Name: _____ Score (Out of 8 points):

1. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function. Fix $x_0 \in X$.

   (a) (4 points) Suppose that f has the property that, for every neighbourhood U of $f(x_0)$, the preimage $f^{-1}(U)$ is open in X. Prove that f is continuous at the point x_0.

   Solution. Recall that, by definition, the function f is continuous at the point x_0 if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.
   So fix $\varepsilon > 0$. Consider the open ball $U = B_\epsilon(f(x_0))$ about $f(x_0)$. We proved on Worksheet #2 that open balls are open, hence, its preimage $f^{-1}\Big(B_\epsilon(f(x_0))\Big)$ is open in X by assumption on f. Its preimage must contain x_0 (by definition of preimage), so x_0 is an interior point of $f^{-1}\Big(B_\epsilon(f(x_0))\Big)$. This means that there exists some $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$. But then

   $$f\Big(B_\delta(x_0)\Big) \subseteq f\Big(f^{-1}(B_\epsilon(f(x_0)))\Big) \subseteq B_\epsilon(f(x_0)),$$

   where the last containment follows from Homework #1 Assignment Problem 1(b). This concludes the proof.

   (b) (4 points) Suppose that f is continuous at the point x_0. Show that, if $(a_n)_{n\in\mathbb{N}}$ is a sequence of points in X converging to x_0, then $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$.

   Solution. Recall that, to show $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$, we must verify that for all $\varepsilon > 0$, there exists some N such that $f(a_n) \in B_\epsilon(f(x_0))$ for all $n \geq N$.
   Let $\varepsilon > 0$. Consider the ball $B_\epsilon(f(x_0))$. The statement that f is continuous at x_0 means that there exists some $\delta > 0$ so that $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$. The statement that $(a_n)_{n\in\mathbb{N}}$ converges to x_0 means that there exists some N so that $a_n \in B_\delta(x_0)$ for all $n \geq N$. But consequently, for all $n \geq N$,

   $$f(a_n) \in f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0)).$$

   We conclude that the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$, as claimed.