Name: _____ Score (Out of 13 points):

1. (5 points) Let X be a set. State the definition of a *metric* on X.

Solution. A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

2. (8 points) The following is a list of spaces X and functions $d : X \times X \to \mathbb{R}$ that **fail** to be a metric on X. For each function d, identify an axiom for a metric that fails to hold, and show by counterexample that the axiom fails.

   (a) $X = \mathbb{N} = \{1, 2, 3, \dots\}$, and d is the function $d(x, y) = |x||x - y|$.

   Sample Solution. The function d fails to be symmetric.
   For example, $d(1, 2) = 1$ but $d(2, 1) = 2$.

   (b) $X = \mathbb{R}$, and d is the constant function $d(x, y) = 10$ for all $x \in X$.

   Sample Solution. The function d fails to satisfy the condition $d(x, x) = 0$ for all x.
   For example, $d(0, 0) = 10$.

   (c) $X = \mathbb{R}$, and d is the function $d(x, y) = |x^2 - y^2|$.

   Sample Solution. There are elements x, y with $d(x, y) = 0$ but $x \neq y$.
   For example, $d(1, -1) = 0$.

   (d) $X = \{a, b, c\}$, and d is the function

$$\begin{aligned} d(a, a) &= d(b, b) = d(c, c) = 0 \\ d(a, b) &= d(b, a) = 2 \\ d(b, c) &= d(c, b) = 3 \\ d(a, c) &= d(c, a) = 6. \end{aligned}$$

   Sample Solution. The function d fails to satisfy the triangle inequality. Observe that

$$d(a, b) + d(b, c) = 3 + 2 = 5 < 6 = d(a, c).$$