1 Sequences in topological spaces

Definition 1.1. (Convergence topological spaces.) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in a topological space $(X, \mathcal{T})$. Then we say that that $(a_n)_{n\in\mathbb{N}}$ *converges* to a point $a_\infty \in X$ if ...

Example 1.2. Let $X = \{0, 1\}$ with the *Sierpiński* topology $\mathcal{T} = \Big\{\varnothing, \{0\}, \{0, 1\}\Big\}$.
Find all limits of the following sequences.

(a) Constant sequence $0, 0, 0, 0, 0, \ldots$

(b) Constant sequence $1, 1, 1, 1, 1, \ldots$

(c) Alternating sequence $0, 1, 0, 1, 0, \ldots$

Example 1.3. Let $X = \mathbb{R}$ with the cofinite topology. Find all limits of the sequence $(n)_{n\in\mathbb{N}}$.

In-class Exercises

1. Consider the following definition.

   Definition 1.4. (Sequential continuity). Let $f : X \to Y$ be a function of topological spaces. Then f is called *sequentially continuous* if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X and any limit x_∞ of the sequence, the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y converges to the point $f(x_\infty)$.

   You proved that on Homework #3 Problem 3 that if X and Y are metric spaces (or, more generally, metrizable topological spaces), then sequential continuity is equivalent to continuity for a function $f : X \to Y$.

   Prove that any continuous function $f : X \to Y$ of topological spaces is sequentially continuous. It turns out that the converse fails to hold for some topological spaces! (See Problem 4.)

2. Prove the following theorem.

Definition 1.5. (Sequential closure). Let X be a topological space and $A \subseteq X$ a subset. The *sequential closure* $\mathrm{sc}(A)$ of A is the set

$$\mathrm{sc}(A) = \{x \in X \mid \text{there exists a sequence of points } (a_n)_{n\in\mathbb{N}} \text{ in } A \text{ converging to } x\}.$$

Theorem 1.6. (Sequential closure vs closure). *Let X be a topological space and $A \subseteq X$.*

(i) If there exists a sequence of points $(a_n)_{n\in\mathbb{N}}$ in A converging to a point $x \in X$, then $x \in \overline{A}$. In other words, $\mathrm{sc}(A) \subseteq \overline{A}$.

(ii) In particular, if A is a closed set, it contains all limits of its convergent subsequences.

(iii) If X is metrizable, then $\mathrm{sc}(A) = \overline{A}$. *In other words, in a metric space, a point $x \in X$ is in $\overline{A}$ if and only if there is a sequence of points in A converging to x.*

It turns out that, in general, $\mathrm{sc}(A)$ can be a **proper** subset of $\overline{A}$. (See Problem 5.)

3. **(Optional).** Let $X = \{0, 1\}$ with the topology $\mathcal{T} = \Big\{\varnothing, \{0\}, \{0, 1\}\Big\}$. Prove that every sequence of points in X converges to 1.

4. **(Optional).** In this problem, we will construct an example of a function of topological spaces that is sequentially continuous but not continuous.

(a) Recall that a set S is *countable* if there exists an injective function $S \to \mathbb{N}$, equivalently, if there is a surjective functions $\mathbb{N} \to S$. Such sets are either finite or countably infinite. Define a topology on $\mathbb{R}$ by $\mathcal{T}_{cc} = \{\varnothing\} \cup \{\, U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable } \}$.
Show that $\mathcal{T}_{cc}$ is indeed a topology on $\mathbb{R}$. It is called the *co-countable topology.*

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathbb{R}, \mathcal{T}_{cc})$. Show that $(a_n)_{n\in\mathbb{N}}$ converges if and only if it is *eventually constant,* i.e., there is some $N \in \mathbb{N}$ and $x \in \mathbb{R}$ so that $a_n = x$ for all $n \geq N$.

(c) Let $\mathcal{T}_{dsc}$ denote the discrete topology on $\mathbb{R}$. Let $I : \mathbb{R} \to \mathbb{R}$ be the identity map. Show that the following map of topological spaces is **not** continuous:

$$I : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_{dsc})$$

(d) Show that $I : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_{dsc})$ is sequentially continuous.

5. **(Optional).** Consider $\mathbb{R}$ with the cocountable topology (Problem 4), and let $A \subseteq \mathbb{R}$.

(a) What is $\overline{A}$ if A is (i) countable, or (ii) uncountable?

(b) Let A be the interval $(0, 1)$, so $\overline{A} = \mathbb{R}$. Show that, for any $x \in \overline{A} \setminus A$, there is **no** sequence of points in A that converge to x.

6. **(Optional).**

Definition (First countable spaces). A topological space $(X, \mathcal{T})$ is called *first countable* if each point $x \in X$ has a *countable neighbourhood basis.* This means, for each $x \in X$, there is a countable collection $\{N_i\}_{i\in\mathbb{N}}$ of neighbourhoods of x with the property that, if N is any neighbourhood of x, then there is some i such that $N_i \subseteq N$.

(a) Let $f : X \to Y$ be a function of topological spaces. Show that, if X is first countable, then f is continuous if and only if it is sequentially continuous.

(b) Let X be a first countable space, and let $A \subseteq X$. Show that, given any $x \in \overline{A}$, there is some sequence of points in A that converges to x. In other words, $\mathrm{sc}(A) = \overline{A}$ for any subset A of a first countable space.