Exercises

1.) Let us fill in the details of the proof of Theorem 6.1:

(a) Check that

$$CS_p(M)_n = \bigoplus_{(f,C)\in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^p,\mathbb{Z}^n)} M_C \cong \mathrm{Ind}_{G_{n-p}}^{G_n} M_{n-p} \cong \mathbb{Z}\,\mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_{n-p}(\mathbb{Z})} M_{n-p}.$$

(b) Prove Shapiro's Lemma: Let H be a subgroup of a group G and let M be a $\mathbb{Z}H$–module. Then

$$H_*(G; \mathrm{Ind}_H^G M) \cong H_*(H; M).$$

(c) Show that every face map $d_i \colon CS_p(M)_n \to CS_{p-1}(M)_n$ induces the stability map

$$\begin{aligned} E^1_{pq} &\cong H_q(\mathrm{GL}_{n-p}(\mathbb{Z}); M_{n-p}) \cong H_q(E_* \mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} CS_p(M)_n \\ &\longrightarrow E^1_{p-1,q} \cong H_q(\mathrm{GL}_{n-p+1}(\mathbb{Z}); M_{n-p+1}) \cong H_q(E_* \mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} CS_{p-1}(M)_n. \end{aligned}$$

As a consequence $d^1 \colon E^1_{pq} \to E^1_{p-1,q}$ is the stability map if p is odd and zero if p is even.

2.) For the spectral sequence given by the double complex

$$E_* \mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_n} \tilde{C}_{*-1}(Y(n)),$$

we want to prove that

$$E^2_{pq} \cong HS_p(H_q(\mathrm{IA}))_n.$$

(a) Show that

$$E_* \mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_{n-p}} \mathbb{Z}$$

is a $\mathbb{Z}\,\mathrm{GL}_{n-p}(\mathbb{Z})$–module.

(b) Find the isomorphism

$$E^0_{pq} \cong E_q \mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_n} \mathrm{Ind}_{\mathrm{Aut}(F_{n-q})}^{\mathrm{Aut}(F_n)} \mathbb{Z} \cong \mathrm{Ind}_{\mathrm{GL}_{n-p}(\mathbb{Z})}^{\mathrm{GL}_n(\mathbb{Z})} E_* \mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_{n-p}} \mathbb{Z}.$$

(c) Prove that face maps of $Y(n)$ precisely induce the face maps of $CS_*(H_*(\mathrm{IA}))_n$.

(d) Finish the proof.