Name: _____ Score (Out of 6 points):

1. (4 points) Let (X, d) be a metric space. Recall that we defined the *distance* between nonempty subsets A, B in X by

$$D(A, B) = \inf_{\substack{a \in A \\ b \in B}} d(a, b).$$

Let $A \subseteq X$. Prove that

$$\overline{A} = \{x \in X \mid D(A, \{x\}) = 0\}.$$

Solution. Recall that we defined the closure $\overline{A}$ of A as

$$\overline{A} = \{x \in X \mid \text{ for every } r > 0 \text{ the ball } B_r(x) \text{ contains a point of } A\}.$$

To prove the result, we first show that $\{x \in X \mid D(A, \{x\}) = 0\} \subseteq \overline{A}$. Suppose that x is a point in X such that $D(A, \{x\}) = 0$. Explicitly, by definition of D, this means

$$\inf_{a \in A} d(a, x) = 0.$$

Fix $r > 0$ and consider the ball $B_r(x)$. Since $\inf_{a \in A} d(a, x) = 0$, there must be some point $a_0 \in A$ with $d(a_0, x) < r$. Hence $a_0 \in B_r(x)$. Since r was arbitrary, we conclude that $x \in \overline{A}$.

We next show that $\overline{A} \subseteq \{x \in X \mid D(A, \{x\}) = 0\}$. Let $x \in \overline{A}$. Consider the set of real numbers

$$\{d(a, x) \mid a \in A\};$$

our goal is to show that its infemum is 0. Since $d(a, x) \geq 0$ for all $a \in A$ by definition of a metric, we know that 0 is a lower bound of this set. To show that 0 is the greatest lower bound, it suffices to show that for any $r > 0$, the number r is not a lower bound. For fixed $r > 0$, the assumption that $x \in \overline{A}$ means that $B_r(x)$ contains a point a_0 in A. Equivalently, there is a point a_0 in A such that $d(a_0, x) < r$. Thus r is not a lower bound, and we conclude that the infemum $\inf_{a \in A} d(a, x)$ is 0 as claimed.