Name: _____ Score (Out of 7 points):

1. (7 points) Prove the following theorem, which gives a new equivalent definition of continuity.

   Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following property: for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

   Solution. We first prove a lemma.

   Lemma. Let $A \subseteq Y$ be any subset. Then

   $$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$$

   In other words, the preimage of the complement of A is the complement of its preimage.

   Proof of lemma. Given a subset $A \subseteq Y$, we proved on Homework #3 Assignment Problem 1(c) that

   $$f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A).$$

   The result then follows from the observation that the preimage $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\}$ is equal to X, since (by definition) the function f maps all points of X to Y. □

   We will now use the lemma to prove the theorem.

   First suppose that $f : X \to Y$ is a function with the property that $f^{-1}(C)$ is closed in X for every closed subset C of Y. We will show that f is continuous. By Worksheet #5 Problem 1, it suffices to show that $f^{-1}(U)$ is open for every open subset U of Y.

   Let U be an open subset of Y. Then $Y \setminus U$ is closed, by definition of closed. It follows from our assumption on f that the preimage $f^{-1}(Y \setminus U)$ is closed in X. But by the lemma, $f^{-1}(Y \setminus U)$ is the complement in X of $f^{-1}(U)$. This means (again by definition of closed) that $f^{-1}(U)$ is open in X, as desired.

   Conversely, suppose that f is continuous. Again by Worksheet #5 Problem 1, this implies that the preimage $f^{-1}(U)$ is open in X for every open subset U of Y. Suppose that $C \subseteq Y$ is any closed subset; our goal is to show that $f^{-1}(C)$ is closed in X.

   By definition of closed, the complement $Y \setminus C$ is open in Y. By continuity, then, its preimage $f^{-1}(Y \setminus C)$ is open in X. But by the lemma, $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$. The complement $f^{-1}(C)$ of the open subset $f^{-1}(Y \setminus C)$ is by definition closed. This concludes the proof.