Name: _____ Score (Out of 5 points):

1. Let (X, d) be a metric space, and let $S \subseteq X$ be a subset.

   (a) (1 point) State the definition of an *accumulation point* of S.

   Solution. A point $x \in X$ is called an *accumulation point* of S if for every $r > 0$ the ball $B_r(x)$ centred at x contains at least one point of S distinct from x. (The point x itself may or may not itself be an element of S.)

   (b) (4 points) Suppose that the set S contains all of its accumulation points. Prove that S is a closed subset of X.

   Solution. To prove that S is a closed subset of X, by definition, we must show that its complement $X \setminus S$ is open in X. This means we must show that every point of $X \setminus S$ is an interior point of $X \setminus S$.

   Fix a point $x \in X \setminus S$. Since x is not contained in S, we know (by assumption) that x is not an accumulation point of S. This means that there must be at least one $r_0 > 0$ such that the ball l $B_{r_0}(x)$ does not contain a point of S distinct from x. But the point x itself is not an element of S, so the ball $B_{r_0}(x)$ is disjoint from S. Equivalently, we have containment $B_{r_0}(x) \subseteq X \setminus S$.

   This shows, by definition, that x is an interior point of $X \setminus S$. Since x was arbitrary, we conclude that every point of $X \setminus S$ is an interior point. Thus $X \setminus S$ is open, and S is closed, as claimed.

   Combined with the result of Homework #1 Question 5(c), this shows that a subset of a metric space is closed if and only if it contains all of its accumulation points!