Exercises

1.) Show that $CS_*(M)_n$ together with ∂ is a chain complex.

2.) A semi-simplicial set is a sequence of sets $(X_p)_{p\in\mathbb{N}_0}$ together with maps $d_i\colon X_p \to X_{p-1}$ for $i = 0, \dots, p$ such that $d_i \circ d_j = d_{j-1} \circ d_i$ for all pairs $i < j$.

(a) Show that every semi-simplicial set is isomorphic to a set of the form, where $X_p \subset X_0^{p+1}$ and $d_i\colon X_p \to X_{p-1}$ will drop the $(i-1)$th entry from the sequence.

(b) Check that $W_p(n) = \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^{p+1}, \mathbb{Z}^n)$ gives a semi-simplicial set.

(c) The simplicial chain complex of a semi-simplicial set X is given by $C_p(X) = \mathbb{Z}X_p$ and $\partial = \sum(-1)^i d_i$. Check that $C_*(W(n)) \cong CS_{*-1}(M(0))_n$.

3.) This exercise shall show that central stability homology gives bounds on the generation degree of syzygies. You may use that $HS_i(M(0))_n = 0$ for all $n > 2i$.

(a) Observe that a $\mathsf{VIC}(\mathbb{Z})$–module M is generated in degrees $\leq d$ if and only if $HS_0(M)_n = 0$ for all $n > d$.

(b) Show $HS_i(M(m))_n = 0$ for all $n > 2i + m$.

(c) Let $d_0, \dots, d_N \in \mathbb{N}_0$ with $d_{i+1} - d_i \geq 2$ and

$$P_N \to \cdots \to P_0 \to M \to 0$$

be a resolution of free $\mathsf{VIC}(\mathbb{Z})$–modules P_i generated in degrees $\leq d_i$. Show that $HS_i(M)_n = 0$ for all $n > d_i$.

(d) Let d_i be as above and $HS_i(M)_n \cong 0$ for all $n > d_i$. Show that there exists a partial resolution

$$P_N \to \cdots \to P_0 \to M \to 0$$

of free $\mathsf{VIC}(\mathbb{Z})$–modules P_i generated in degrees $\leq d_i$.