r + 1 - g$. To actually identify the fibers scheme-theoretically, we use a trick. Let $\mathcal{L} \in \mathrm{Pic}(C \times \underline{\mathrm{Pic}}(C)_r)$ be the universal degree r line bundle trivialized along $\{x\} \times \underline{\mathrm{Pic}}(C)_r$ for a fixed base point $x \in C(k)$. For each $\ell \in \underline{\mathrm{Pic}}(C)_r$ corresponding to a degree r line bundle L on C, we have $\mathcal{L}|_{C \times \{\ell\}} \simeq L$ by the universal property of $\mathcal{L}$. By cohomology and base change (and our assumption $r > 2g-2$), it follows that $\mathcal{E} := pr_{2,*}\mathcal{L}$ is a vector bundle of rank $r + 1 - g$. Using the easy direction of the characterization of Remark 9.1.2, one can produce a map $h : \mathbf{P}(\mathcal{E}) \to \mathrm{Sym}^r(C)$ over $\underline{\mathrm{Pic}}(C)_r$ that is a bijection on all the fibers over $\underline{\mathrm{Pic}}(C)_r$, and thus a bijection. As both $\mathbf{P}(\mathcal{E})$ and $\mathrm{Sym}^r(C)$ are normal and k has characteristic 0, it follows that h is an isomorphism, which proves everything.

4. Now assume $r \leq g$. To show birationality of $\mathrm{Sym}^r(C) \to W_r$, it is enough to exhibit a single point of $\underline{\mathrm{Pic}}(C)_r$ over which the fiber of σ_r is set-theoretically a singleton. Indeed, by the irreducibility of W_r, this would ensure that $\mathrm{Sym}^r(C) \to W_r$ is generically finite; one uses that W_r stratified according to the fiber dimension of this map. But since the geometric generic fibres are projective spaces (at first set-theoretically, but then also scheme-theoretically by generic smoothness), the generic finiteness would then also ensure that $\mathrm{Sym}^r(C) \to W_r$ is birational. To find this point, we claim the following:

   Lemma 9.1.4. *For $r \leq g$, there is a non-empty open $U \subset C^r$ such that $h^0(\mathcal{O}_C(\sum_i [x_i])) = 1$ for $(x_1, .., x_r) \in U(k)$.*

Proof. Consider the following assertion:

$(*)$: If D is an effective divisor on C with $h^1(D) > 0$, then there exists a non-empty open set $V \subset C$ such that $h^1(D+P) = h^1(D) - 1$ for $P \in U(k)$.

Granting $(*)$, we may prove the lemma as follows. Consider the trivial effective divisor D_0 corresponding to the line bundle $\mathcal{O}_X$. Then $h^1(D_0) = g > 0$. Applying $(*)$ inductively r times, we obtain an open subset $U \subset C^r$ such that $h^1(\sum_i P_i) = h^1(D_0 + P_1 + ... + P_r) = g - r$ for $(P_1, ..., P_r) \in C^r(k)$. By Riemann-Roch, this implies that $h^0(\sum_i P_i) = r + (1-g) + (g-r) = 1$, as wanted.

It remains to prove $(*)$. By Riemann-Roch and Serre duality, this is equivalent to finding an open set $V \subset C$ such that $h^0(K_C(-D-P)) = h^0(K_C(-D)) - 1$. Now, by assumption, $h^0(K_C(-D)) > 0$, so there exists a dense open $V \subset C$ such that the canonical map $H^0(K_C(-D)) \otimes \mathcal{O}_C \to K_C(-D)$ is surjective over V. Taking any point $P \in V(k)$, it follows that applying $H^0(C, -)$ to the standard sequence

$$0 \to K_C(-D-P) \to K_C(-D) \to K_C(-D)|_P \to 0$$

gives an exact sequence. In particular,

$$H^0(C, K_C(-D)) \to K_C(-D)|_P$$

is surjective, and hence $h^0(K_C(-D-P)) = h^0(K_C(-D)) - 1$. □

5. Recall that $\mathrm{Jac}(C)$ was defined as the identity component of $\underline{\mathrm{Pic}}(C)$ and is thus clearly contained in $\underline{\mathrm{Pic}}(C)_0$. It is therefore enough to show that $\underline{\mathrm{Pic}}(C)_0$ is connected. But if we choose some $x \in C(k)$, then the map $L \mapsto L(r[x])$ gives an isomorphism $\underline{\mathrm{Pic}}(C)_0 \simeq \underline{\mathrm{Pic}}(C)_r$, so it is enough to prove $\underline{\mathrm{Pic}}(C)_r$ is connected for some r. Thus follows from (2) and the connectedness of $\mathrm{Sym}^r(C)$.

□

Construction 9.1.5 (The infinite symmetric product)**.** For $r, s \geq 0$, we have an obvious isomorphism $C^r \times C^s \simeq C^{r+s}$. There is an evident $S_r \times S_s$-action on $C^r \times C^s$, and an S_{r+s}-action on C^{r+s}. The preceding isomorphism is equivariant for these actions with respect to the obvious inclusion $S_r \times S_s \subset S_{r+s}$. Passing to the quotients, this gives a map $\mathrm{Sym}^r(C) \times \mathrm{Sym}^s(C) \to \mathrm{Sym}^{r+s}(C)$. Set $\mathrm{Sym}^0(C) = \mathrm{Spec}(k)$ and $\mathrm{Sym}(C) = \sqcup_{r \geq 0} \mathrm{Sym}^r(C)$. We call $\mathrm{Sym}(C)$ the *infinite symmetric product* of C. The maps defined above fit together to give $\mathrm{Sym}(C)$ the structure of a commutative monoid scheme. There is an obvious map $C \to \mathrm{Sym}(C)$ given by $C \simeq \mathrm{Sym}^1(C)$. The maps $\sigma_r : \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}(C)_r$ fit together to give a map $\mathrm{Sym}(C) \to \underline{\mathrm{Pic}}(C)$ of commutative monoids. Thus, we have constructed maps

$$C \to \mathrm{Sym}(C) \to \underline{\mathrm{Pic}}(C)$$

where the second term is a commutative monoid scheme, and the third term is a commutative group scheme. We shall show that these maps are universal with these properties. For now, simply note that the composite map $C \simeq \mathrm{Sym}^1(C) \to \underline{\mathrm{Pic}}^1(C)$ is given by $x \mapsto \mathcal{O}_C([x])$.

SymUnivProp

Corollary 9.1.6. *The natural map $C \to \mathrm{Sym}(C)$ is the universal map from C into a commutative monoid scheme.*

Proof. Say G is any commutative monoid scheme equipped with a map $f : C \to G$. Then f defines maps $C^r \xrightarrow{f^r} G^r \xrightarrow{m} G$, where m is the multiplication on G. As G is commutative, this map is S_r-equivariant, and thus factors to give a map $\mathrm{Sym}^r(C) \to C$. Putting these together gives a map $\mathrm{Sym}(C) \to G$ which agrees with f on the component $C \simeq \mathrm{Sym}^1(C) \subset \mathrm{Sym}(C)$. This gives the extension, and the uniqueness is proven similarly. □

PicUnivProp

Corollary 9.1.7. *The natural map $C \to \underline{\mathrm{Pic}}(C)$ is the universal map from C into a locally finitely presented commutative group scheme whose identity component is an abelian variety.*

We shall implicitly use that any locally finitely presented k-scheme X has a discrete set of connected components, and that each connected component is clopen.

Proof. Let $a : C \to G$ be a map from C to a commutative group scheme G. By Corollary 9.1.6, a extends unique to a map $b : \mathrm{Sym}(C) \to G$ of commutative monoid schemes. As abelian varieties admit no non-trivial maps from rational curves by Corollary 1.3.15, it follows from Theorem 9.1.3 that the restriction of b to $\sqcup_{r \geq 2g-2} \mathrm{Sym}^r(C) \subset \mathrm{Sym}(C)$ factors over

$$\mathrm{Sym}^{\geq 2g-2}(C) := \sqcup_{r \geq 2g-2} \mathrm{Sym}^r(C) \to \underline{\mathrm{Pic}}^{\geq 2g-2}(C) := \sqcup_{r \geq 2g-2} \underline{\mathrm{Pic}}^r(C).$$

to give a map

$$c : \underline{\mathrm{Pic}}^{\geq 2g-2}(C) \to G.$$

The map c intertwines the composition law on the LHS (inherited from the group law on $\underline{\mathrm{Pic}}(C)$) with the group law on the RHS. Using the fact that C is a group (and not just a monoid), it is easy to see that c extends uniquely to a homomorphism $\widetilde{c} : \underline{\mathrm{Pic}}(C) \to G$ of group schemes; explicitly, if $x \in \underline{\mathrm{Pic}}^a(C)(T)$, then the extension is determined by the requirement

$$\widetilde{c}(x) = c(x+y) - c(y)$$

for any $y \in \underline{\mathrm{Pic}}^b(C)(T)$ with $b \geq (2g-2) - a$. $\square$

Remark 9.1.8. The assumption that the identity component is an abelian variety in Corollary 9.1.7 is not necessary; one argues as in Remark 1.3.16.

JacAlb **Corollary 9.1.9** (Jacobian as an Albanese)**.** *Fix a base point $P \in C(k)$, and let $f : C \to \mathrm{Jac}(C) \subset \underline{\mathrm{Pic}}(C)$ be the map defined by $Q \mapsto \mathcal{O}_C([Q]-[P])$. This map carries P to e, and identifies $\mathrm{Jac}(C)$ with the Albanese variety of C, i.e., the natural map $\mathrm{Alb}(C) \to \mathrm{Jac}(C)$ is an isomorphism. This map is also identified with $f^* : \mathrm{Jac}(C)^t \to \underline{\mathrm{Pic}}^0(C)$ under the identification $\mathrm{Alb}(C) := \mathrm{Jac}(C)^t$ and $\underline{\mathrm{Pic}}^0(C) =: \mathrm{Jac}(C)$.*

Proof. Let A be an abelian variety, and let $g : C \to A$ be a map with $g(P) = e$. By Corollary 9.1.7, we get a unique extension of g to a homomorphism $h : \underline{\mathrm{Pic}}(C) \to A$ of group schemes. Restricting this map gives a map $H : \mathrm{Jac}(C) = \underline{\mathrm{Pic}}^0(C) \to A$. Note that $h(f(P)) = e$ and that translating by $f(P)$ gives an isomorphism $\underline{\mathrm{Pic}}^0(C) \simeq \underline{\mathrm{Pic}}^1(C)$. As h is a group homomorphism, it immediately follows that $g = H \circ f$, and that H is unique with this property.

The identification $\mathrm{Alb}(C) \simeq \mathrm{Jac}(C)$ coming from the previous paragraph is the unique one that carries $Q \in C(k)$ to $\mathcal{O}_C([Q]-[P]) \in \mathrm{Jac}(C)(k)$ via the canonical map $C \to \mathrm{Alb}(C)$. We wish to identify this map with f^*. The map $f : C \to \mathrm{Jac}(C) = \underline{\mathrm{Pic}}^0(C)$ corresponds, under the moduli description of $\mathrm{Jac}(C)$ as the space of degree 0 line bundles on C trivialized at P, to the line bundle $\mathcal{P}_C := \mathcal{O}_{C \times C}(\Delta) \otimes p_1^* \mathcal{O}_C(-P) \otimes p_2^* \mathcal{O}_C(-P)$, i.e, $\mathcal{P}_C$ is the pullback of the Poincare bundle on $\mathrm{Jac}(C) \times C$ along $C \times C \xrightarrow{f,id} \mathrm{Jac}(C) \times C$. The pullback f^* sends $x \in \mathrm{Jac}(C)^t$ corresponding to $\mathcal{P}|_{\mathrm{Jac}(C) \times \{x\}} \in \underline{\mathrm{Pic}}^0(\mathrm{Jac}(C))$ to the line bundle $f^*(\mathcal{P}|_{\mathrm{Jac}(C) \times \{x\}})$. In particular, if $x \in \mathrm{Jac}(C)^t$ comes from $Q \in C(k)$ via the Albanese, then it follows that $f^*(\mathcal{P}|_{\mathrm{Jac}(C) \times \{x\}}) \simeq \mathcal{P}_C|_{C \times \{Q\}} \simeq \mathcal{O}_C([Q]-[P])$.

$\square$

As a consequence of the previous discussion, we have constructed an isomorphism $\mathrm{Jac}(C) \simeq \mathrm{Jac}(C)^t$ depending on the choice of a base point $P \in C(k)$. One can show directly that this is principal polarization, and that the corresponding Θ-divisor is given by the image $W^{g-1} \subset \underline{\mathrm{Pic}}^{g-1}(C)$ of σ_{g-1} under some standard isomorphism $\underline{\mathrm{Pic}}^{g-1}(C) \simeq \underline{\mathrm{Pic}}^0(C)$. This is essentially done in the next section, using the language of determinants of cohomology and the Fourier transform.

9.1.3 The principal polarization

We shall use without proof that the notion of determinants extends to the derived category:

Proposition 9.1.10 (Mumford-Knudsen)**.** *Fix a smooth k-scheme X. The functor* $\det(-)$ *on the category of vector bundles on X extends to a symmetric monoidal functor $D^b_{coh}(X) \to \mathrm{Pic}(X)$ by passage to resolutions. For any exact triangle*

$$K \to L \to M$$

in $D^b_{coh}(X)$, there is a canonical isomorphism $\det(L) \simeq \det(K) \otimes \det(M)$.

More generally, the same assertions hold true for any scheme X provided we replace $D^b_{coh}(X)$ with the derived category $D_{perf}(X)$ of perfect *complexes on X, i.e., those complexes that locally can be represented by a finite complex of finite free $\mathcal{O}_X$-modules.*

Using the determinant of cohomology, one can often construct sections:

DetCoh **Proposition 9.1.11.** *Let S be a connected k-scheme, and let $f : X \to S$ be a proper smooth relative curve. Let M be a vector bundle on X such that Rf_*M is generically acyclic. Then the line bundle $L := \det(Rf_*M)$ comes equipped with a canonical nonzero section $\theta \in H^0(S, L)$ such that $Z(\theta)$ coincides set-theoretically with*

$$S^i(S, Rf_*M) := \{s \in S \mid H^i(X_s, M|_{X_s}) \neq 0\}$$

for $i = 0, 1$.

Proof. We first note that $S^0(S, Rf_*M) = S^1(S, Rf_*M)$. Indeed, by assumption on generic acyclicity, we know that $\chi(X_s, M|_{X_s}) = 0$ for $s \in S$ generic; by connectedness of S, the same holds for any $s \in S$. Thus, for any $s \in S$, we have $H^0(X_s, M|_{X_s}) \neq 0$ exactly when $H^1(X_s, M|_{X_s}) \neq 0$, which gives the claim.

To construct θ, we make choices; the section is independent of these choices, but we do not explain that here. Choose an effective divisor $D \subset X$ that is finite flat over S with sufficiently large degree on the fibers. We have an exact sequence

$$0 \to M \to M(D) \to M|_D \to 0$$

on X. For D sufficiently positive, by semicontinuity, we may assume that $R^1f_*M(D) = 0$ and that $f_*M(D)$ is a vector bundle on S. As $D \to S$ is finite flat, we also know that $R^1f_*(M|_D) = 0$ and $f_*(M|_D)$ is a vector bundle. It follows that Rf_*M is computed by the two-term complex

$$f_*M(D) \xrightarrow{\eta} f_*(M|_D)$$

of vector bundles on S. The generic acyclicity of Rf_*M ensures that η is an isomorphism generically; in particular, both vector bundles appearing above have the same rank. Taking determinants, it follows that

$$L := \det(Rf_*M) \simeq \det(f_*M(D)) \otimes \det(f_*(M|_D)).$$

The determinant of η gives a section $\theta \in H^0(S, L)$. It is clear from the definition that for any $s \in S$, the section $\theta \otimes \kappa(s) \in L \otimes \kappa(s)$ is nonzero exactly when $\eta \otimes \kappa(s)$ is an isomorphism, which happens exactly when $R\Gamma(X_s, M_s) \simeq 0$. This implies that $Z(\theta)$ coincides with $S^i(S, Rf_*M)$ for $i = 0, 1$, as wanted. $\square$

Theta1 **Proposition 9.1.12.** *Let $a : C \to A$ be a finite map to an abelian variety; we view coherent sheaves on C as coherent sheaves on A via pushforward along a. For a line bundle L on C, write $d(L) := \det(\Phi_A(L)) \in \mathrm{Pic}(A^t)$. The map $\phi_{d(L)} : A^t \to A$ is independent of L, and coincides with the map*

$$A^t = \underline{\mathrm{Pic}}^0(A) \xrightarrow{a^*} \underline{\mathrm{Pic}}^0(C) \simeq \mathrm{Jac}(C) \xrightarrow{a_*} A,$$

where a_* is the map induced by the universal property of $\underline{\mathrm{Pic}}(C)$ from Corollary 9.1.7.

Proof. As all objects in sight are smooth k-varieties, it is enough to show the claim on k-points. Fix a point $x \in A^t$. If $\mathcal{P}$ be the Poincare bundle on $A \times A^t$, then $\mathcal{P}_x = \mathcal{P}_{A\times\{x\}}$ is the corresponding line bundle on A under the identification $A^t(k) \simeq \mathrm{Pic}^0(A)$. We then compute

$$\phi_{d(L)}(x) = t_x^* d(L) \otimes d(L)^{-1} = t_x^* \det(\phi_A(L)) \otimes \det(\phi_A(L))^{-1} \simeq \det(\phi_A(L \otimes \mathcal{P}_x)) \otimes \det(\phi_A(L))^{-1} \simeq d(L \otimes a^* \mathcal{P}_x) \otimes d(L)^{-1}.$$

where the third equality uses that det commutes with pullback and that ϕ_A intertwines translations on A^t with tensor products on A by Lemma 6.3.3. We must show that this line bundle, viewed as an element of $\mathrm{Pic}^0(A^t) \simeq (A^t)^t(k)$, agrees with $a_* a^* \mathcal{P}_x \in A(k)$ via the canonical isomorphism $A \simeq (A^t)^t$. Recall that the latter is characterized by the requirement that $p \in A(k)$ corresponds to $\mathcal{P}|_{\{p\}\times A^t} \in (A^t)^t(k)$. The claim now follows from the following more general assertion applied to a divisor D representing $a^*\mathcal{P}_x \in \mathrm{Pic}^0(C)$.

Lemma 9.1.13. *For any divisor D on C, we have $d(L(D)) \simeq d(L) \otimes \mathcal{P}|_{\{a_* D\}\times A^t}$, where $a_* : \underline{\mathrm{Pic}}(C) \to A$ is the map induced by the universal property of Corollary 9.1.7.*

Proof. By linearity, it is enough to show the claim when $D = [p]$ for a single point $p \in C(k)$. In this case, we have the exact sequence

$$0 \to L \to L(p) \to L(p)|_p \simeq \kappa(p) \to 0.$$

Applying Φ_A gives

$$\phi_A(L) \to \phi_A(L(p)) \to \mathcal{P}|_{\{p\}\otimes A^t}.$$

Taking determinants then shows

$$d(L(p)) = d(L) \otimes \mathcal{P}|_{\{p\}\otimes A^t}.$$

□

□

To explain the principal polarization on $\mathrm{Jac}(C)$, we recall some definitions. Recall that a *polarization* on an abelian variety A is defined to a map $\phi : A \to A^t$ of the form ϕ_L for some (unspecified) ample line bundle L; such a map is necessarily finite. If ϕ is an isomorphism, then we say that ϕ is a *principal polarization*. In this case, we have $\dim H^0(A, L) = 1$ by Corollary 7.3.8. We shall write $\Theta_L \subset A$ for the zero locus of any nonzero section of L, and call it the θ-divisor attached to L. Given a principal polarization ϕ, the following facts can checked:

1. The collection of L with $\phi_L = \phi$ forms naturally a torsor X_ϕ for $A^t(k)$. This amounts to the assertion that $\mathrm{Pic}(A) \xrightarrow{M \mapsto \phi_M} \mathrm{Hom}(A, A^t)$ has kernel $A^t(k)$ (as well as the fact that ampleness is invariant under tensoring with a degree 0 line bundle).

2. For each choice of $L \in X_\phi$, we get a θ-divisor $\Theta_L \subset A$. Given $x \in A(k)$, we have $x + \Theta_L = \Theta_{L\otimes\phi(-x)}$: both sides are effective Cartier divisors attached to the line bundle $t_x^* L$, which has a unique nonzero global section up to scaling. Now given $L, M \in X_\phi$, we can write $L \simeq M \otimes \phi(-x)$ for a unique $x \in A(k)$ as ϕ is an isomorphism. It follows that $\Theta_L = x + \Theta_M$ for a unique $x \in A(k)$. Note that $x + \Theta_M = \Theta_M$ exactly when $x = e$ (as ϕ is an isomorphism). In other words, we may regard Θ as an abstract defined variety equipped with an A-torsor Y_ϕ of "standard" embeddings $\Theta \hookrightarrow A$. There is a natural identification $Y_\phi = X_\phi$ of A-torsors given by sending $L \in X_\phi$ to $\Theta = \Theta_L \hookrightarrow A$.

We now explain how to construct the principal polarization on $\mathrm{Jac}(C)$.

JacPPAV **Corollary 9.1.14.** *Let $a : C \to \mathrm{Jac}(C)$ be the Albanese map defined using a base point $P \in C(k)$ as in Corollary 9.1.9.*

1. *The map ϕ considered in Proposition 9.1.12 (for some choice of L, which is irrelevant by the lemma) is an isomorphism.*

2. *The map ϕ gives a principal polarization on* $\mathrm{Jac}(C)$. *More precisely,* $d(L)$ *is ample for any degree* $g-1$ *line bundle* L.

3. *Choose L to be a degree $g-1$ line bundle on C. The θ-divisor $\Theta \subset \mathrm{Jac}(C)$ attached to ϕ_L coincides with the image $W^{g-1} \subset \underline{\mathrm{Pic}}^{g-1}(C)$ of $\mathrm{Sym}^{g-1}(C)$ under the isomorphism $\mathrm{Jac}(C) \xrightarrow{M \mapsto L \otimes M} \underline{\mathrm{Pic}}^{g-1}(C)$. In particular, $\Theta(k) \subset \mathrm{Pic}^0(C)$ consists of those degree 0 line bundles M such that $H^0(C, L \otimes M) \neq 0$*

Proof. For (1), by Proposition 9.1.12, it is enough to show that the map

$$\mathrm{Jac}(C)^t = \underline{\mathrm{Pic}}^0(\mathrm{Jac}(C)) \xrightarrow{a^*} \underline{\mathrm{Pic}}^0(C) \xrightarrow{a_*} \mathrm{Jac}(C)$$

is an isomorphism. The map a_* is exactly the map realizing the isomorphism between $\mathrm{Alb}(C) := \mathrm{Jac}(C)^t$ and $\mathrm{Jac}(C)$ coming from Corollary 9.1.9, while a^* is also an isomorphism by the second part of Corollary 9.1.9.

Having checked that ϕ is an isomorphism, to show (2), it is now enough to show that $d(L)$ is effective (by Corollary 3.2.14) for some choice of L. We shall check this in a manner that also establishes parts of (3). Choose L to be a line bundle of degree $g-1$. Then we claim that $d(L) = \det(\Phi_A(L))$ is effective. If we write $\mathcal{P}$ for the Poincare bundle on $C \times \mathrm{Jac}(C)$, the $\Phi_A(L)$ is given by $Rpr_{2,*}(pr_1^* L \otimes \mathcal{P})$. In particular, for any point $x \in Jac(C)(k)$, the fibre $\Phi_A(L) \otimes \kappa(x)$ identifies with $R\Gamma(C, L \otimes \mathcal{P}|_{C \times \{x\}})$; as x varies through all points of $\mathrm{Jac}(C)(k)$, the line bundle $L \otimes \mathcal{P}|_{C \times \{x\}}$ varies through all points of $\underline{\mathrm{Pic}}^{g-1}(C)(k)$. In particular, for generic x, we have $R\Gamma(C, L \otimes \mathcal{P}|_{C \times \{x\}}) = 0$ by Theorem 9.1.3. Applying Proposition 9.1.11 to the map $pr_2 : C \times \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ with $M = pr_1^* L \otimes \mathcal{P}$ then shows that $d(L) = \det(Rpr_{2,*}(pr_1^* L \otimes \mathcal{P}))$ admits a canonical section θ such that $Z(\theta)$ coincides with the set of all degree 0 line bundles $M \in \underline{\mathrm{Pic}}^0(C)(k)$ such that $H^0(C, L \otimes M) \neq 0$. This establishes (2), as well the set-theoretic variant of (3).

To finishing proving (3), we must check that $Z(\theta)$ as defined above coincides with W^{g-1} as schemes. As both sides are irreducible Cartier divisor and W^{g-1} is reduced, it is enough to check that $Z(\theta)$ is reduced. But this follows from the following general fact:

Lemma 9.1.15. *Let A be an abelian variety, and let L be an ample line bundle giving a principal polarization. If the θ-divisor $\Theta \subset A$ is irreducible, it is reduced.*

Proof. If Θ was not reduced, then we could write $\Theta = k * D$ for some $k > 1$ and divisor D. It then follows that $L = M^k$ for some ample line bundle M and $k > 1$. But it is easy to see that $K(M^k)$ contains $A[k]$: the theorem of the square implies that the map $M \mapsto \phi_M$ gives a group homomorphism $\mathrm{Pic}(A) \to \mathrm{Hom}(A, \underline{\mathrm{Pic}}^0(A))$, and hence ϕ_{M^k} is divisible by k as a map, and thus kills $A[k]$. Since ϕ_L is an isomorphism by assumption, we get a contradiction.

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9.1.4 The Torelli theorem

The statement

Let A be an abelian variety over k equipped with a principal polarization $\phi : A \to A^t$, and let $\Theta \subset A$ be some effective divisor inducing ϕ. Set

$$Z(A, \phi) = \mathrm{coker}(\mathrm{Pic}(A) \to \mathrm{Pic}(\Theta^{ns})).$$

As different choices of Θ differ by translations, one checks that $Z(A, \phi)$ depends only on ϕ and not on Θ. Assume now that Θ is symmetric, i.e., $[-1]^*\Theta = \Theta$; such divisors always exist[2], and two choices differ by

[2] Indeed, if $\phi = \phi_L$ for some L (so $L = \mathcal{O}_A(\Theta)$), then Θ is symmetric exactly when $[-1]^*L = L$. Now given any L with $\phi = \phi_L$, the difference $L^{-1} \otimes [-1]^*L$ is a degree 0 line bundle, and thus of the form M^2 for some degree 0 line bundle M. As $[-1]^*M = M^{-1}$, it follows that $L \otimes M$ is symmetric; moreover, we have $\phi_L = \phi_{L \otimes M}$. Thus, we have constructed a symmetric line bundle L inducing the principal polarization. It is clear from this construction that the set of all such L's forms a torsor for $A[2](k)$. This reasoning also shows that the set of symmetric Θ-divisors for ϕ forms a torsor for $A[2](k)$.

translations by a unique $x \in A[2](k)$. In particular, we may regard Θ as a symmetric subscheme of A that is well-defined (i.e., depends only on ϕ) up to translations by $A[2](k)$. Consider set $P(A, \phi) \subset \mathrm{Pic}(\Theta^{ns})$ of all line bundles M with the following two properties:

1. $M \otimes [-1]^*M \simeq \omega_{\Theta^{ns}}$.
2. M generates $Z(A, \phi)$.

We can now formulate the Torelli theorem:

Theorem 9.1.16. *Consider the principally polarized abelian variety $(\mathrm{Jac}(C), \phi)$ constructed in Corollary 9.1.14. Pick a symmetric Θ-divisor $\Theta \subset \mathrm{Jac}(C)$, and let $j : \Theta^{ns} \to \Theta$ be the inclusion. Then:*

1. *The set $P(\mathrm{Jac}(C), \phi)$ is non-empty.*
2. *For any $M \in P(\mathrm{Jac}(C), \phi)$, the complex $\Phi_{\mathrm{Jac}(C)}(j^{ns}_* M)$ has form $F[1-g]$ for a coherent sheaf F on $\mathrm{Jac}(C)^t \overset{\phi}{\simeq} \mathrm{Jac}(C)$. The scheme-theoretic support of F is isomorphic to the curve C, and F is a line bundle of degree $g-1$ on C.*

A sketch of the proof

We sketch roughly the main steps of the proof. As the Torelli theorem is clear in genus 2 (though one must also check the above formulation), we restrict to $g \geq 3$.

1. By carefully studying the geometry of the θ-divisor $W^{g-1} \subset \underline{\mathrm{Pic}}^{g-1}(C)$ via the map $\mathrm{Sym}^{g-1}(C) \to W^{g-1}$ one shows the following:

   Proposition 9.1.17. *If C is not hyperelliptic, then $Z(\mathrm{Jac}(C), \phi) \simeq \mathbf{Z}$. If C is hyperelliptic of genus ≥ 3, then $Z(\mathrm{Jac}(C), \phi) \simeq \mathbf{Z}/2$. In both cases, the restriction map $\mathrm{Pic}(\underline{\mathrm{Pic}}^{g-1}(C)) \to \mathrm{Pic}(\Theta^{ns})$ is injective.*

   The proof uses many classical facts about curves (such as Clifford's theorem and Maarten's theorem).

2. Consider the semidirect product $G := \mathrm{Jac}(C)^t \rtimes \mathbf{Z}/2$, where the $\mathbf{Z}/2$ acts via $[-1]$ on $\mathrm{Jac}(C)^t$. There is a natural G-action on the set $P(\mathrm{Jac}(C), \phi)$: the $\mathbf{Z}/2$ acts via $[-1]^*$ (which makes sense because Θ is symmetric), while a point $\xi \in \mathrm{Jac}(C)^t(k)$ corresponding to a degree 0 line bundle M on $\mathrm{Jac}(C)$ acts via tensor product with $M|_{\Theta^{ns}}$. One then proves the following:

   Lemma 9.1.18. *The G-action on $P(\mathrm{Jac}(C), \phi)$ is transitive. More precisely, this holds true for any ppav (A, ϕ) with symmetric θ-divisor Θ as long as $\mathrm{Pic}(A) \to \mathrm{Pic}(\Theta^{ns})$ is injective.*

   As a consequence, if one understands the Fourier transform of one particular $M \in P(\mathrm{Jac}(C), \phi)$, one understands the Fourier transform of every such M: tensor products with degree 0 line bundles as well as $[-1]^*$ behave predictably under the Fourier transform.

3. We must show that the set $P(\mathrm{Jac}(C), \phi)$ is non-empty, and that that $\Phi_{\mathrm{Jac}(C)}(M)$ verifies (2) in the theorem for any $M \in P(\mathrm{Jac}(C), \phi)$. By the previous reduction, it suffices to prove non-emptyness, and verify (2) for a single $M \in P(\mathrm{Jac}(C), \phi)$. Both of these follow from the involutivity of the Fourier transform, and the following:

   Proposition 9.1.19. *Let $\Psi : D(C) \to D(\underline{\mathrm{Pic}}^{g-1}(C))$ be the transform defined by the Poincare bundle on $C \times \underline{\mathrm{Pic}}^{g-1}(C)$. Then $\Psi(\mathcal{O}_C)[1] \simeq j^{ns}_* M$ for some $M \in P(\underline{\mathrm{Pic}}^{g-1}(C), \phi)$.*

   Beyond standard cohomological machinery, the main non-trivial ingredient that goes into this proof is the following: the singular locus of Θ has codimension ≥ 3. This assertion ensures that $j^{ns}_*(F|_{\Theta^{ns}}) \simeq F$ for any sheaf F that admits a two term resolution by vector bundles.