Midterm Exam Solutions

Math 490
21 October 2025
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Name: _____

Instructions: This exam has 4 questions for a total of 35 points.

The exam is closed-book. No books, cell phones, calculators, or other devices are permitted.

Each student may bring in one double-sided standard-size (8.5 in $\times$ 11 in) sheet of notes, which they must prepare themselves. Notes may be handwritten or typed with font size at least 12.

Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	14	
2	8	
3	6	
4	7	
Total:	35	

1. (14 points) For each of the following: If the statement is true, write "`True`". Otherwise, state a specific counterexample. **No further justification needed.**

   If the statement is not true, you may receive partial credit for writing "`False`" without a counterexample.

   (a) Let X be a metric space. If $U \subseteq X$ is an open subset, then every point of U is an accumulation point of U.

   Counterexample. Let $X = (\mathbb{N}, \text{Euclidean})$. Then $U = \{1\}$ is open, but it has no accumulation points. (In fact, X has the discrete topology. Every subset is open, and no subset has accumulation points).

   (b) Let X be a metric space and $S \subseteq X$. The set S' of accumulation points of S is closed in X.

   True. *Hint:* See Homework #2 Problem .

   (c) Let X be a metric space and $S \subseteq X$. Then the set of isolated points of S is closed in X.

   Counterexample. Let $X = (\mathbb{R}, \text{Euclidean})$, and let $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Then S is equal to its set of isolated point, but it is not a closed subset, since its closure contains $0 \notin S$.

   (d) If a metric space X is bounded, then for any $\epsilon > 0$, the space X is the union of a finite number of open balls of radius ϵ.

   Counterexample. Let $X = \mathbb{R}$ with the discrete metric, and $\epsilon = \frac{1}{2}$. The space X is bounded since $d(x, y) \leq 1$ for all $x, y \in X$, but for any $x \in X$, $B_{\frac{1}{2}}(x) = \{x\}$, so the infinite space X is not the union of a finite number of balls of the form $B_{\frac{1}{2}}(x)$.

   (e) Let X be a metric space. If, for some $\epsilon > 0$, the space X is the union of a finite number of open balls of radius ϵ, then X is bounded.

   True.

   Hint: Suppose there are elements $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_\epsilon(x_i)$. Then the triangle inequality implies that X has diameter at most $2\epsilon + \max_{i,j=1,\ldots,n} d(x_i, x_j)$.

   (f) Let X be a metric space with at least two elements. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A that is **not** Cauchy.

   True. *Hint:* If $x, y \in X$, consider the sequence $x, y, x, y, x, y, x, y, \ldots$.

(g) Let X and Y be nonempty metric spaces with the **discrete** metrics. Then the product metric $d_{X\times Y}$ on $X \times Y$ is topologically equivalent to the discrete metric.

True.

Hint: We must verify that $d_{X\times Y}$ induces the discrete topology on $X \times Y$. It suffices to check that points are open. But for any $(x, y) \in X \times Y$, the set $\{x\}$ is open in X and $\{y\}$ is open in Y, hence $\{(x, y)\} = \{x\} \times \{y\}$ is open in $X \times Y$ by Homework #2 Problem 6(b).

(h) If A and B are dense subsets of a metric space X, then their intersection $A \cap B$ is dense in X.

Counterexample. Consider $X = \mathbb{R}$ with the Euclidean metric. Let A be the rationals and B the irrationals. Then both A and B are dense, but $A \cap B = \varnothing$ is not dense.

(i) Let $f : X \to Y$ be a continuous function of metric spaces. If x is an accumulation point of a set A, then $f(x)$ is an accumulation point of $f(A)$.

Counterexample. Let $A = X = \mathbb{R}$ with the Euclidean metric, and let $x = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function mapping to 1. Then $f(A) = \{1\}$ has no accumulation points.

(j) Let $f : X \to Y$ be a continuous **injective** function of metric spaces. If x is an accumulation point of a set A, then $f(x)$ is an accumulation point of $f(A)$.

True.

Hint: Let U be an open neighbourhood of $f(x)$ in Y. Then $f^{-1}(U)$ is an open neighbourhood of x in X. It contains a point $y \in A$ distinct from x. Then $f(y) \in U \cap f(A)$. Since f is injective, $f(y)$ is distinct from $f(x)$.

(k) Let X be a topological space and $S \subseteq X$ a subspace. If $U \subseteq S$ is open in the subspace topology, then U is open in X.

Counterexample. Let $S = [0, 1]$ in $\mathbb{R}$ with the Euclidean topology. Then $U = S = [0, 1]$ is open in S but not in $\mathbb{R}$.

(l) Let X be a topological space, and $A \subseteq X$ a subset. Then $\overline{A} = A \cup A'$.

True.

Hint: Suppose that x is an element of $\overline{A}$ that is not in A. Since $x \in \overline{A}$, any neighbourhood U of x must contain a point $a_U \in A$. Since $x \notin A$, the point a_U must be distinct from x. Thus $x \in A'$. Conversely, both A and A' are contained in $\overline{A}$.

(m) Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. Let $A \subseteq X$ be a subset. Then $\mathrm{Int}(A) = \bigcup_{\substack{B \in \mathcal{B} \\ \text{with } B \subseteq A}} B$.

True.

Hint: Any such B is contained in $\mathrm{Int}(A)$ by Worksheet #12 Theorem 1.7 (vii). Conversely, by definition of interior, any point a of $\mathrm{Int}(A)$ has an open neighbourhood U with $U \subseteq A$. Hence (by definition of a basis) there exists some $B \in \mathcal{B}$ with $a \in B \subseteq U \subseteq A$.

(n) Let $f : X \to Y$ be a continuous function of metric spaces, and $A \subseteq Y$. Then $f^{-1}(\mathrm{Int}(A)) \subseteq \mathrm{Int}\Big(f^{-1}(A)\Big)$.

True.

Hint: $\mathrm{Int}(A)$ is open by Worksheet #12 Theorem 1.7 (iv), so $f^{-1}(\mathrm{Int}(A))$ is open since f is continuous. Since $\mathrm{Int}(A) \subseteq A$, $f^{-1}(\mathrm{Int}(A)) \subseteq f^{-1}(A)$. An open subset $f^{-1}(\mathrm{Int}(A))$ contained in $f^{-1}(A)$ is contained in $\mathrm{Int}\Big(f^{-1}(A)\Big)$ by Worksheet #12 Theorem 1.7 (vii).

2. (a) (4 points) Let U_i, $i \in I$, be a (possibly infinite) collection of subsets of a metric space X. Consider the following statement.

If the subset U_i is _____ *for all $i \in I$, then so is their union* $\bigcup_{i \in I} U_i$.

Circle all properties that truthfully fill in the blank. **No justification needed.**

(open) closed sequentially compact non-sequentially-compact

complete non-complete bounded (unbounded)

(By "U is complete" we mean U is a complete metric space with respect to the subspace metric)

(b) (4 points) Let $U_1, \ldots, U_n$ be a **finite** collection subsets of a metric space X. Consider the following statement.

If the subset U_i is _____ *for all i, then so is their union* $\bigcup_{i=1}^{n} U_i$.

Circle all properties that truthfully fill in the blank. **No justification needed.**

(open) (closed) (sequentially compact) non-sequentially-compact

(complete) non-complete (bounded) (unbounded)

3. (6 points) Let (X, d) be a metric space. Prove that, if X has two distinct accumulation points x_0 and y_0, then X contains a subset that is neither open nor closed.

Solution. Let $R = d(x_0, y_0)$. Let

$$S = \left(B_{\frac{R}{2}}(x_0) \setminus \{x_0\}\right) \cup \{y_0\}.$$

We claim that S is neither open nor closed. We first prove a lemma:

> **Lemma.** The open balls $B_{\frac{R}{2}}(x_0)$ and $B_{\frac{R}{2}}(y_0)$ are disjoint. In particular $B_{\frac{R}{2}}(y_0) \setminus \{y_0\}$ is disjoint from S.
>
> *Proof of lemma.* Suppose that there were some point $z \in B_{\frac{R}{2}}(x_0) \cap B_{\frac{R}{2}}(y_0)$. Then by the triangle inequality,
>
> $$d(x_0, y_0) \leq d(x_0, z) + d(z, y_0) < \frac{R}{2} + \frac{R}{2} = R,$$
>
> which contradicts our definition of R as the value $R = d(x_0, y_0)$. $\square$

We first show that S is not open, by showing that y_0 is not an interior point. Fix $\epsilon > 0$ and consider $B_\epsilon(y_0)$. Let $r = \min\left(\epsilon, \frac{R}{2}\right)$. Since y_0 is an accumulation point of X, there exists some point $y \in B_r(y_0)$ distinct from y_0. Because $B_r(y_0) \subseteq B_{\frac{R}{2}}(y_0)$ is disjoint from $B_{\frac{R}{2}}(x_0)$ by the lemma, we know that $y \notin S$. But $y \in B_r(y_0) \subseteq B_\epsilon(y_0)$, so we deduce that $B_\epsilon(y_0)$ is not contained in S. Thus y_0 is not an interior point of S.

We next show that S is not closed, that is, its complement $X \setminus S$ is not open. We will show that x_0 is not an interior of $X \setminus S$. Again fix $\epsilon > 0$ and consider $B_\epsilon(x_0)$. Again let $r = \min\left(\epsilon, \frac{R}{2}\right)$. Since x_0 is an accumulation point of X, the set $B_r(x_0) \setminus \{x_0\}$ contains a point $x \in X$. But since

$$B_r(x_0) \setminus \{x_0\} \quad \subseteq \quad B_{\frac{R}{2}}(x_0) \setminus \{x_0\} \quad \subseteq \quad S,$$

we infer that $x \in S$. Since $x \in B_r(x_0) \subseteq B_\epsilon(x_0)$, we see that $B_\epsilon(x_0)$ is not disjoint from S. We conclude that x_0 is not an interior of $X \setminus S$, and therefore S is not closed.

Alternate approach. Again let $S = \left(B_{\frac{R}{2}}(x_0) \setminus \{x_0\}\right) \cup \{y_0\}$.

Construct a sequence of points in $B_{\frac{R}{2}}(x_0) \setminus \{x_0\}$ converging to x_0, and a sequence of points in $B_{\frac{R}{2}}(y_0) \setminus \{y_0\}$ converging to y_0. Then by Homework #4 Problem 3 neither S nor its complement is closed.

4. (7 points) Suppose that (X, d_X) and (Y, d_Y) are nonempty, complete metric spaces. Show that their product $X \times Y$, with the product metric $d_{X\times Y}$, is a complete metric space.

Solution.

Suppose that $((x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $X \times Y$. Our goal is show that it converges. We first prove a lemma.

> **Lemma.** Let x and x' be any points of X. Then, for any $y, y' \in Y$,
>
> $$d_X(x, x') \le d_{X\times Y}((x, y), (x', y')).$$

Similarly $d_Y(y, y') \le d_{X\times Y}((x, y), (x', y'))$.

Proof of lemma.

$$\begin{aligned} d_{X\times Y}((x, y), (x', y')) &= \sqrt{d_X(x, x')^2 + d_Y(y, y')^2} \\ &\ge \sqrt{d_X(x, x')^2 + 0} \quad \text{since the function } \sqrt{} \text{ is increasing, and } d_Y(y, y')^2 \ge 0 \\ &= d_X(x, x'). \end{aligned}$$

We next claim that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Fix $\epsilon > 0$. Since the sequence $((x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy, there exists some N so that $d_{X\times Y}((x_n, y_n), (x_m, y_m)) < \epsilon$ for all $m, n \ge N$. But then by the lemma,

$$d_X(x_n, x_m) \le d_{X\times Y}((x_n, y_n), (x_m, y_m)) < \epsilon \qquad \text{for all } m, n \ge N.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy as claimed.

But the space X is complete, so this implies that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to some point x_∞ in X. Since the statement of the problem is symmetric in X and Y, the same argument shows that $(y_n)_{n\in\mathbb{N}}$ is Cauchy and therefore converges to a point y_∞ in Y.

We will now verify that the Cauchy sequence $((x_n, y_n))_{n\in\mathbb{N}}$ converges to (x_∞, y_∞). Let W be an open neighbourhood of (x_∞, y_∞). By Homework #2 Problem 6(c), there exists an open neighbourhood U_x of x_∞ in X and an open neighbourhood U_y of y_∞ in Y such that $(x_\infty, y_\infty) \subseteq U_x \times U_y \subseteq W$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x_∞, there exists some $N_x \in \mathbb{N}$ so that $x_n \in U_x$ for all $n \ge N_x$. Similarly there is some $N_y \in \mathbb{N}$ such that $y_n \in U_y$ for all $n \ge N_y$. But then, for all $n \ge N = \max(N_x, N_y)$, we have $(x_n, y_n) \in U_x \times U_y \subseteq W$. We conclude that $((x_n, y_n))_{n\in\mathbb{N}}$ converges to (x_∞, y_∞).