Name: _____ Score (Out of 6 points):

1. (6 points) Let X be a topological space with the T_1 property ("points are closed"). Let $S \subseteq X$ be a subset, and $x \in X$ an accumulation point of S. Show that every open neighbourhood of x contains infinitely many points of S.

Solution. Recall that, by definition, a space X has the T_1 property if for all $x, y \in X$, there exists an open neighbourhood U of y that does not contain x. We proved that this is equivalent to the condition that, for all elements $x \in X$, the singleton set $\{x\}$ is closed in X.

Proof. Let X be a topological space with the T_1 property. Let $S \subseteq X$ be a subset, and $x \in X$ an accumulation point of S. This means, by definition, that every open neighbourhood of x contains a point of S distinct from x.

Let W be an open neighbourhood of x. Our goal is to show that W contains infinitely many points of S. We will construct a sequence of distinct points $s_1, s_2, s_3, \ldots$ in $W \cap S$.

We proceed by induction. Since x is an accumulation point of S, the open subset W must contain a point s_1 in S that is distinct from x. Now suppose by induction that we have identified n distinct points $s_1, s_2, \ldots, s_n$ of S in W that are distinct from x.

Since each singleton set $\{s_1\}$ is closed in X, the finite union $\{s_1, \ldots, s_n\}$ is necessarily closed in X. Let $U = X \setminus \{s_1, \ldots, s_n\}$ be its (open) complement.

The intersection $U \cap W$ is an open neighbourhood of x, and therefore must contain a point s_{n+1} in S distinct from x. The point s_{n+1} is contained in $(W \cap U) \subseteq W$, and since s_{n+1} is contained in U it must be distinct from $s_1, s_2, \ldots, s_n$. This completes the inductive step, and the proof.