1 The interior, the closure, and the boundary of a set

Definition 1.1. (Interior of a set). Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. A point $a \in A$ is called an *interior point* of A if there exists an open neighbourhood U of a with $U \subseteq A$. The *interior of* A, denoted $\text{Int}(A)$ or $\mathring{A}$, is defined to be the set

$$\text{Int}(A) = \{a \in A \mid a \text{ is an interior point of } A\}.$$

We will see that $\text{Int}(A)$ is an open set, and it is in a sense the "largest" open subset of A.

Definition 1.2. (Closure of a set). Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then the *closure of* A, denoted $\overline{A}$, is defined to be the set

$$\overline{A} = \{x \in X \mid \text{every open neighbourhood } U \text{ of } x \text{ contains a point of } A\}.$$

We will see that $\overline{A}$ is a closed set, and that in a sense it is the "smallest" closed set containing A. The concept of closure is closely related to the concept of accumulation points.

Definition 1.3. (Accumulation points of a set). Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then the *accumulation points of* A, denoted A', are defined to be the elements

$$A' = \{x \in X \mid \text{ every open neighbourhood } U \text{ of } x \text{ contains a point of } A \textit{ distinct from } x\}.$$

Definition 1.4. (Boundary of a set). Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then the *boundary* of A, denoted ∂A, is the set

$$\partial A = \{x \in X \mid \text{ every open neighbourhood } U \text{ of } x \text{ contains a point in } A \text{ and a point in } X \setminus A\}.$$

Theorem 1.5. *Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then every point of X in contained in precisely one of the following sets.*

(i) interior points of A;

(ii) interior points of $X \setminus A$;

(iii) points in the (common) boundary of A and $X \setminus A$.

Example 1.6. For the following topological spaces and subsets A, find $\text{Int}(A)$, $\overline{A}$, A', and ∂A.

(i) Let $X = \mathbb{R}$ with the standard Euclidean topology. Let $A = [0, 1)$.

(ii) Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$. Let $A = \{b, c\}$.

(iii) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$. Let $A = \{0, 1\}$.

We will prove the following results in the Worksheet Problems and on our Homework.

Theorem 1.7. *Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.*

(i) $\text{Int}(A) \subseteq A$

(ii) If $A \subseteq B$ then $\text{Int}(A) \subseteq \text{Int}(B)$

(iii) $\text{Int}(A) = \bigcup_{U \text{ open in } X,\ U \subseteq A} U$.

(iv) $\text{Int}(A)$ *is open in* X

(v) A is open if and only if $A = \text{Int}(A)$

(vi) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$

(vii) $\text{Int}(A)$ *is the largest open subset of A, in the following sense: If $U \subseteq A$ is any open subset of A, then $U \subseteq \text{Int}(A)$.*

Theorem 1.8. *Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.*

(i) $A \subseteq \overline{A}$

(ii) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$

(iii) $\overline{A} = \bigcap_{C \text{ closed in } X,\ A \subseteq C} C$.

(v) $\overline{A}$ *is closed in* X

(iii) A is closed if and only if $A = \overline{A}$

(iv) $\overline{\overline{A}} = \overline{A}$

(vi) $\overline{A}$ *is the smallest closed set containing A, in the following sense: If $A \subseteq C$ for some closed set C, then $\overline{A} \subseteq C$.*

Theorem 1.9. *Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.*

(i) $\partial A = \partial(X \setminus A)$

(ii) $\partial A = \overline{A} \setminus \text{Int}(A)$

(iii) $\text{Int}(A) = \overline{A} \setminus \partial A$

(iv) $\partial A = \overline{A} \cap (\overline{X \setminus A})$

(v) ∂A *is closed in* X.

In-class Exercises

1. Prove Theorem 1.7.
2. **(Optional).** Let A be a subset of a topological space X. Consider the sets

$$\text{Int}(X \setminus A) \qquad X \setminus \text{Int}(A) \qquad \overline{X \setminus A} \qquad X \setminus \overline{A}$$

   Determine which of these sets are necessarily equal or necessarily subsets of one another. Give counterexamples to show where equality or containment fails.

3. **(Optional).** Let A_i, $i \in I$, be a collection of subsets of a topological space. For each of the following statements, either prove the statement, or construct a counterexample.

   (a) $\text{Int}\left(\bigcup_{i \in I} A_i\right) \subseteq \bigcup_{i \in I} \text{Int}(A_i)$

   (b) $\text{Int}\left(\bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} \text{Int} A_i$

   (c) $\text{Int}\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} \text{Int}(A_i)$

   (d) $\text{Int}\left(\bigcap_{i \in I} A_i\right) \supseteq \bigcap_{i \in I} \text{Int}(A_i)$

   (e) $\overline{\bigcup_{i \in I} A_i} \subseteq \bigcup_{i \in I} \overline{A_i}$ (f) $\overline{\bigcup_{i \in I} A_i} \supseteq \bigcup_{i \in I} \overline{A_i}$ (g) $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$ (h) $\overline{\bigcap_{i \in I} A_i} \supseteq \bigcap_{i \in I} \overline{A_i}$

4. **(Optional).** Prove the following equivalent definition of continuity.

   Theorem (An equivalent definition of continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f : X \to Y$ is continuous if and only if

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \qquad \text{for every subset } A \subseteq X.$$