Name: _____ Score (Out of 7 points):

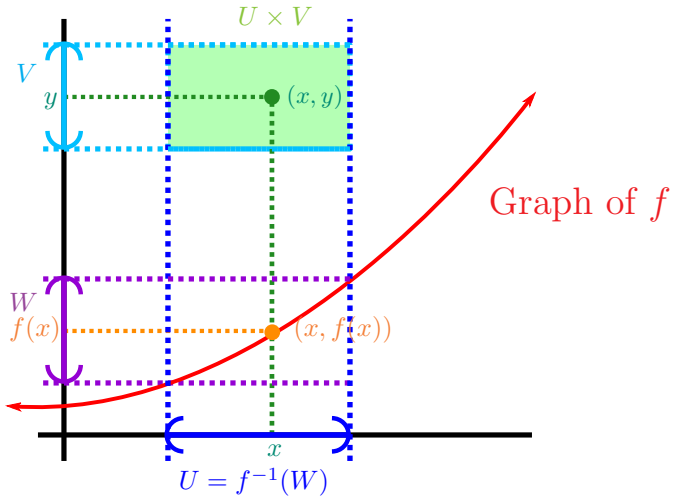
1. (7 points) Let $f : X \to Y$ be a function of topological spaces. Recall that the *graph* of f is defined to be the subset of $X \times Y$

$$\{ (x, f(x)) \in X \times Y \mid x \in X \}.$$

Suppose that Y is Hausdorff. Show that, if f is continuous, then the graph of f is a closed subset of $X \times Y$ with respect to the product topology on $X \times Y$.

Solution: Denote the graph by $G \subseteq X \times Y$. To show that G is closed, we wish to show that its complement is open. To do this, it suffices to show that any point (x, y) in the complement of G has an open neighbourhood that is contained in the complement of G.

Let (x, y) be a point in the complement of G. Then $y \neq f(x)$. Since Y is Hausdorff, we can therefore find disjoint open neighbourhoods V of y and W of $f(x)$ in Y. Since f is continuous, the subset $U = f^{-1}(W)$ is open in X. Because $f(x) \in W$, we know x is contained in its preimage U.



We claim that the subset $U \times V$ is an open neighbourhood of (x, y) in $X \times Y$ contained in the complement of G. The subset $U \times V$ contains (x, y) by construction, and it is open in $X \times Y$ by the definition of the product topology. So it suffces to show that $U \times V$ is contained in the complement of G.

Consider $(u, v) \in U \times V$. Then $f(u) \in W$ by definition of U. But $v \in V$ and V is disjoint from W by construction, so $v \neq f(u)$. We conclude that $(u, v) \notin G$, and that $U \times V$ is contained in the complement of G as claimed.