1 Cauchy sequences and complete metric spaces

Definition 1.1. Cauchy sequences. Let (X, d) be a metric space. Then a sequence $(a_n)_{n\in\mathbb{N}}$ of points in X is called a *Cauchy sequence* if ...

Example 1.2. Verify that the sequence $\left(\frac{1}{n}\right)_n$ in $\mathbb{R}$ is Cauchy.

Definition 1.3. Complete metric spaces. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

Later, using results from the homework, we will prove the following theorem.

Theorem 1.4. ($\mathbb{R}^n$ is complete). *The space $\mathbb{R}^n$ is complete with respect to the Euclidean metric.*

In-class Exercises

1. Prove that every convergent sequence in X is a Cauchy sequence.
2. Give an example of a metric space (X, d) and a sequence $(a_n)_{n\in\mathbb{N}}$ in X that is Cauchy but does not converge. Fully justify your solution!
3. **(Optional)** Let (X, d) be a nonempty set with the discrete metric $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$ Under what conditions is X complete?
4. **(Optional)** Let X and Y be metric spaces. Suppose that X is complete.
   (a) Let $f : X \to Y$ be a continuous function. Must its image $f(X)$ be complete?
   (b) Suppose that $f : X \to Y$ is an isometric embedding. Must $f(X)$ be complete?

5. **(Optional)** A subset A of a metric space X is *dense* if every nonempty open subset of X contains a point of A. Recall that an *isometry* $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is a bijective isometric embedding, i.e., a bijective map that preserves distances in the sense that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

Consider the following definition.

> **Definition (Completion).** Let X be a metric space. The *completion* of X is a complete metric space Y along with an isometric embedding $h : X \to Y$ such that $h(X)$ is dense in Y.

In this question, we will construct the completion of X, and verify that it is unique up to isometry.

(a) Let A be a dense subset of metric space Z. Show that, if every Cauchy subsequence in A converges in Z, then Z is complete.

(b) Let (X, d) be a metric space. Let $\tilde{X}$ denote the set of Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in X. Let Y denote the equivalences classes defined by the equivalence relation on $\tilde{X}$,

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \qquad \Longleftrightarrow \qquad d(x_n, y_n) \xrightarrow{n\to\infty} 0.$$

Verify that this is indeed an equivalence relation (i.e. it is reflexive, symmetric, transitive).

(c) For $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, let $[\mathbf{x}]$ and $[\mathbf{y}]$ denote the corresponding equivalence classes. Define

$$D : Y \times Y \to \mathbb{R}$$
$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Show that D is well-defined, that is, its value does not depend on the choice of representative of the equivalence class.

(d) Show that D defines a metric on Y.

(e) Define

$$h : X \to Y$$
$$x \mapsto [(x)_{n\in\mathbb{N}}]$$

sending a point x to the constant sequence $(x)_{n\in\mathbb{N}}$. Show that h is an isometric embedding.

(f) Show that, for any Cauchy sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the sequence $(h(x_n))_{n\in\mathbb{N}}$ converges in Y to $[\mathbf{x}]$.

(g) Conclude from part (f) that $h(X)$ is dense in Y.

(h) Further conclude that every Cauchy sequence in $h(X)$ must converge in Y, and thus by part (a) the space (Y, D) is complete. This shows that Y is the completion of X.

(i) Show that this completion is unique, in the sense of the following theorem.

> **Theorem (The completion of X is unique up to isometry).** Let $h : X \to Y$ and $h' : X \to Y'$ be isometric embeddings of the metric space (X, d) into complete metric spaces (Y, D) and (Y', D'), respectively, with dense image. Then there is an isometry of $(\overline{h(X)}, D)$ and $(\overline{h'(X)}, D')$ that equals $h' \circ h^{-1}$ when restricted to the subspace $h(X)$.

(j) Explain how we could define the real numbers $\mathbb{R}$ as the completion of the rationals $\mathbb{Q}$.