1 Homeomorphisms of metric spaces

Definition 1.1. (Homeomorphism). Let X and Y be metric spaces. Then a map $f : X \to Y$ is a *homeomorphism* if

- f is continuous;
- f has an inverse f^{-1};
- f^{-1} is continuous.

The metric space X is called *homeomorphic* to Y if there exists a homeomorphism $f : X \to Y$.

Example 1.2. Consider the function

$$f : (\mathbb{R},\ \text{discrete metric}) \to (\mathbb{R},\ \text{Euclidean metric})$$
$$f(x) = x$$

Prove that f is continuous and invertible, but its inverse f^{-1} is not continuous.

Remark: Note the contrast to other mathematical fields, such as linear algebra: if a linear map has an inverse, then the inverse is automatically linear. This exercise shows that this is not true for continuous maps!

In-class Exercises

1. (a) Let X be a metric space. Show that X is homeomorphic to itself.
   (b) Let $f : X \to Y$ a homeomorphism of metric spaces. Explain why $f^{-1} : Y \to X$ is also a homeomorphism. Conclude that X is homeomorphic to Y if and only if Y is homeomorphic to X. (We simply call them "homeomorphic" metric spaces).
   (c) Let X, Y, Z be topological spaces. Show that, if X is homeomorphic to Y, and Y is homeomorphic to Z, then X is homeomorphic to Z.

   This exercise shows that homeomorphism defines an *equivalence relation* on metric spaces.

2. Determine which of the following properties are preserved by homeomorphism. In other words, suppose X and Y are homeomorphic metric spaces. For each of the following properties P, prove or give a counterexample to the statement "X has property P if and only if Y has property P".

   (i) sequentially compact (ii) complete (iii) bounded

   Properties that are preserved by homeomorphisms are called *homeomorphism invariants*, *topological invariants*, or *topological properties* of a metric space.

3. For each of the following pairs of metric spaces, either prove they are homeomorphic, or prove they are not homeomorphic.

   Hint: One way to prove two spaces are homeormorphic is to find a homeomorphism. One way to prove two spaces are not homeomorphic is to study their topological invariants (Problem 2).

   (a) $\mathbb{R}$ and $(0,1)$ (with the Euclidean metric)

   (b) $(0,1)$ and $[0,1]$ (with the Euclidean metric)

   (c) $[0,1] \subseteq \mathbb{R}$ with the Euclidean metric, and $[0,1]$ with the discrete metric.

   (d) $\mathbb{N} \subseteq \mathbb{R}$ with the Euclidean metric, and $\mathbb{N}$ with the discrete metric.

4. **(Optional).** Let X be a set and d_1, d_2 be two metrics on X. Show that d_1 and d_2 are equivalent if and only if the identity map $(X, d_1) \to (X, d_2)$ is a homeomorphism.

5. **(Optional).** Let $f : X \to Y$ be a homeomorphism of metric spaces. Show that

$$\begin{aligned} \{U \subseteq Y \mid U \text{ is open}\} &\longrightarrow \{V \subseteq X \mid V \text{ is open}\} \\ U &\longmapsto f^{-1}(U) \end{aligned}$$

   defines a bijection between the collection of all open subsets of Y, and the collection of all open subsets of X.
   Hint: To show it is a bijection, you could check that the assignment $V \mapsto f(V)$ is its inverse.

6. **(Optional).** Let $f : X \to Y$ be a homeomorphism, and let $A \subseteq X$. Prove that f restricts to a homeomorphism $f|_A : A \to f(A)$ between the subspaces A and $f(A)$.

7. **(Optional).** Prove that two spaces X and Y with the discrete metric are homeomorphic if and only if they have the same cardinality.

8. **(Optional).** Let $X \times Y$ be the product of a metric space X and a nonempty metric space Y, endowed with the product metric. Fix $y_0 \in Y$. Prove that X is homeomorphic to the subspace $X \times \{y_0\} \subseteq X \times Y$.

9. **(Optional).** Let X and Y be metric spaces, and $F : X \to Y$ a continuous function. Recall that the *graph* G of F is the set

$$G = \{(x, f(x)) \mid x \in X\}$$

   viewed as a subspace of $X \times Y$ with the product metric. Prove that G is homeomorphic to X.

10. **(Optional).** All spaces in this problem have the Euclidean metric.

    (a) **(Challenge).** Prove that the unit interval $[0,1]$ is not homeomorphic to the unit circle $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

    (b) **(Challenge).** Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

    (c) **(Even harder challenge).** Prove that for $n \neq m$ the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.