Name: _____ Score (Out of 4 points):

1. (4 points) Let (X, d) be a metric space, and let $A, B \subseteq X$. Show that, if A and B are sequentially compact, then so is $A \cap B$.

Solution: Suppose that A is sequentially compact. This means that every sequence in A has a subsequence that converges to a point of A. Assume similarly that B is sequentially compact.

Our goal is to show that $A \cap B$ is sequentially compact. So let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in $A \cap B$. Since A is sequentially compact, there is some subsequence $(a_{n_i})_{i\in\mathbb{N}}$ converging to a point $a \in A$. But, since B is sequentially compact, we have proved (Worksheet #8 Problem 1) that B is closed. Since $(a_{n_i})_{i\in\mathbb{N}}$ is a convergent sequence in the closed set B, by Homework #4 Problem 3 its limit $a = \lim_{i\in\mathbb{N}} a_{n_i}$ must be in B.

So $a \in A \cap B$. Thus we have found a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ converging to a point of $A \cap B$. We conclude that $A \cap B$ is sequentially compact.

Alternate Solution: As above, our goal is to show that $A \cap B$ is sequentially compact. So let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in $A \cap B$. Since A is sequentially compact, there is some subsequence $(a_{n_i})_{i\in\mathbb{N}}$ converging to a point $a \in A$. Since B is sequentially compact, by definition, the subsequence $(a_{n_i})_{i\in\mathbb{N}}$ has a (sub)subsequence that converges to a point of B. But by Worksheet #8 Proposition 1.2, every subsequence of $(a_{n_i})_{i\in\mathbb{N}}$ converges to its limit a. This implies that that $a \in B$.

So again $a \in A \cap B$. Thus we have found a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ converging to a point of $A \cap B$. We conclude that $A \cap B$ is sequentially compact.