Name: _____ Score (Out of 6 points):

1. (6 points) Prove the following theorem.

   Theorem. Let X be a compact topological space. If A is an infinite subset of X, then A has an accumulation point in X.

   Solution. Let X be a compact topological space. We will prove the contrapositive: if a subset A has no accumulation points in X, then A must be finite.

   So suppose some set $A \subseteq X$ has no accumulation points.

   Given any point $x \in X \setminus A$, since x is not an accumulation point of A, the point x must have an open neighbourhood that contains no points of A. This means that x is an interior point of $X \setminus A$, and we deduce that $X \setminus A$ is open.

   Given any point $a \in A$, the point a is not an accumulation point of A. This means that there exists some open neighbourhood U_a of a in X such that $U_a \cap A = \{a\}$.

   Observe that the sets $\mathcal{U} = \{U_a \mid a \in A\} \cup \{X \setminus A\}$ form an open cover of X. Since X is compact, this open cover must have a finite subcover.

   But any point $a \in A$ is only contained in the single open subset U_a in the open cover $\mathcal{U}$, which means any subcover of $\mathcal{U}$ must contain all the subsets $\{U_a \mid a \in A\}$. We conclude that A is finite, as claimed.

   Alternate Solution. Let X be a topological space. We will prove that, if A is an infinite subset of X with no accumulation points, then X is not compact.

   First we claim that, for any $a \in A$, the subset $V_a := (X \setminus A) \cup \{a\}$ is open in X. We will check that every point is an interior point. Let $x \in X \setminus A$. Since x is not an accumulation point of A, it must have some neighbourhood U_x contained in $(X \setminus A) \subseteq V_a$, hence X is an interior point of V_a. Now consider the point a. Since a is not an an accumulation point of A, it must have some neighbourhood U that contains no points of A distinct from a. Hence $U \subseteq V_a$, and we conclude that a is also an interior point of V_a. This implies that V_a is open.

   Now, the infinite collection of subsets $\mathcal{V} = \{V_a\}_{a \in A}$ is an open cover of X. Since the point $a \in A$ is only contained in the single element V_a of $\mathcal{V}$, the cover $\mathcal{V}$ does not have an proper subcovers. This implies that X is not compact, as claimed.