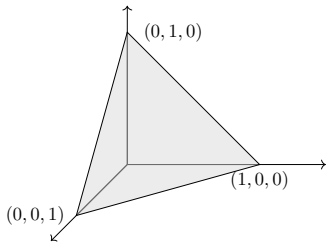
1 (Generalized) simplicial complexes

Definition I. For $n \geq 1$ we define *the standard n-simplex* Δ^n to be the convex hull of the $(n+1)$ standard basis vectors in $\mathbb{R}^{n+1}$. Concretely,

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} t_i \geq 0 \text{ for all } i, \\ t_0 + t_1 + \cdots + t_n = 1 \end{array} \right\}.$$

We define Δ^0 to be a point.
More generally, an *n-simplex* is a closed n-disk identified homeomorphically with the standard n-simplex.



This parameterization of an n-disk allows us to study linear maps between simplices. For an alternative coordinate system on an n-simplex see Exercise 2.

Definition II. Given a subset $A \subseteq \{0, 1, 2, \ldots, n\}$ of cardinality $(m+1)$ for $m \geq 0$, the subset

$$\{(t_0, t_1, \ldots, t_n) \in \Delta^n \mid t_i = 0 \text{ for all } i \notin A\}$$

is called a *face* or *m-face* of Δ^n. The 0-faces, called the *vertices* of Δ^n, are the standard basis elements $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, 0, \ldots, 0), \ldots$. The 1-faces are called *edges*. The *boundary* of the simplex is the union of its proper faces and is denoted $\partial\Delta^n$.

Proposition III. *For fixed m, n, let f be a map from the vertex set $v_0, v_1, \ldots, v_m$ of Δ^m to the vertex set of Δ^n. Then f extends uniquely to a linear map*

$$\begin{aligned} f : \Delta^m &\longrightarrow \Delta^n \\ t_0 v_0 + t_1 v_1 + \cdots + t_m v_m &\longmapsto t_0 f(v_0) + t_1 f(v_1) + \cdots + t_m f(v_m) \end{aligned}$$

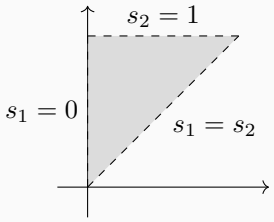
The m-faces of a n-simplex are linearly isomorphic to m-simplices by a map of this form.

Exercise 1. (Bonus) Verify that Δ^1 is a closed line segment, Δ^2 is a triangle, and Δ^3 is a tetrahedron. Can you describe Δ^4?

Exercise 2. (Bonus)

Some sources define the standard n-simplex as the subset of $\mathbb{R}^n$ (not $\mathbb{R}^{n+1}$) given by

$$\Delta^n_* = \{(s_1, s_2, \ldots s_n) \mid 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1\}$$



Unlike Definition I, this construction is not symmetric under permutations of the coordinates.

(a) Verify that the map $s_i = t_0 + t_1 + \cdots + t_i$ defines a linear isomorphism between the two definitions of an n-simplex.

(b) Describe the m-dimensional faces of Δ^n_*, and verify that they are linearly isomorphic to m-simplices.

(c) Describe how a map between vertex sets of two simplices extends to a linear map of simplices with this parameterization.

Exercise 3. (Bonus) Verify Proposition III.

Exercise 4. (Bonus) Check that the Cartesian product of simplices does not in general inherit a natural simplex structure. Can you devise a convention for subdividing the product $\Delta^n \times \Delta^m$ into a union of simplices?

Definition IV. A *generalized simplicial complex* is a CW complex X for which the n-disks are parameterized as n-simplices Δ^n_α, and the attaching maps $\phi^n_\alpha : \partial\Delta^n_\alpha \to X^{(n-1)}$ restrict to a linear isomorphism from each $(n-1)$-face of Δ^n_α to an $(n-1)$-simplex Δ^{n-1}_β (in the sense of Proposition III) composed with a characteristic map $\Delta^{n-1}_\beta \to X^{(n-1)}$.

Our definition of a generalized simplicial complex is not standard. This notion is more general than the concept of a Δ-*complex* (also called a *semisimplicial set*, *triangulated space*, or *trisp*). In a Δ-complex, each simplex has a distinguished orientation, and the attaching maps are required to respect orientation.

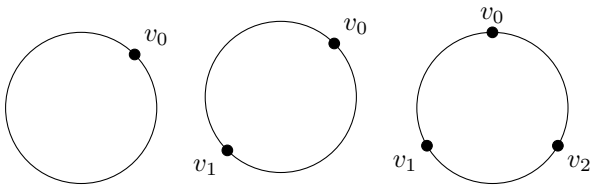
Definition V. A *simplicial map* of generalized simplicial complexes $f : X \to Y$ is a map that restricts on each simplex of X to a linear map of simplices in the sense of Proposition III.

By Worksheet #2 Corollary V, simplicial maps are continuous.

Definition VI. A generalized simplicial complex is a *simplicial complex* if it satisfies the additional conditions,

- Each n-simplex has $(n+1)$ distinct vertices.
- Any $(n+1)$ vertices span at most one n-simplex.

Example VII. The following figure shows three generalized simplicial structures on the circle S^1. Only the third is a simplicial complex structure.



An (open) cell in a generalized simplicial complex is sometimes called an *open simplex*. A closed n-cell in a simplicial complex is an embedded n-simplex, though in a generalized simplicial complex a closed n-cell may be a quotient of a n-simplex with one or more of its faces glued together via linear isomorphisms.

Exercise 5. (Bonus) Show that an n-simplex has the structure of a simplicial complex, in which the m-skeleton is the union of the m-faces.

Exercise 6. (Bonus) Show that for a simplicial complex—but not necessarily for a generalized simplicial complex—each n-simplex Δ^n_α embeds in the complex.

Exercise 7. (Bonus) Find a simplicial structure on the torus. Any simplicial structure includes at least fourteen 2-simplices.

Exercise 8. (Bonus) Some sources define a simplicial complex as a union of simplices $\mathcal{K}$ subject to the conditions:

- Every face of a simplex in $\mathcal{K}$ is a simplex in $\mathcal{K}$.
- The intersection of two simplices σ_1, σ_2 in $\mathcal{K}$ must either be empty or equal a common face of both σ_1 and σ_2.

Reconcile this definition with our definition.