Final Exam

Math 592
29 April 2025
Jenny Wilson

Name: _____

Instructions: This exam has 6 questions for a total of 30 points.

The exam is **closed-book**. No books, notes, cell phones, calculators, or other devices are permitted.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting, and explicitly check that the hypotheses hold.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	3	
2	4	
3	3	
4	3	
5	7	
6	10	
Total:	30	

Notation

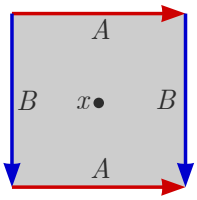
- $I = [0, 1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
  (unit n-sphere)
  (we may view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n \geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

1. (3 points) Let $T = S^1 \times S^1$ be the torus. Suppose that $f : T \to T$ is a continuous map that is **not** surjective. Show that the induced map $f_* : H_2(T) \to H_2(T)$ is zero.

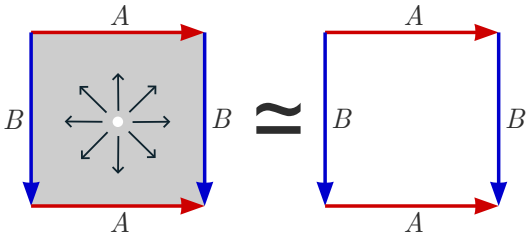
Solution. Let $x \in T$ denote a point that is not in the image of f. Then f factors through the inclusion $\iota : T \setminus \{x\} \hookrightarrow T$,

$$T \xrightarrow{\overline{f}} T \setminus \{x\} \xrightarrow{\iota} T, \qquad f = \iota \circ \overline{f}$$

We first claim that $T \setminus \{x\}$ is homotopy equivalent to $S^1 \vee S^1$. We observe that all points of T are indistinguishable up to self-homeomorphisms of T, for example, there is a transitive action on T by rotation in each factor S^1. So we can consider the standard CW complex structure on T shown below, and assume without loss of generality that x is the midpoint of the 2-cell.



The punctured torus $T \setminus \{x\}$ deformation retracts onto its 1-skeleton $S^1 \vee S^1$, for example, by the straight-line homotopy along radial lines in the 2-cell to the image of its boundary, as illustrated.



But the degree-2 homology of $S^1 \vee S^1$ vanishes. This follows, for example, by considering cellular homology: it is admits a CW complex structure with no 2-cells. Since homology is a homotopy invariant, we deduce that $H_2(T \setminus \{x\}) \cong 0$.

By functoriality of homology, the induced map $f_* : H_2(T) \to H_2(T)$ factors

$$H_2(T) \xrightarrow{\overline{f}_*} H_2(T \setminus \{x\}) \xrightarrow{\iota_*} H_2(T). \qquad f_* = \iota_* \circ \overline{f}_*$$

Thus f_* factors through the zero group, and we conclude that it is the zero map.

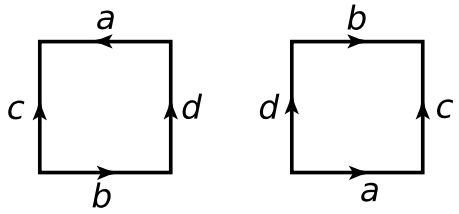
2. (a) (1 point) State the classification theorem for closed surfaces.

Solution. By a *closed surface*, we mean a compact 2-dimensional manifold without boundary. The classification theorem states that every connected closed surface is homeomorphic to precisely one of the following:

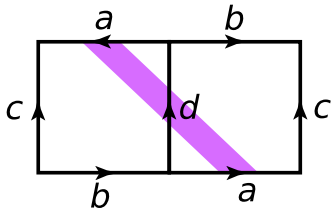
- the orientable genus-g surface Σ_g,
  i.e., the connected sum of g tori, for some integer $g \geq 1$,
- the non-orientable genus-g surface,
  i.e, the connected sum of g real projective planes, for some integer $g \geq 1$,
- the 2-sphere S^2 (also called the orientable genus-0 surface)

In particular, this means that every closed surface is determined up to homeomorphism by (i) whether it is orientable, and (ii) its Euler characteristic.

(b) (3 points) A certain surface X is constructed by gluing together two squares by the edge identifications shown. Determine (with justification) which surface it is, in the sense of the classification theorem. You do not need to verify that X is a surface.



Solution. We first observe that the surface is non-orientable, since it contains an embedded Mobius band (shaded purple).



We next compute its Euler characteristic. When we trace through the edge identifications, we see that all 8 vertices of the two squares are identified to a single point. Thus we can interpret the squares-with-edge-identifications as a CW complex structure with one vertex, four edges a, b, c, d, and two 2-cells. The Euler characteristic is

$$\chi(X) \quad = \quad \#\text{2-cells} - \#\text{edges} + \#\text{vertices} \quad = \quad 2 - 4 + 1 \quad = \quad -1.$$

A non-orientable genus-g surface has Euler characteristic $2 - g$, so (solving for g) we find that X has genus 3. We conclude

$$X \cong \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2.$$

3. (3 points) Let $p : \widetilde{X} \to X$ be a (surjective) covering space map. Suppose that $\widetilde{X}$ is a finite tree, that is, a tree with finitely many vertices and edges. Prove that the covering space is trivial in the sense that p must be 1-sheeted.

Solution. Choose a basepoint x_0 in X and a point $\widetilde{x_0}$ in its preimage $p^{-1}(x_0) \subseteq \widetilde{X}$. If $\widetilde{X}$ is a finite tree, it is contractible, in particular, its fundamental group $\pi_1(\widetilde{X}, \widetilde{x_0})$ is zero. The image $H := p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) \subseteq \pi_1(X, x_0)$ is the trivial subgroup, and so is necessarily normal. We proved that normality of H is equivalent to the statement that p is a regular cover, in the sense that the deck group of p acts transitively on the fibre $p^{-1}(x_0)$. Thus to prove that p is 1-sheeted—that is, $p^{-1}(x_0) = \{\widetilde{x_0}\}$—it suffices to prove that its deck group is trivial.

Suppose that $f : \widetilde{X} \to \widetilde{X}$ is a deck transformation. The tree $\widetilde{X}$ is contractible, so its only nonzero unreduced homology group is a copy of $\mathbb{Z}$ in degree 0. The induced map f_* is the 1×1 identity matrix in degree zero, as f must map the single path-component of $\widetilde{X}$ to itself. The Lefschetz number of f is therefore

$$\tau(f) = \sum (-1)^i \text{Trace}\left(f_* : H_i(\widetilde{X}) \to H_i(\widetilde{X})\right) = \text{Trace}\left(f_* : H_0(\widetilde{X}) \to H_0(\widetilde{X})\right) = 1.$$

Because $\widetilde{X}$ is a finite CW complex, the Lefschetz Fixed Point Theorem applies. It states that, since $\tau(f)$ is nonzero, the map f must have a fixed point.
(Instead of computing the Lefschetz number, it would be enough for the solution to simply cite our fixed-point theorem for compact, contractible CW complexes.)

However, we proved (by the uniqueness theorem for lifts of maps to covering spaces) that any deck transformation of a connected cover is determined by the image of a single point. In particular, the only deck transformation of $\widetilde{X}$ with a fixed point is the identity. We can therefore conclude that the cover p has trivial deck group, which proves the result.

We note that, in contrast, infinite trees can be nontrivial covers – we have proved that the universal cover of any non-contractible graph is an infinite tree.

4. (3 points) Let $p : E \to X$ be a covering space map of topological spaces that have the T_1 property (this means "points are closed"). Prove that p induces isomorphisms on local homology groups, that is, for all $e \in E$ and integers $q \geq 0$,

$$p_* : H_q(E, E \setminus \{e\}) \xrightarrow{\cong} H_q(X, X \setminus \{p(e)\}).$$

Solution. Fix $e \in E$. Let U be an open neighbourhood of $p(e)$ that is evenly covered by p, and let V be the slice of $p^{-1}(U)$ containing e. This means in particular that the restriction $p|_V : V \to U$ is a homeomorphism.

To study the local homology groups, we will use the Excision Theorem.

> **Excision Theorem.** Given subspaces $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_q(X \setminus Z, A \setminus Z) \to H_q(X, A)$ for all q.

We will first apply this result to the subsets of X given by $A = X \setminus \{p(e)\}$ and its subset $Z = X \setminus U$. We note that $Z = X \setminus U$ is closed by construction, and $A = X \setminus \{p(e)\}$ is open by the assumption that X is T_1. Thus $\mathrm{cl}(Z) = Z \subseteq A = \mathrm{int}(A)$, and the hypotheses of the Excision Theorem hold. We deduce that

$$H_q(X, X \setminus \{p(e)\}) \cong H_q(U, U \setminus \{p(e)\}).$$

The same argument implies

$$H_q(E, E \setminus \{e\}) \cong H_q(V, V \setminus \{e\}).$$

Now, since $p|_V : V \to U$ is a homeomorphism, by Homework #10 Problem 5(c) it must induce isomorphisms

$$H_q(V, V \setminus \{e\}) \xrightarrow{\cong} H_q(U, U \setminus \{p(e)\}).$$

Thus we have isomorphisms

$$H_q(E, E \setminus \{e\}) \cong H_q(V, V \setminus \{e\}) \xrightarrow[\cong]{(p|_V)_*} H_q(U, U \setminus \{p(e)\}) \cong H_q(X, X \setminus \{p(e)\}).$$

We check explicitly that the composition of these isomorphisms is indeed the map p_*:

- Choose a relative homology class σ in $H_q(E, E \setminus \{e\})$. Since the first isomorphism is induced by the inclusion of pairs $(V, V \setminus \{e\}) \hookrightarrow (E, E \setminus \{e\})$, to apply the first isomorphism we choose a chain in $V \subseteq E$ representing σ.
- The second isomorphism is performed by applying p to this chain.
- The third isomorphism is induced by the inclusion of pairs $(U, U \setminus \{p(e)\}) \hookrightarrow (X, X \setminus \{p(e)\})$.

Thus the composition of isomorphisms coincides with the map $\sigma \mapsto p_*(\sigma)$, as desired.

5. Fix $n \geq 1$ and consider complex projective space $\mathbb{C}\mathrm{P}^n$.

   (a) (2 points) Fix a point $x_0 \in \mathbb{C}\mathrm{P}^n$. Show that punctured complex projective space $\mathbb{C}\mathrm{P}^n \setminus \{x_0\}$ deformation retracts onto $\mathbb{C}\mathrm{P}^{n-1}$.

   Solution. We first will check that, up to homeomorphism, all points $x_0 \in \mathbb{C}\mathrm{P}^n$ are indistinguishable. We observe that (for example) the action of $\mathrm{GL}_{n+1}(\mathbb{C})$ acts transitively on the nonzero vectors in $\mathbb{C}^{n+1}$ by linear maps, which induces a transitive action on the space of complex lines $\mathbb{C}\mathrm{P}^n$ by homeomorphisms. Thus it suffices to check the claim for any specific point $x_0 \in \mathbb{C}\mathrm{P}^n$.

   Recall from Homework #2 Problem 1(c) that there is a CW complex structure on $\mathbb{C}\mathrm{P}^n$ with precisely one cell in every even dimension, constructed inductively so that the $(2k)$-skeleton of $\mathbb{C}\mathrm{P}^n$ is $\mathbb{C}\mathrm{P}^k$ for all $k \leq n$. Thus (by our definition of a CW complex) $\mathbb{C}\mathrm{P}^n$ is a quotient of the disjoint union $\mathbb{C}\mathrm{P}^{n-1} \sqcup D^{2n}$ by gluing the closed 2-disk D^{2n} along its boundary to $\mathbb{C}\mathrm{P}^{n-1}$ by its attaching map.

   Then punctured $\mathbb{C}\mathrm{P}^n$ is the quotient of $\mathbb{C}\mathrm{P}^{n-1} \sqcup (D^{2n} \setminus \mathbf{0})$, where we puncture the disk D^{2n} (viewed as the unit ball in $\mathbb{R}^{2n}$) at the origin. Consider the deformation retraction of $D^{2n} \setminus \mathbf{0}$ to its boundary (say, by the radial straight-line homotopy). We proved on Homework #1 Problem 3(c) that a homotopy $(X/\sim) \times I \to Y$ on a quotient space is continuous if and only if it arises from a continuous map $X \times I \to Y$ which is, for each fixed $t \in I$, constant on equivalence classes in X. Since the deformation retraction of $D^{2n} \setminus \mathbf{0}$ is stationary on the boundary ∂D^{2n}, it follows that the deformation retraction of $\mathbb{C}\mathrm{P}^{n-1} \sqcup (D^{2n} \setminus \mathbf{0})$ onto $\mathbb{C}\mathrm{P}^{n-1} \sqcup \partial D^{2n}$ induces a deformation retraction of the quotient $\mathbb{C}\mathrm{P}^n$ onto the quotient of $\mathbb{C}\mathrm{P}^{n-1} \sqcup \partial D^{2n}$, which is the $(2n-2)$-skeleton $\mathbb{C}\mathrm{P}^{n-1}$.

   (b) (5 points) Compute the homology of the connected sum $\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n$.

   Recall that the connected sum of two m-dimensional (oriented) manifolds is obtained by deleting an open m-ball from each, and gluing together the boundaries of these balls via an (orientation-reversing) homeomorphism.

Solution. We will use the Mayer–Vietoris long exact sequence.

Consider the connected sum $\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n$. Since the connected sum is (up to homeomorphism) independent of the choice of open $2n$-balls in $\mathbb{C}\mathrm{P}^n$, and $\mathbb{C}\mathrm{P}^n$ is locally Euclidean, we can choose an open ball B so that its boundary ∂B is a deformation retract of some small neighbourhood N of ∂B in $\mathbb{C}\mathrm{P}^n \setminus B$.

Let U be the open subset of $\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n$ given by the union of the first copy of $\mathbb{C}\mathrm{P}^n \setminus B$ with the subset N of the second copy of $\mathbb{C}\mathrm{P}^n \setminus B$. Let V be the union of the second copy of $\mathbb{C}\mathrm{P}^n \setminus B$ with the subset N of the first copy. Then by construction U and V are open, and each is homotopy equivalent to $\mathbb{C}\mathrm{P}^n \setminus B$.

Since the punctured ball B deformation retracts onto ∂B, the space $\mathbb{C}\mathrm{P}^n \setminus B$ is a deformation retract of punctured $\mathbb{C}\mathrm{P}^n$, which by part (a) is homotopy equivalent to $\mathbb{C}\mathrm{P}^{n-1}$.

Extra space for Problem 5.

The intersection $U \cap V$ deformation retracts onto $\partial B \cong S^{2n-1}$.

Because the sets U, V form an open cover of $\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n$, the Mayer–Vietoris sequence applies.

$$\cdots \longrightarrow \widetilde{H}_i(U \cap V) \xrightarrow{\Phi} \widetilde{H}_i(U) \oplus \widetilde{H}_i(V) \xrightarrow{\Psi} \widetilde{H}_i(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) \xrightarrow{\delta} \widetilde{H}_{i-1}(U \cap V) \longrightarrow \cdots$$

$$\cdots \longrightarrow \widetilde{H}_0(X) \longrightarrow 0.$$

We know

$$\widetilde{H}_i(U \cap V) \cong \widetilde{H}_i(S^{2n-1}) \cong \begin{cases} \mathbb{Z}, & i = 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\widetilde{H}_i(U) \cong \widetilde{H}_i(V) \cong \widetilde{H}_i(\mathbb{C}\mathrm{P}^{n-1}) \cong \begin{cases} \mathbb{Z}, & 0 < i < 2n, \ i \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the long exact sequence simplifies, for $i = 2n, 2n-1$,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \widetilde{H}_{2n}(U) \oplus \widetilde{H}_{2n}(V) & \longrightarrow & \widetilde{H}_{2n}(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) & \xrightarrow{\delta} & \widetilde{H}_{2n-1}(U \cap V) & \longrightarrow & \widetilde{H}_{2n-1}(U) \oplus \widetilde{H}_{2n-1}(V) & \\ & \| & & & & \| & & \| & \\ & 0 & & & & \mathbb{Z} & & 0 & \end{array}$$

$$\longrightarrow \widetilde{H}_{2n-1}(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) \xrightarrow{\delta} \widetilde{H}_{2n-2}(U \cap V) \longrightarrow \cdots$$
$$\|$$
$$0$$

which implies $\widetilde{H}_{2n}(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$ and $\widetilde{H}_{2n-1}(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) \cong 0$.
For $i < 2n-1$,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \widetilde{H}_i(U \cap V) & \longrightarrow & \widetilde{H}_i(U) \oplus \widetilde{H}_i(V) & \longrightarrow & \widetilde{H}_i(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) & \xrightarrow{\delta} & \widetilde{H}_{i-1}(U \cap V) & \longrightarrow \cdots \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

which implies that $\widetilde{H}_i(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) \cong \widetilde{H}_i(U) \oplus \widetilde{H}_i(V) \cong \widetilde{H}_i(\mathbb{C}\mathrm{P}^{n-1}) \oplus \widetilde{H}_i(\mathbb{C}\mathrm{P}^{n-1})$ for $i < 2n-1$. Thus,

$$\widetilde{H}_{2n-1}(\mathbb{C}\mathrm{P}^n \# \mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & i = 2n \\ \mathbb{Z}^2, & 0 < i < 2n, \ i \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

6. (10 points) For each of the following statements: if the statement is true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let $\mathcal{F}$ be a functor from the category of topological spaces and continuous maps, to the category of abelian groups and group homomorphisms. Then $\mathcal{F}$ must take the coproduct of spaces $X \coprod Y$ to the coproduct $\mathcal{F}(X) \coprod \mathcal{F}(Y)$ of abelian groups.

   False. Recall from Homework #2 Problem 3 that the coproduct of topological spaces is their disjoint union (with the disjoint union topology), and the coproduct of abelian groups is the direct sum. One counterexample is the constant functor at some nonzero abelian group (say, $\mathbb{Z}/2\mathbb{Z}$) taking all morphisms to the identity map. Another counterexample is the reduced homology group $\widetilde{H}_0$.

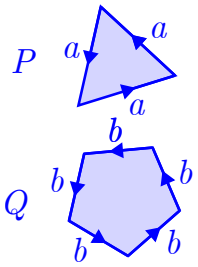
   (b) Let T denote the torus. For every $n \geq 1$, any continuous map $f : \mathbb{C}\mathrm{P}^n \to T$ is nullhomotopic.

   True.

   Hint: Since $\mathbb{C}\mathrm{P}^n$ is simply connected, we can use the lifting criterion for covering spaces to argue that f factors through the universal cover $\mathbb{R}^2 \to T$. Since $\mathbb{R}^2$ is contractible we deduce that f is nullhomotopic.

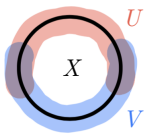
   (c) There does not exist a CW complex with fundamental group $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

   False. For example, take the Cartesian product of the following two quotients spaces P and Q of the polygons via the edge identifications shown. Then $\pi_1(P) \cong \mathbb{Z}/3\mathbb{Z}$ and $\pi_1(Q) \cong \mathbb{Z}/5\mathbb{Z}$ by our results on the fundamental group of CW complexes (Homework #4 Problem 1(a)), and the claim follows from our result that $\pi_1(P \times Q) \cong \pi_1(P) \times \pi_1(Q)$.



   (d) Suppose that a space X is the union of two open subsets U, V with nonempty intersection. If U and V are simply connected, then X must be simply connected.

   False. For example, consider the circle $X = S^1$. Let U be a small neighbourhood of the "upper hemisphere" and V a small neighbourhood of the "lower hemisphere", as shown. Then U and V intersect, but their union $X = U \cup V$ has nontrivial fundamental group.
   In contrast, if we knew the intersection $U \cap V$ were path-connected, the Seifert–Van Kampen Theorem would imply that X is simply connected.

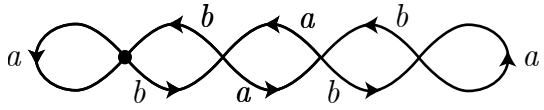


   (e) Suppose that a certain based map $f : (X, x_0) \to (Y, f(x_0))$ of path-connected spaces X, Y induces the zero map on fundamental group. Then $f_* : H_1(X) \to H_1(Y)$ must be the zero map.

   True.

   Hint: This follows from the naturality of the Hurewicz homomorphism (Homework #10 Problem 4(h)).

(f) Let $X = S^1 \vee S^1$ be the wedge of two circles, with wedge point x_0. We label the two oriented edges by a and b, respectively, and (by abuse of notation) also denote by a and b the corresponding elements of $\pi_1(X, x_0)$, so we can identify $\pi_1(X, x_0)$ with the free group $F_{\{a,b\}}$. Let H be the subgroup of $F_{\{a,b\}}$ associated with the based covering space $(\widetilde{X}, \widetilde{x_0})$ of X shown. Then every element of its normalizer $N(H)$ in $F_{\{a,b\}}$ is in fact contained in H.



False. For example, the element $bab \in F_{\{a,b\}}$ is in $N(H)$ but not H.

Recall that we showed that an element in $\pi_1(X, x_0)$ is contained in H if and only if its lift to $\widetilde{X}$ starting at $\widetilde{x_0}$ is a loop, and it is contained in the normalizer $N(H)$ if and only if its the endpoint of this lift is in the orbit of $\widetilde{x_0}$ under the deck group. This cover has one nontrivial deck transformation—rotation by 180°. Thus the elements of $N(H)$ not contained in H are precisely the images in $\pi_1(X, x_0)$ of any path in $\widetilde{X}$ from the basepoint $\widetilde{x_0}$ to the fourth vertex of $\widetilde{X}$ (read from left to right).

(g) Let $(\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and $H \subseteq F_{\{a,b\}}$ be as in part (f). Then every element of $F_{\{a,b\}}$ is contained in $N(H)$.

False. For example, the element $b \in F_{\{a,b\}}$ is not in $N(H)$.

Observe that the lift of b starting at $\widetilde{x_0}$ ends at the second vertex, which is not in the orbit of $\widetilde{x_0}$ under the deck group.

(h) Let $p : S^3 \to S^3$ be a map of degree $d > 1$. Then p cannot be a covering space map.

True.

Hint: Since S^3 is a simply connected CW complex, by the classification of covering spaces, the only possible covering maps are 1-sheeted covers, i.e., homeomorphisms. But we saw that the degree of a homeomorphism must be a unit ± 1.
(Contrast to the case of S^1, which has degree-d covers $S^1 \to S^1$ for all $d \geq 1$!)

(i) Let X be a CW complex that has cells only in dimensions $0, 1, 2, 4$. Then $H_2(X)$ is a free abelian group (possibly zero).

True.

Hint: Since the cellular chain group $C_3(X)$ is zero, $H_2(X)$ is by definition the kernel of the differential $\partial_2 : C_2(X) \to C_1(X)$. We deduce that $H_2(X)$ is a subgroup of the free abelian group $C_2(X)$ and therefore free abelian.

(j) Let $A \subseteq X$ be topological spaces. If A is contractible, then $H_q(X, A) \cong \widetilde{H}_q(X/A)$ for all q.

False. For example, consider the case that $X = S^1$ and $A \subseteq X$ is the complement of a point. Homework #2 Problem 2(g) proved that X/A is contractible. However, since A is contractible, the long exact sequence of a pair implies that $H_q(X, A)$ coincides with $\widetilde{H}_q(S^1)$ for all q. In particular $\widetilde{H}_1(X/A) \cong 0$ but $H_1(X, A) \cong \mathbb{Z}$.