Exercises

1.) The free group: Let S be a set and S^{-1} the symbols of inverses of S. Adding and removing ss^{-1} or $s^{-1}s$ for $s \in S$ defines an equivalence relation on the set of words of $S \cup S^{-1}$. The *free group* F_S is the set of equivalence classes with concatenation as group multiplication.

   (a) Prove that in every equivalence class there is exactly one fully canceled word, i.e. one word that doesn't contain an ss^{-1} or $s^{-1}s$ for $s \in S$.

   (b) Prove that F_S is a group.

   (c) Prove the universal property of F_S: Let G be a group. For every set map $f\colon S \to G$, there is a unique group homomorphism $F_S \to G$ extending f.

2.) The generators of $\mathrm{Aut}(F_n)$: Let $S = \{x_1, \dots, x_n\}$ and denote F_S by F_n. A group homomorphism $f\colon F_n \to F_n$ is given by the images $f(x_1), \dots, f(x_n)$. Define the *length* $|f|$ of f be the sum of the lengths of (the completely canceled words) $f(x_i)$.

   (a) Prove that $|f| \geq n$ if $f \in \mathrm{Aut}(F_n)$.

   (b) Observe that every permutation $\sigma \in S_n$ defines an automorphism $x_i \mapsto x_{\sigma(i)}$.

   (c) Let inv_i be the automorphism of F_n defined by $x_j \mapsto x_j$ for $j \neq i$ and $x_i \mapsto x_i^{-1}$. Prove that if $|f| = n$ and $f \in \mathrm{Aut}(F_n)$, then f is generated by permutations and inv_1.

   (d) Let $\mathrm{leftmul}_{ij}$ be the automorphism of F_n defined by $x_k \mapsto x_k$ for $k \neq j$ and $x_j \mapsto x_i x_j$. Let $\mathrm{rightmul}_{ij}$ be the automorphism of F_n defined by $x_k \mapsto x_k$ for $k \neq j$ and $x_j \mapsto x_j x_i$. Observe that the permutations, inv_1, and $\mathrm{leftmul}_{12}$ generate all $\mathrm{leftmul}_{ij}$ and $\mathrm{rightmul}_{ij}$.

   (e) Let f be an automorphism of F_n with $|f| > n$. Let w_i, w_i' be the reduced words defined by $f(x_i), f^{-1}(x_i)$, resp. By replacing the x_i's in w_j' with w_i, we get a word that cancels to x_j. Observe that if $|w_j'| > 1$, one of the ${w_i}^{\pm 1}$ must be completely canceled only by its neighbors.

   (f) If a ${w_i}^{\pm 1}$ is canceled completely by its neighbors where one neighbor cancels more letters than the other, use $\mathrm{leftmul}_{ij}$ or $\mathrm{rightmul}_{ij}$ to reduce the length of f.

   (g) If all ${w_i}^{\pm 1}$ that are canceled completely by its neighbors are canceled exactly in the middle, let ${w_i}^{\pm 1}$ be one of these with minimal length and let ${w_i}^{\pm 1} = ab$ with $|a| = |b|$, use $\mathrm{leftmul}_{ij}$ or $\mathrm{rightmul}_{ij}$ to replace b^{-1}'s in the beginning and b's in the end of a w_j by a's and a^{-1} respectively. Prove that this can only be done finitely many times before the length of f can be reduced using (f).

   (h) Conclude that there is a four element generator set of $\mathrm{Aut}(F_n)$.

3.) Prove that abelianizing is functorial. That means it is a functor from the category of groups to the category of abelian groups. Most importantly, for group homomorphisms $G \to H$, there exist homomorphisms between the abelianizations that behave well under composition.

4.) Prove the universal property of the abelianization: Let G be a group and A be an abelian group. Every group homomorphism $G \to A$ factors uniquely through the abelianization of G.

5.) Show that the abelianization of F_n is $\mathbb{Z}^n$.

6.) Observe that the group of group automorphisms of $\mathbb{Z}^n$ is precisely $\mathrm{GL}_n(\mathbb{Z})$.