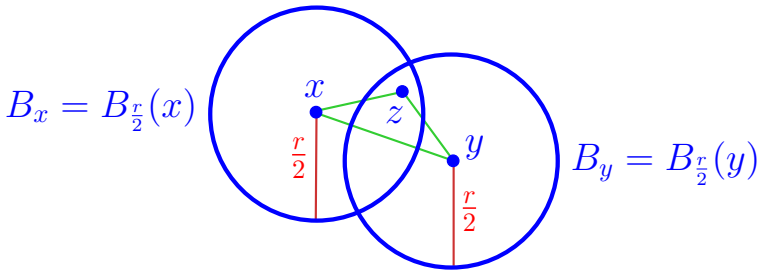
Name: _____ Score (Out of 8 points):

1. (4 points) Let (X, d) be a metric space. Suppose that x, y are distinct points in X. Show that there exist **disjoint** open balls B_x, B_y in X that contain x and y, respectively.

Solution. Suppose that X contains the two distinct elements x and y, and suppose that $d(x, y) = r$. Then $r > 0$ by definition of a metric, and so the sets $B_x = B_{\frac{r}{2}}(x)$ and $B_y = B_{\frac{r}{2}}(y)$ are open balls around x and y, respectively. We will show that these two open balls are disjoint.

Suppose (for the sake of eventual contradiction) that z were an element in B_x and B_y.



The assumption that $z \in B_x = B_{\frac{r}{2}}(x)$ implies that $d(x, z) < \frac{r}{2}$. Similarly the assumption that $z \in B_y = B_{\frac{r}{2}}(y)$ implies that $d(y, z) < \frac{r}{2}$. But then, by the triangle inequality axiom for d,

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r$$

which contradicts our premise that $d(x, y) = r$. Thus no such element z can exist, and we conclude that $B_x \cap B_y = \varnothing$ as claimed.

2. (4 points) Let (X, d) be a metric space, and $x \in X$. Show that the set $\{x\}$ is a closed subset of X.

Solution. To show that $\{x\}$ is a closed subset of X, we must show that $X \setminus \{x\}$ is open. This means, for every point $y \in X \setminus \{x\}$, we must find an open ball B_y about y that is contained in $X \setminus \{x\}$. In other words, for every point y in X that is distinct from x, we must find an open ball B_y about y that does not contain x.

So fix a point $y \neq x$ in X. Consider the balls B_y and B_x as constructed in the Problem 1. We proved these balls are disjoint, in particular, B_y does not contain the point $x \in B_x$. Thus y is an interior point of $X \setminus \{x\}$. Since y was arbitrary, we conclude that $X \setminus \{x\}$ is open in X, and $\{x\}$ is closed.