Name: _____ Score (Out of 4 points):

1. (4 points) For each of the following: If the statement is true, write "`True`". Otherwise, state a specific counterexample. **No further justification needed.**

   If the statement is not true, you may receive partial credit for writing "`False`" without a counterexample.

   Let $f : X \to Y$ be a continuous function of metric spaces.

   (a) If $C \subseteq Y$ is closed, then $f^{-1}(C) \subseteq X$ is closed.

   True. *Hint:* See Quiz #5.

   (b) If $U \subseteq Y$ is unbounded, then $f^{-1}(U) \subseteq X$ is unbounded.

   Counterexample. Let

   $$f : ((0,\infty), \text{Euclidean}) \longrightarrow ((0,\infty), \text{Euclidean}))$$
   $$x \longmapsto \frac{1}{x}$$

   Then $U = (1,\infty)$ is unbounded, but its preimage $f^{-1}(U) = (0,1)$ is bounded.

   Alternate ounterexample. Consider

   $$f : (\mathbb{N}, \text{discrete metric}) \longrightarrow (\mathbb{N}, \text{Euclidean})$$
   $$x \longmapsto x$$

   Then $U = \mathbb{N}$ is unbounded with respect to the Euclidean metric, but its preimage $f^{-1}(U) = \mathbb{N}$ is bounded with respect to the discrete metric.

   (c) If X is complete, then the metric subspace $f(X)$ of Y is complete.

   Counterexample. Consider

   $$f : (\mathbb{R}, \text{Euclidean}) \longrightarrow \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{Euclidean}\right)$$
   $$x \longmapsto \arctan(x)$$

   Then $X = \mathbb{R}$ is complete, but $f(X) = (-\frac{\pi}{2}, \frac{\pi}{2})$ is not complete with respect to the Euclidean metric (consider a sequence of real numbers in $(-\frac{\pi}{2}, \frac{\pi}{2})$ converging to $\frac{\pi}{2}$).

   (d) If $C \subseteq Y$ has no accumulation points, then $f^{-1}(C) \subseteq X$ has no accumulation points.

   Counterexample. Consider the constant function

   $$f : (\mathbb{R}, \text{Euclidean}) \longrightarrow (\mathbb{R}, \text{Euclidean})$$
   $$x \longmapsto 0$$

   The subset $\{0\} \subseteq \mathbb{R}$ has no accumulation points, but its preimage $f^{-1}(\{0\}) = \mathbb{R}$ has many.