Exercises

1.) Prove that $\mathrm{End}_{\mathsf{VIC}(R)}(R^n) = \mathrm{Aut}_{\mathsf{VIC}(R)}(R^n) \cong \mathrm{GL}_n(R)$.

2.) Show that $\mathrm{Hom}_{\mathsf{VIC}(R)}(R^m, R^n) \cong \mathrm{GL}_n(R)/\mathrm{GL}_{n-m}(R)$ as a $\mathrm{GL}_n(R)$-set.

3.) Let $F\colon \mathsf{VIC}(R) \to \mathsf{Ab}$ be a functor.

(a) Show that $M_n := F(R^n)$ is a $\mathrm{GL}_n(R)$–representation.

(b) Show that $(f, C) \in \mathrm{Hom}_{\mathsf{VIC}(R)}(R^n, R^{n+1})$ given by $f(e_i) = e_i$ for $1 \le i \le n$ and $C = \mathrm{span}(e_{n+1})$ induces a $\mathrm{GL}_n(R)$–equivariant map $\phi_n\colon M_n \to M_{n+1}$.

(c) Show that $\mathrm{GL}_m(R)$ included into $\mathrm{GL}_{n+m}(R)$ by the block inclusion

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

acts trivially on the image of the composition

$$M_n \xrightarrow{\phi_n} M_{n+1} \xrightarrow{\phi_{n+1}} \cdots \xrightarrow{\phi_{n+m-1}} M_{n+m}.$$

(d) Conversely, let $(M_n)_{n\in\mathbb{N}_0}$ be a sequence of $\mathrm{GL}_n(R)$–representations and let there be $\mathrm{GL}_n(R)$–equivariant maps $\phi_n\colon M_n \to M_{n+1}$. If $\mathrm{GL}_m(R)$ acts trivially on the image of the composition $M_n \to M_{n+m}$, there is a $\mathsf{VIC}(R)$–module $F\colon \mathsf{VIC}(R) \to \mathsf{Ab}$ such that $F(R^n) = M_n$ and $(f, C) \in \mathrm{Hom}_{\mathsf{VIC}(R)}(R^n, R^{n+1})$ given by $f(e_i) = e_i$ for $1 \le i \le n$ and $C = \mathrm{span}(e_{n+1})$ induces $\phi_n\colon M_n \to M_{n+1}$.

4.) Let $M(m) := \mathbb{Z}\,\mathrm{Hom}_{\mathsf{VIC}(R)}(R^m, -)$ define a free $\mathsf{VIC}(R)$–module.

(a) Show that $M(m)$ is generated by one element.

(b) Show that $\mathrm{Hom}_{\mathsf{VIC}(R)-\mathsf{mod}}(M(m), M) \cong M_m$.

(c) Show that if M is generated in degrees $\le d$, there is a set I, numbers $m_i \le d$ for $i \in I$ and a surjection $\bigoplus_{i\in I} M(m_i) \to M$.

5.) The following functors from $\mathsf{VIC}(R)$–modules can be considered forgetful functors. Find their left adjoints.

(a) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \mathsf{Set}$ be the functor sending M to the underlying set of M_m.

(b) Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \mathsf{Set}^{\mathbb{N}_0}$ be the functor sending M to the sequence of underlying sets of $(M_m)_{m\in\mathbb{N}_0}$.

(c) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \mathrm{GL}_m(R) - \mathsf{Set}$ be the functor sending M to the underlying $\mathrm{GL}_m(R)$–set of M_m.

(d) Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \prod_{m\in\mathbb{N}_0} \mathrm{GL}_m(R) - -\mathsf{Set}$ be the functor sending M to the sequence of underlying $\mathrm{GL}_m(R)$–sets of $(M_m)_{m\in\mathbb{N}_0}$.

(e) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \mathsf{Ab}$ be the functor sending M to the underlying abelian group M_m.

(f) Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \mathsf{Ab}^{\mathbb{N}_0}$ be the functor sending M to the sequence of underlying abelian groups $(M_m)_{m\in\mathbb{N}_0}$.

(g) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \mathrm{GL}_m(R){-}\mathsf{mod}$ be the functor sending M to the $\mathrm{GL}_m(R)$–representation M_m.

(h) Let $\mathsf{VIC}(R){-}\mathsf{mod} \to \prod_{m\in\mathbb{N}_0} \mathrm{GL}_m(R){-}\mathsf{mod}$ be the functor sending M to the sequence of $\mathrm{GL}_m(R)$–modules $(M_m)_{m\in\mathbb{N}_0}$.