Exercises

1.) Let G be a group and let M and N be $\mathbb{Z}G$–modules. Let $P_* \to M \to 0$ and $Q_* \to N \to 0$ be projective G–resolutions. Given a G–equivariant map $M \to N$, show that up to chain homotopy there exists a unique G–equivariant map of chain complexes $P_* \to Q_*$ inducing the given map.

2.) Let G be a group. Show $H_1(G;\mathbb{Z}) \cong G^{\mathrm{ab}}$.

3.) Let H be a subgroup of G. Show that a projective resolution $E_*G \to \mathbb{Z} \to 0$ of $\mathbb{Z}G$–modules is also a projective resolution of $\mathbb{Z}H$–modules.

4.) Let

$$1 \to K \to G \to Q \to 1$$

be a short exact sequence of groups. This exercise proves that Q acts on the homology of K.

(a) G acts on K by conjugation. Let $E_*G \to \mathbb{Z} \to 0$ be a projective (right) G–resolution of the trivial representation. Check that multiplication by g^{-1} induces a map of chain complexes $E_*G \otimes_K \mathbb{Z} \to E_*G \otimes_K \mathbb{Z}$. Thus induces an action of G on the homology of K.

(b) Check that K acts trivially through this action and deduce that Q acts.

5.) We want to show that there is a $\mathsf{VIC}(\mathbb{Z})$–module structure on the sequence $(H_i(\mathrm{IA}_n))_{n\in\mathbb{N}_0}$ for every fixed $i \in \mathbb{N}_0$. The $\mathrm{GL}_n(\mathbb{Z})$–action on $H_i(\mathrm{IA}_n)$ follows from the previous exercise.

(a) The inclusion $\mathrm{IA}_n \subset \mathrm{IA}_{n+1}$ induces a map on homology. Check that this map is $\mathrm{GL}_n(\mathbb{Z})$–equivariant.

(b) Show that $\mathrm{GL}_m(\mathbb{Z})$ acts trivially on the image of $H_i(\mathrm{IA}_n) \to H_i(\mathrm{IA}_{n+m})$.

6.) We want to prove Corollary 3.3: Recall that the Johnson homomorphism sends $f \in \mathrm{IA}_n$ to $x \cdot F_n' \mapsto f(x)x^{-1} \cdot [F_n, F_n']$ which is a map in $\mathrm{Hom}(\mathbb{Z}^n, \bigwedge^2 \mathbb{Z}^n)$. Show that this gives a morphism of $\mathsf{VIC}(\mathbb{Z})$–modules.