8 Central stability homology for polynomial $\mathsf{VIC}(\mathbb{Z})$–modules

Proof of Theorem 7.5. We prove the theorem by a double induction over r and i. If $r = -\infty$ or $i < 0$ the theorem is true. We thus may assume that if M has polynomial degree $\leq s$ in ranks $> d$,

$$HS_q(M)_n \cong 0 \qquad \text{for } n > \max(d+q, 2q+s)$$

as long as $s < r$ or $q < i$.

Consider two double complexes:

$$\begin{aligned} X_{pq} &= \bigoplus_{(f,C)\in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^p,\mathbb{Z}^n)} \bigoplus_{(g,D)\in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^q,C)} M_{\mathrm{im}\, f\oplus D} \\ &\cong CS_p(CS_q(\Sigma^p M))_n \\ &\cong CS_q(CS_p(M(0))\otimes M)_n \end{aligned}$$

and

$$\begin{aligned} Y_{pq} &= \bigoplus_{(f,C)\in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^p,\mathbb{Z}^n)} \bigoplus_{(g,D)\in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^q,C)} M_D \\ &\cong CS_p(CS_q(M))_n \\ &\cong CS_q(CS_p(M))_n. \end{aligned}$$

Let

$$E^1_{pq} = CS_p(HS_q(\Sigma^p M))_n$$

denote the spectral sequence associated to X. It converges to zero in the range $n > 2(p+q)$.

Let us denote the spectral sequence associated to Y by $\widehat{E}^r_{pq}$. It turns out that $d^1\colon \widehat{E}^1_{1,q} \to \widehat{E}^1_{0,q}$ is always the zero map.

The map of double complexes

$$Y_{pq} \longrightarrow X_{pq}$$

induces maps

$$\widehat{E}^1_{pq} \longrightarrow E^1_{pq}$$

that are surjective for $n > \max(d+p+q-1, p+2q+r-1)$ and injective for $n > \max(d+p+q, p+2q+r+1)$. This uses the induction hypothesis.

Therefore

$$E^2_{0,i}(M)_n = E^1_{0,i} \cong HS_i(M)_n \qquad \text{for } n > \max(d+i, 2i+r).$$

The theorem follows because by induction

$$E^1_{pq} \cong CS_p(HS_q(\Sigma^p M))_n \cong 0$$

for $q < i$ and $n > \max(d+q, p+2q+r)$. This implies that

$$HS_i(M)_n \cong E^1_{0,1} \cong E^2_{0,i} \cong E^\infty_{0,i}$$

in the given range, which vanishes for $n > 2i$. □