Name: _____ Score (Out of 8 points):

1. (8 points) Let X and Y be metric spaces, and the consider the product $X \times Y$ as a metric space with the product metric.

   Consider a sequence of points $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ in $X \times Y$. Show that this sequence converges if and only if $(x_n)_{n\in\mathbb{N}}$ converges in X, and $(y_n)_{n\in\mathbb{N}}$ converges in Y.

   Solution.

   Recall that we proved that a sequence $(a_n)_{n\in\mathbb{N}}$ in a metric space Z converges if and only if there is a point $a_\infty \in Z$ with the following property: for every open neighbourhood U of a_∞ there exists some integer N such that $a_n \in U$ for all $n \geq N$.

   So first assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges in X, and $(y_n)_{n\in\mathbb{N}}$ converges in Y. Let $x_\infty \in X$ and $y_\infty \in Y$ denote their respective limits. We will show that $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ converges to $(x_\infty, y_\infty) \in X \times Y$. Let U be an open neighbourhood of (x_∞, y_∞) in $X \times Y$. We proved on Homework #2 Problem 4(c) that there exists a neighboorhood U_X of x_∞ in X and U_Y of y_∞ in Y such that $U_X \times U_Y \subseteq U$.

   Since $(x_n)_{n\in\mathbb{N}}$ converges to x_∞ by assumption, there exists some number N_X such that $x_n \in U_X$ for all $n \geq N_X$. Similarly, there exists some N_Y such that $y_n \in U_Y$ for all $n \geq N_Y$. Let $N = \max(N_X, N_Y)$. Then for all $n \geq N$ we have $x_n \in U_X$ and $y_n \in U_Y$, hence for all $n \geq N$ we have $(x_n, y_n) \in U_X \times U_Y \subseteq U$. Thus we have established that $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ converges to (x_∞, y_∞), as desired.

   Conversely, let's assume that $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ converges in $X \times Y$. Let (x_∞, y_∞) be its limit. We will show that $(x_n)_{n\in\mathbb{N}}$ converges in X to x_∞, and $(y_n)_{n\in\mathbb{N}}$ converges in Y to y_∞. Let U be an open neighbourhood of x_∞. Then $U \times Y$ contains the point (x_∞, y_∞), and we proved on Homework #2 Problem 4(b) that the set $U \times Y$ is open in $X \times Y$. Thus, since $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ converges by assumption, there must be some number N such that $(x_n, y_n) \in U \times Y$ for all $n \geq N$. In particular, $x_n \in U$ for all $n \geq N$. This proves that $(x_n)_{n\in\mathbb{N}}$ converges to x_∞, as claimed. The same argument (reversing the roles of X and Y) proves that $(y_n)_{n\in\mathbb{N}}$ converges in Y to y_∞. This conclude the proof.

   Alternate Solution (Converse). Let's assume that $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ converges in $X \times Y$. Let (x_∞, y_∞) denote its limit. We proved on Homework #2 Problem 4(d) that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous. We proved moreover on Homework #3 Problem 3 that continuous functions map convergent sequences to convergent sequences, and the limit of the image is the image of the limit. Thus,

   $$\lim_{\mathbf{n}\to\infty} x_n = \lim_{\mathbf{n}\to\infty} \pi_X(x_n, y_n) = \pi_X\left(\lim_{n\to\infty}(x_n, y_n)\right) = \pi_X(x_\infty, y_\infty) = x_\infty$$

   and similarly $\lim_{\mathbf{n}\to\infty} y_n = y_\infty$. This concludes the proof.