Inducing t-structures on semiorthogonal components

Alexander Perry
(joint work with Alexander Kuznetsov, Shengxuan Liu)

Let $\mathcal{C} \subset \mathrm{D}^{\mathrm{b}}(X)$ be a semiorthogonal component of the bounded derived category of coherent sheaves on a smooth projective variety. Such a category $\mathcal{C}$ is often considered as a model for a "noncommutative algebraic variety". Indeed, on the one hand, $\mathcal{C}$ can be expressed as the derived category of a noncommutative DG algebra. On the other, $\mathcal{C}$ exhibits many of the same structures as (the derived category of) an actual variety, including:

- a theory of Serre duality, encoded by the Serre functor;
- motivic invariants, such as K-theory and Hochschild homology; and
- a moduli stack parameterizing its objects.

One of the most fundamental features of $\mathrm{D}^{\mathrm{b}}(X)$ is that it admits a bounded t-structure. This motivates the following question.

Question 1. *If $\mathcal{C}$ is a semiorthogonal component as above, does it admit a bounded t-structure?*

No negative results are known, while positive results were previously only known for a handful of Kuznetsov components of Fano varieties [1, 12, 10]. In these positive cases, the t-structures were also used to construct stability conditions, which is an important underlying motivation for the question.

Our main result, joint with Kuznetsov and Liu, provides a new approach to the problem.

Theorem 2 ([8]). *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{C} \rangle$ be a semiorthogonal decomposition of a triangulated category. Let $\tau = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$ such that:*

(i) *τ is noetherian, i.e. its heart $\mathcal{D}^{\heartsuit} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is a noetherian abelian category.*

(ii) *τ restricts to a t-structure on $\mathcal{B}$, i.e. the pair $\tau_{\mathcal{B}} = (\mathcal{D}^{\leq 0} \cap \mathcal{B}, \mathcal{D}^{\geq 0} \cap \mathcal{B})$ is a t-structure on $\mathcal{B}$.*

Then there exists a t-structure $\tau_{\mathcal{C}}$ on $\mathcal{C}$, called the induced t-structure*, which is bounded if τ is bounded, and given explicitly by the pair*

$$
\begin{aligned}
\mathcal{C}^{\leq 0} &= \mathcal{D}^{\leq 0} \cap \mathcal{C}, \\
\mathcal{C}^{\geq 0} &= \{ C \in \mathcal{C} \mid \mathrm{Hom}(C'[1], C) = 0 \text{ for all } C' \in \mathcal{C}^{\leq 0} \}.
\end{aligned}
\tag{1}
$$

The proof can be roughly summarized in four steps. Let $\gamma \colon \mathcal{C} \to \mathcal{D}$ be the inclusion, $\gamma^!$ its right adjoint, and $\mathrm{pr}_{\mathcal{C}} = \gamma \circ \gamma^! \colon \mathcal{D} \to \mathcal{D}$ the projection functor onto $\mathcal{C}$. First, it is easy to see that assumption (ii) implies $\mathrm{pr}_{\mathcal{C}}$ has right t-amplitude ≤ 1 with respect to τ. Second, in the main step, using assumption (i) we construct via tilting a new t-structure $\tau^\sharp = (\mathcal{D}^{\sharp, \leq 0}, \mathcal{D}^{\sharp, \geq 0})$ on $\mathcal{D}$ with the property that $\mathrm{pr}_{\mathcal{C}}$ is right t-exact with respect to $\tau^\sharp$. Third, it is easy to see that this right t-exactness implies $\gamma^! \tau^\sharp = (\gamma^!(\mathcal{D}^{\sharp, \leq 0}), \gamma^!(\mathcal{D}^{\sharp, \geq 0}))$ is a t-structure on $\mathcal{C}$. Finally, we check that $\gamma^! \tau^\sharp$ coincides with the prescription for $\tau_{\mathcal{C}}$ above.

Remark 3. Our proof suggests a general approach to a positive answer to Question 1. For $\mathcal{C} \subset \mathrm{D^b}(X)$ as in the question, the projection functor $\mathrm{pr}_{\mathcal{C}}$ has bounded t-amplitude with respect to the standard t-structure τ_X on $\mathrm{D^b}(X)$, so we may try to iteratively tilt τ_X to reduce the right t-amplitude of $\mathrm{pr}_{\mathcal{C}}$ to zero, in which case the above argument applies. The issue with this approach is that tilting may destroy the noetherianity of the t-structure, and without this property there is no guarantee that further tilts exist.

We call the t-structure $\tau^\sharp$ from the above construction the *perverse t-structure*, in view of the following example.

Example 4. Let $f\colon X \to Y$ be a proper surjective morphism of varieties with $\mathrm{R}f_*\mathcal{O}_X = \mathcal{O}_Y$. Then the pullback functor $\mathrm{L}f^*\colon \mathrm{D}^-(Y) \to \mathrm{D}^-(X)$ between bounded above derived categories of coherent sheaves is fully faithful, and there is a semiorthogonal decomposition $\mathrm{D}^-(X) = \langle \mathcal{B}, \mathcal{C} \rangle$, where

$$\mathcal{B} = \ker(\mathrm{R}f_*) \qquad \text{and} \qquad \mathcal{C} = \mathrm{L}f^*\mathrm{D}^-(Y).$$

If the fibers of f have dimension ≤ 1, e.g. if f is a threefold flopping contraction, then it is easy to see that the standard t-structure on $\mathrm{D}^-(X)$ satisfies assumption (ii) of Theorem 2. In this case, the induced t-structure $\tau_{\mathcal{C}}$ is just the standard t-structure on $\mathrm{D}^-(Y)$, but the t-structure $\tau^\sharp$ is very interesting — it coincides with (one of) Bridgeland's perverse t-structures used in his proof of the D-equivalence conjecture for Calabi–Yau threefolds [3].

We prove criteria for checking assumption (ii), leading in particular to the following version of Theorem 2.

Theorem 5. *Let $\mathcal{D}$ be a k-linear triangulated category, where k is a field. Let τ be a noetherian bounded t-structure on $\mathcal{D}$. Let $E_1, \dots, E_n \in \mathcal{D}^\heartsuit$ be an exceptional sequence of objects in the heart of τ which satisfies*

$$\mathrm{Hom}(E_i, E_j) = 0 \quad \textit{for all } i < j. \tag{2}$$

Let

$$\mathcal{D} = \langle E_1, E_2, \dots, E_n, \mathcal{C} \rangle$$

be the corresponding semiorthogonal decomposition of $\mathcal{D}$. Then τ induces a bounded t-structure $\tau_{\mathcal{C}}$ on $\mathcal{C}$ given by (1).

The forward Hom-vanishing condition (2) is fairly restrictive, but it is still satisfied by many interesting examples. For example, it is automatic when the exceptional collection has length $n = 1$. This applies, for instance, to the semiorthogonal complement of the structure sheaf of any Fano variety, as well as to a curious semiorthogonal component of a quiver with relations recently shown in [5] not to admit a stability condition.

Perhaps the most interesting application of Theorem 5 is to phantom categories. A nonzero semiorthogonal component $\mathcal{P} \subset \mathrm{D^b}(X)$ of a smooth projective variety is called a *phantom* if its Grothendieck group and Hochschild homology vanish. Although such categories were originally conjectured not to exist, there is now a growing list of examples [4, 2, 7, 6, 9], suggesting that they may be ubiquitous.

A phantom category cannot admit a stability condition, due to the vanishing of its Grothendieck group; in fact, by [11], it cannot admit a noetherian bounded t-structure. This suggests that phantoms form a critical test case for Question 1. As an application of Theorem 5, we construct bounded t-structures on almost all currently known phantom categories.

Theorem 6. *Let* $\mathcal{P} \subset \mathrm{D^b}(X)$ *be one of the phantom categories constructed in* [2, 7, 6, 9]. *Then the standard t-structure on* $\mathrm{D^b}(X)$ *induces a bounded t-structure on* $\mathcal{P}$.

Among the questions suggested by our results, we highlight one.

Question 7. *Does there exist an extension of the usual theory of stability conditions that applies, in particular, to phantom categories and gives rise to proper moduli spaces of objects in the hearts of the induced t-structures above?*

References

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