2 VIC–modules

Definition 2.1. Let $\mathcal{C}$ be a category whose isomorphism classes of objects from a set. A $\mathcal{C}$–module is a functor from $\mathcal{C}$ to the category of abelian groups Ab. The category $\mathcal{C}-\mathsf{mod}$ of $\mathcal{C}$–modules has natural transformations as morphisms.

Example 2.2. Let G be a group and $\mathcal{C}$ be the one-object category whose morphisms are given by G. Then $\mathcal{C}$–modules is the same as $\mathbb{Z}G$–modules (which we will also call G–representations). Let $F\colon \mathcal{C} \to \mathsf{Ab}$ be a functor and let $*$ denote the single object of $\mathcal{C}$. Then $F(*)$ is an abelian group and for every $g \in G$, we get an endomorphism of $F(*)$ given by $F(g)$.

Example 2.3. Let $\mathcal{C}$ be a groupoid, i.e. all morphisms are isomorphisms. Let F be a $\mathcal{C}$–module. If objects C_1 and C_2 are isomorphic, so are $F(C_1)$ and $F(C_2)$. Thus, the category of $\mathcal{C}$–modules is equivalent to the product category of $\mathbb{Z}\operatorname{Aut}(C)$–modules for all C in a set of representatives of the isomorphism classes of objects of $\mathcal{C}$. That is the same as a collection of $\mathbb{Z}\operatorname{Aut}(C)$–modules.

For example, let R be a commutative ring and let $\mathcal{C}$ be the groupoid of all finitely generated free abelian groups$R$–modules and isomorphisms. Then $C-\mathsf{mod}$ is equivalent to the category of sequences $(M_n)_{n\in\mathbb{N}_0}$ where M_n is a $\mathbb{Z}\operatorname{GL}_n(R)$–modiule.

Definition 2.4. Let R be a commutative ring. Define $\mathsf{VIC}(R)$ to be the category whose objects are finitely generated free R–modules and whose morphisms are

$$\operatorname{Hom}_{\mathsf{VIC}(R)}(V,W) := \{(f,C) \mid f\colon V \hookrightarrow W, C \text{ is free}, \operatorname{im} f \oplus C = W\}.$$

For $(f,C) \in \operatorname{Hom}_{\mathsf{VIC}(R)}(V,W)$ and $(g,D) \in \operatorname{Hom}_{\mathsf{VIC}(R)}(U,V)$, the composition is given by

$$(f,C)\circ(g,D) := (f\circ g, C \oplus f(D)) \in \operatorname{Hom}_{\mathsf{VIC}(R)}(U,W).$$

We want to make some easy observations:

- $\mathsf{VIC}(R)$ is equivalent to the induced subcategory on only the objects R^n for $n \in \mathbb{N}_0$.
- The endomorphisms $\operatorname{Hom}_{\mathsf{VIC}(R)}(R^n,R^n)$ are all isomorphisms and $\operatorname{Aut}_{\mathsf{VIC}(R)}(R^n) \cong \operatorname{GL}_n(R)$.
- Let M be a $\mathsf{VIC}(R)$–module, then it gives rise to a sequence $(M_n)_{n\in\mathbb{N}_0}$ of $\mathbb{Z}\operatorname{GL}_n(R)$–modules.
- The standard decomposition $R^n \oplus R \to R^{n+1}$ induces a $\operatorname{GL}_n(R)$–equivariant map $\phi_n\colon M_n \to M_{n+1}$.

Proposition 2.5. *A sequence $(M_n)_{n\in\mathbb{N}_0}$ of $\mathbb{Z}\operatorname{GL}_n(R)$–modules together with $\operatorname{GL}_n(R)$–equivariant maps $\phi_n\colon M_n \to M_{n+1}$ comes from a $\mathsf{VIC}(R)$–module if and only if $\operatorname{GL}_m(R)$ acts trivially on the image of $\phi_{n+m-1}\circ\cdots\circ\phi_n\colon M_n \to M_{n+m}$. Such a $\mathsf{VIC}(R)$–module is then uniquely determined.*

Definition 2.6. Let $M(m)$ denote the representable functor $\mathbb{Z}\operatorname{Hom}_{\mathsf{VIC}(R)}(R^m,-)$. We call a direct sum of representable functors *free*.

More about free $\mathsf{VIC}(R)$–modules in the exercises.