Name: _____ Score (Out of 7 points):

1. (7 points) For each of the following statements: either give a proof of the statement, or state a specific counterexample. If the statement is false, you can simply state the counterexample without further justification.

   (a) Let (X, d) be a metric space, and $A, B \subseteq X$ be subsets. If a point $x \in X$ is an accumulation point of A and is an accumulation point of B, then x must be an accumulation point of the intersection $A \cap B$.

   False.

   Counterexample: Consider the case $X = \mathbb{R}$ with the usual Euclidean metric. Consider the open intervals $A = (0, 1)$ and $B = (-1, 0)$. Then 0 is an accumulation point of both A and B, but the intersection $A \cap B$ is empty and has no accumulation points.

   (b) Let (X, d) be a metric space, and $A, B \subseteq X$ be subsets. If $x \in X$ is an accumulation point of the union $A \cup B$, then x must be an accumulation point of at least one of A and B.

   True.

   Proof: We will prove the contrapositive: if x is not an accumulation point of A or B, then x cannot be an accumulation point of the union $A \cup B$.
   By definition, the statement that x is not an accumulation point of A means that there exists some radius $R_A > 0$ such that the punctured ball $B_{R_A}(x) \setminus \{x\}$ is disjoint from A. Similarly, there exists some radius $R_B > 0$ such that the punctured ball $B_{R_B}(x) \setminus \{x\}$ is disjoint from B.
   Let $R = \min\{R_A, R_B\}$, and consider the punctured ball $B_R(x) \setminus \{x\}$. Since $R \leq R_A$, there is containment

   $$B_R(x) \setminus \{x\} \subseteq B_{R_A}(x) \setminus \{x\}.$$

   But this means that $B_R(x) \setminus \{x\}$ must be disjoint from A. Similarly

   $$B_R(x) \setminus \{x\} \subseteq B_{R_B}(x) \setminus \{x\},$$

   so $B_R(x) \setminus \{x\}$ must be disjoint from B. But this means that $B_R(x) \setminus \{x\}$ does not contain any points of $A \cup B$. We conclude that x is not an accumulation point of $A \cup B$ as claimed.