Math 54 Final Review

Several of the questions below are from Professor Nadler's Fall 2014 final exam.

1. Find a solution to the following system of linear equations. (Hint: if you notice something about the columns of the matrix you can calculate the answer pretty quickly.)

$$\begin{array}{rcrcrcr} x_1 & + & 6x_2 & + & 2x_3 & = & 23 \\ 2x_1 & - & x_2 & + & x_3 & = & 1 \\ 3x_1 & & & - & 16x_3 & = & -29 \\ 4x_1 & - & x_2 & + & 11x_3 & = & 23 \end{array}$$

2. Find a solution to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & -1 & 1 \\ 3 & 0 & -16 \\ 4 & -1 & 11 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}$$

   (a) How many pivots does A have?

   (b) What is $\dim(\operatorname{Null} A)$?

   (c) What is $\operatorname{rank} A$?

   (d) What is $\dim(\operatorname{Row} A)$?

   (e) Is the linear transformation given by $\mathbf{v} \mapsto A\mathbf{v}$ one-to-one? Onto?

   (f) Find a matrix B such that $BA = I$.

3. Find a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that is equal to $\mathbf{b}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -16 \\ 11 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}$$

4. For which pair of real numbers a and b is the following matrix rank one:

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & a & 1 \\ 3 & -6 & b \end{bmatrix}$$

5. Let A be a 2×2 matrix with real entries and eigenvalues 2 and -3. What is the characteristic polynomial of A? What if 2 is the only eigenvalue? What if $2 - 2i$ is an eigenvalue?

6. Find a nontrivial solution $u(x,t)$ to the heat equation ($\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial^2 x}$) with the usual boundary values ($u(0,t) = u(L,t) = 0$) such that $u(L/27, 0) = 0$.

7. Let $f\colon [0, L] \to \mathbb{R}$ be a function such that $f(0) = f(L) = 0$ and f' is continuous. Suppose that the n^{th} coefficient of the fourier sine series for f is $\frac{1}{2^n}$.

(a) Find $f(\frac{L}{2})$

(b) Find the fourier sine series for $f(x)\cos\left(\frac{n\pi x}{L}\right)$. Hint: use the fact that for any A and B, $\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B)+\sin(A+B))$.

8. The following assertions are FALSE. For each one, provide a counterexample to show it is false.

(a) If A is a 2×2 symmetric matrix with positive integer entries then any eigenvalue of A is positive or zero.

(b) Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a one-to-one linear transformation and let $\mathcal{B}$ be a basis for $\mathbb{R}^3$. Then there is a basis $\mathcal{C}$ of $\mathbb{R}^2$ such that

$$_{\mathcal{B}}[T]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(c) Let f and g be two differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If the Wronskian $W[f,g](t)$ is always 0 then f and g are linearly dependent.

(d) If the columns of a matrix are orthonormal then so are the rows (hint: this is true if the matrix is square).

9. Mark each of the following statements true or false. For each statement, either give a proof that it is always true or give a counterexample to show it can be false.

(a) For any inner product space V, if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $||\mathbf{u}|| = 1, ||\mathbf{v}|| = 1$ and $||\mathbf{u}-\mathbf{v}|| = \sqrt{2}$ then $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

(b) If A is symmetric and U is orthogonal then UAU^{-1} is symmetric.

(c) If the characteristic polynomial of an $n\times n$ matrix A has a nonzero constant coefficient then A is invertible.

(d) Let V be the vector space of differentiable functions on $\mathbb{R}$. The linear transformation $\frac{d^2}{dt^2} - 2\frac{d}{dt} + 2I$ is one-to-one.