5 Highly connected simplicial complexes

Definition 5.1. An (abstract) simplicial complex X on a vertex set V is a set of nonempty subsets of V that is closed under subsets and contains all singletons. We call a subset in X a simplex of X. If a simplex has $(p+1)$ elements it is called an p-simplex or p-dimensional. A proper subset of a simplex is called a face.

A (topological) p-simplex is the topological space given by the convex hull of the standard basis vectors in $\mathbb{R}^{p+1}$. The simplex spanned by a proper subset of standard basis vectors is called a face. The (topological) realization $|X|$ of an abstract simplicial complex X is the space of topological simplicies for each simplex in X glued along their faces.

Definition 5.2. An (abstract) Δ-complex X (or semisimpicial set) is a sequence of sets $(X_p)_{p\in\mathbb{N}_0}$ together with face maps $d_i\colon X_{p+1} \to X_p$ for each $i \in \{0,\dots,p+1\}$ and $p \geq 1$, such that

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for } i < j.$$

The (topological) realization $|X|$ of an abstract Δ-complex X is the space of topological p-simplicies for each element in X_p for all $p \geq 1$ glued together along the face maps.

Exercise 5.3. Given an abstract simplicial complex, find an abstract Δ-complex with the same realization.

Definition 5.4. The simplicial chain complex $C_*(X)$ of a Δ-complex X is given by $C_p(X) = \mathbb{Z}X_p$ and the boundary map $\partial = \sum(-1)^i d_i$. Denote the homology of this chain complex by $H_*(X)$. (It is isomorphic to the (singular) homology of the realization.)

Definition 5.5. A simplicial map $X \to Y$ between simplicial complexes is a map between the vertex sets such that the image of a simplex of X is a simplex of Y.

For a simplicial complex X, let $[S^p, X]$ be the set of equivalence classes of all simplicial maps $Y \to X$ for all simplicial complexes Y whose realization is homeomorphic to the p-sphere S^p under the following equivalence relation. $f_1\colon Y_1 \to X$ and $f_2\colon Y_2 \to X$ are (freely homotopy) equivalent if there is a simplicial complex Z whose realization is homeomorphic to $S^p \times [0,1]$ and whose two boundaries are Y_1 and Y_2 together with a simplicial map $Z \to X$ that restricts to f_1 and f_2 on the boundary. ($[S^p, X]$ is in bijection to the set of free homotopy classes of continuous maps $S^p \to |X|$.)

A simplicial complex X is called n-connected if $[S^p, X]$ contains only the trivial class for all $p \leq n$.

Theorem 5.6 (Hurewicz)**.** *If a simplicial complex is n-connected than $\tilde{H}_i(X) \cong 0$ for all $i \leq n$.*

Definition 5.7. Let X be a simplicial complex. The link of a simplex σ in X is the union of all simplicies that are disjoint from σ and whose union with σ is also a simplex in X. It is denoted by $\mathrm{Lk}_X(\sigma)$.

A simplicial simplex X is called weakly Cohen-Macaulay of dimension n if X is $(n-1)$-connected and $\mathrm{Lk}_X(\sigma)$ is $(n-p-2)$-connected for every p-simplex σ of X.

Definition 5.8. Let PB_n be the partial basis complex of $\mathbb{Z}^n$, i.e. a set of nonzero vectors in $\mathbb{Z}^n$ form a simplex if they can be completed to a basis of $\mathbb{Z}^n$.

Theorem 5.9 (Maazen 1979)**.** *PB_n is $(n-2)$–connected.*

Proof. Exercise. □

Definition 5.10. Let us define the simplicial complex PBC_n. Its vertex set contains all pairs (v, H), where $v \in \mathbb{Z}^n$ is nonzero and $H \subset Z^n$ is a summand such that $\mathrm{span}(v) \oplus H = \mathbb{Z}^n$. The subset $\{(v_0, H_0), \dots, (v_p, H_p)\}$ is a simplex if $\{v_0, \dots, v_p\}$ is a partial basis of $\mathbb{Z}^n$ and $v_i \in H_j$ for all $i \neq j$.

Definition 5.11. A join complex over a simplicial complex X is a simplicial complex Y together with a simplicial map $\pi\colon Y \to X$, satisfying the following properties:

1. π is surjective.
2. π is simplexwise injective.
3. A collection of vertices $y_0, \ldots, y_p$ spans a simplex of Y whenever there exists simplices $\theta_0, \ldots, \theta_p$ such that for all i, y_i is a vertex of θ_i and the simplex $\pi(\theta_i)$ has vertices $\pi(y_0), \ldots, \pi(y_p)$.

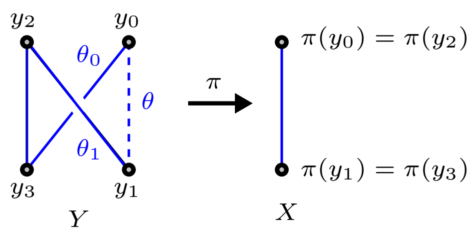

Figure 1: The map π does not exhibit Y as a join complex over X unless θ is a simplex of Y.

Theorem 5.12 (Hatcher–Wahl 2010)**.** *Let Y be a join complex over X via $\pi\colon Y \to X$. Assume X is weakly Cohen–Macaulay of dimension n. Further assume that for all p-simplices τ of Y, the image of the link $\pi(\mathrm{Lk}_Y(\tau))$ is weakly Cohen–Macaulay of dimension $(n-p-2)$. Then Y is $\frac{n-2}{2}$–connected.*

Theorem 5.13 (Randal-Williams–Wahl 2017)**.** PBC_n *is $\frac{n-3}{2}$–connected.*

Proof. In the exercises, it is shown that PBC_n is a join complex over PB_n. The other conditions for the previous theorem are also shown. □

Definition 5.14. Let X be a simplicial complex. Define $X^{\mathrm{ord}} = (X_p^{\mathrm{ord}})_{p\in\mathbb{N}_0}$ to be the Δ-complex whose p-simplices are

$$X_p^{\mathrm{ord}} = \{(x_0, \ldots, x_p) \in X_0^{p+1} \mid \{x_0, \ldots, x_p\} \text{ is a } p\text{-simplex in } X\}.$$

Proposition 5.15 (Randal-Williams–Wahl 2017)**.** *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n then X^{ord} is $(n-1)$–connected.*

Proof. Exercise. □

Corollary 5.16. $HS_i(M(0))_n \cong 0$ *for all* $n > 2i$.

Proof. Exercise. □