Name: _____ Score (Out of 5 points):

1. (5 points) For each of the following functions $f : X \to Y$ of topological spaces: If the function is continuous, write "`continuous`". Otherwise, state an example of an open subset of Y whose preimage is not open, or a closed subset of Y whose preimage is not closed. **No further justification needed.**

   Note: if the function is not continuous, you may receive partial credit for writing "`not continuous`" without writing a specific subset.

   (a) $X = (\mathbb{R}, \text{Euclidean})$, $Y = (\mathbb{R}, \text{cofinite})$,

   $$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto \sin(x) \end{aligned}$$

   Continuous.
   Hint: Given an open subset U of Y, the set U is also open in the Euclidean topology (the Euclidean topology is *finer* than the cofinite topology). Since $f(x) = \sin(x)$ is continuous as a function from $(\mathbb{R}, \text{Euclidean})$ to $(\mathbb{R}, \text{Euclidean})$, the preimage of U must be open in $X = (\mathbb{R}, \text{Euclidean})$.

   (b) $X = (\mathbb{R}, \text{cofinite})$, $Y = (\mathbb{R}, \text{Euclidean})$,

   $$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto \sin(x) \end{aligned}$$

   Not continuous.
   The subset $\{0\}$ is closed in the Euclidean topology, but its preimage $\{0, \pi, -\pi, 2\pi, -2\pi, \dots\}$ is an infinite proper subset of $\mathbb{R}$ and therefore not closed in the cofinite topology.

   (c) $X = (\mathbb{R} \setminus \{0\}, \text{indiscrete})$, $Y = \Big(\mathbb{R}, \{U \mid 0 \in U\} \cup \{\varnothing\}\Big)$,

   $$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto x \end{aligned}$$

   Not continuous.
   For example, the subset $\{0, 1\}$ is open in the Y, but its preimage $\{1\}$ is not open in the indiscrete topology on $\mathbb{R} \setminus \{0\}$.

(d) $X = \Big(\{0,1\}, \Big\{\varnothing, \{1\}, \{0,1\}\Big\}\Big)$, $Y = \Big(\{a,b,c\}, \Big\{\varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}\Big\}\Big)$,

$$\begin{aligned} f : X &\longrightarrow Y \\ 0 &\longmapsto a \\ 1 &\longmapsto c \end{aligned}$$

Not continuous. The preimage of $\{a,b\}$ is $\{0\}$, which is not open.

(e) $X = ([0,1], \text{Euclidean})$, $Y = \Big(\{a,b,c\}, \Big\{\varnothing, \{a\}, \{a,b\}, \{a,b,c\}\Big\}\Big)$,

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto \begin{cases} a, & x \in \left[0, \frac{1}{2}\right) \\ b, & x \in \left[\frac{1}{2}, 1\right) \\ c, & x = 1 \end{cases} \end{aligned}$$

Continuous.
Hint: We can check directly that the preimage of every open subset is open in $[0,1]$,

$$\begin{aligned} f^{-1}(\varnothing) &= \varnothing \\ f^{-1}(\{a\}) &= \left[0, \frac{1}{2}\right) \\ f^{-1}(\{a,b\}) &= [0,1) \\ f^{-1}(\{a,b,c\}) &= [0,1] \end{aligned}$$