Midterm Exam Solutions

Math 490
22 October 2024
Jenny Wilson

Name: _____

Instructions: This exam has 4 questions for a total of 25 points.

The exam is closed-book. No books, cell phones, calculators, or other devices are permitted.

Each student may bring in one double-sided standard-size (8.5 in $\times$ 11 in) sheet of notes, which they must prepare themselves. Notes may be handwritten or typed with font size at least 12.

Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	11	
2	5	
3	4	
4	5	
Total:	25	

1. (11 points) For each of the following: If the statement is true, write "`True`". Otherwise, state a specific counterexample. **No further justification needed.**

   If the statement is not true, you may receive partial credit for writing "`False`" without a counterexample.

   (a) Let $f : X \to Y$ be a function of sets. If the subsets $A, B \subseteq X$ are disjoint, then $f(A)$ and $f(B)$ are disjoint.

   Counterexample. $X = Y = \mathbb{R}$ and f is the constant function that maps $\mathbb{R}$ to 0. Then $A = \{0\}$ and $B = \{1\}$ are disjoint, but $f(A) = f(B) = \{0\}$ are not disjoint.

   (b) Let $f : X \to Y$ be a function of sets. If the subsets $C, D \subseteq Y$ are disjoint, then $f^{-1}(C)$ and $f^{-1}(D)$ are disjoint.

   True.

   Hint: Any point in X can map to at most one of the sets C or D.

   (c) Let X be a metric space and $A \subseteq X$. If A has no accumulation points, then it is closed.

   True.

   Hint: Let A' be the set of accumulation points of A. We proved on Worksheet #5 that $\overline{A} = A \cup A'$ and that A is closed if and only if $A = \overline{A}$. Consequently, A is closed if and only if it contains its accumulation points A'. This must be true in particular when $A' = \varnothing$.

   (d) Let X be a metric space and $A \subseteq X$ a nonempty open subset. Then every point of A is an accumulation point of A.

   Counterexample. $X = (\mathbb{R}, \text{discrete})$. $A = \{0\}$.
   Then A (and, in fact, any subset of X) is open but has no accumulation points.

   (e) Let A, B be subsets of a metric space X. Then $\partial(A \cap B) = \partial A \cap \partial B$.

   Counterexample. $X = (\mathbb{R}, \text{Euclidean})$. Let A be the rationals and B be the irrationals. Then $\partial(A \cap B) = \partial(\varnothing) = \varnothing$, but $\partial A \cap \partial B = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

(f) Let A be a bounded subset of a metric space X. Then $\overline{A}$ is bounded.

True.

Hint: If A is contained in the ball $B_R(x)$ for some $x \in X$ and $R > 0$, then it is contained in the closed and bounded set $C = \{y \in X \mid d(x, y) \leq R\}$. Since C is a closed set containing A, it contains $\overline{A}$ by Worksheet #5 Theorem 1.7.

(g) Let X be a metric subspace and $A \subseteq X$ is subset such that every point of A is an isolated point of A. Then A is closed.

Counterexample. $X = (\mathbb{R}$, Euclidean), and $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

(h) Let $f : X \to Y$ is a continuous function of metric spaces. If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, then $(f(a_n))_{n\in\mathbb{N}}$ is Cauchy in Y.

Counterexample. $X = Y = ((0, \infty)$, Euclidean). Let $a_n = \frac{1}{n}$, and $f(x) = \frac{1}{x}$.

(i) Let $f : X \to Y$ be a continuous function of metric spaces. If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence in X, then $(f(a_n))_{n\in\mathbb{N}}$ is convergent in Y.

True.

Hint: We proved this result on Homework #3 Problem 3.

(j) Let $f : X \to Y$ be a continuous function of metric spaces. If $A \subseteq X$ is closed and bounded, so is $f(A)$.

Counterexample. $X = A = (\mathbb{R}$, discrete), $Y = (\mathbb{R}$, Euclidean).
Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map.

(k) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function of Euclidean metric spaces. If $A \subseteq \mathbb{R}^n$ is closed and bounded, so is $f(A)$.

True.

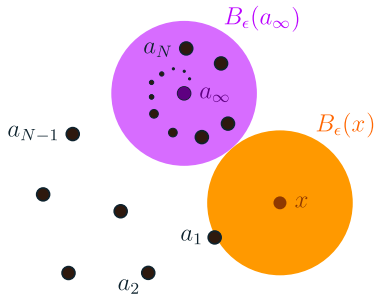
Hint: If $A \subseteq \mathbb{R}^n$ is closed and bounded, then it is sequentially compact. Its image is sequentially compact, hence, closed and bounded.

2. (5 points) Let (X, d) be a metric space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to a point $a_\infty \in X$. Show that the set $A = \{a_n \mid n \in \mathbb{N}\} \cup \{a_\infty\}$ is a closed subset of X.

Solution. To show that A is closed, we will show that its complement is open. Let $x \in X$ be a point with $x \notin A$. Our goal is to show that x is an interior point of $X \setminus A$, that is, we must find $r > 0$ such that the ball $B_r(x)$ contains no points of A.

Let $\epsilon = \frac{1}{2}d(a_\infty, x)$. We observe that the balls $B_\epsilon(x)$ and $B_\epsilon(a_\infty)$ are disjoint: for, if there existed a point $y \in B_\epsilon(x) \cap B_\epsilon(a_\infty)$, then the triangle inequality would imply the contradictory statement that $d(a_\infty, x) \leq d(a_\infty, y) + d(y, x) < \epsilon + \epsilon = d(a_\infty, x)$.

Since the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a_∞, by definition of convergence, there must exist some N such that $a_n \in B_\epsilon(a_\infty)$ for all $n \geq N$. In particular, by disjointness of the balls, $a_\infty, a_n \notin B_\epsilon(x)$ for all $n \geq N$. In other words, we are guaranteed that $d(x, a_\infty) \geq \epsilon$ and $d(x, a_n) \geq \epsilon$ for all $n \geq N$.



Let $r = \min\{\epsilon, d(a_1, x), d(a_2, x), \ldots, d(a_{N-1}, x)\}$. Since r is the minimum of a finite list of positive numbers, $r > 0$. We will show that the ball $B_r(x)$ is disjoint from A.

We know that $r \leq \epsilon \leq d(x, a_\infty)$, and $r \leq \epsilon \leq d(x, a_n)$ for all $n \geq N$. In addition, by choice of r, we know $r \leq d(a_n, x)$ for all $n < N$.

Thus $B_r(x)$ contains no points of A. We conclude that $B_r(x) \subseteq X \setminus A$, which demonstrates that x is an interior point of $X \setminus A$, as desired.

Alternate Solution. We saw on Worksheet #5 that A is closed if and only if $A = \overline{A}$, and that $\overline{A}$ is the union of A and the set of accumulation points of A. Thus, it suffices to show that any point $x \in X \setminus A$ cannot be an accumulation point of A.

So fix $x \in X \setminus A$ and suppose for the sake of contradiction that it is an accumulation point of A. We proved on Homework #1 Problem 5(d) that for all $\epsilon > 0$, the ball $B_\epsilon(x)$ contains infinitely many points of A. Construct a sequence as follows. Let a_{n_1} be any point in $A \cap B_1(x)$. Consider $B_{\frac{1}{2}}(x)$. Since it contains infinitely many points of A, and there are only finitely many indices $1, 2, \ldots n_1$, we can choose a_{n_2} so that $a_{n_2} \in A \cap B_{\frac{1}{2}}(x)$ and $n_2 > n_1$. In general, let a_{n_i} be a point in $A \cap B_{\frac{1}{i}}(x)$ such that $n_i > n_{i-1}$. In this way, we have constructed a subsequence of $(a_n)_{n\in\mathbb{N}}$ that converges to x. This is a contradiction; we proved on Worksheet #6 Proposition 1.2 that all subsequences of $(a_n)_{n\in\mathbb{N}}$ converge to a_∞.

We conclude that there are no accumulation points of A contained in the complement of A, and thus that A is closed.

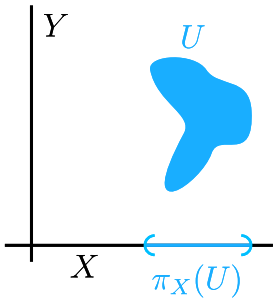
3. (4 points) We call a function $f : W \to Z$ of metric spaces *open* if the **image** of every open subset of W is open in Z.

   Let X and Y be metric spaces. Let $X \times Y$ be the Cartesian product with the product metric, and let

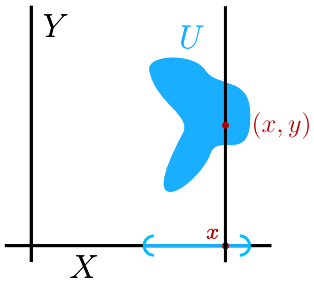
   $$\pi_X : X \times Y \longrightarrow X$$
   $$(x, y) \longmapsto x$$

   be the projection map onto X. Show that π_X is an open map.

Solution. Let U be an open subset of $X \times Y$. Our goal is to show that its image $\pi_X(U)$ is open in X.



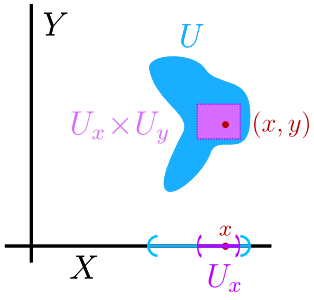
Let x be a point of $\pi_X(U)$; we will show that x is an interior point of $\pi_X(U)$. The statement that x is in the image of U means that there must exist some $y \in Y$ so that $(x, y) \in U$.



We proved on Homework #2 Problem 4 that there exists open neighbourhoods U_x of x in X and U_y of y in Y such that $U_x \times U_y \subseteq U$. But then

$$U_x = \pi_X(U_x \times U_y) \subseteq \pi_X(U)$$

.



Thus we have found an open neighbourhood U_x of x satisfying $U_x \subseteq \pi_X(U)$. We conclude that x is an interior point of $\pi_X(U)$, and hence that $\pi_X(U)$ is open in X.

(Note that the pictures in this proof are not rigorous and only represent one particular open set in one particular metric space. The written proof is self-contained and complete without the pictures. I've included the pictures for expository purposes, as a visual aid to illustrate the steps in the written proof.)

4. (5 points) Let A be a **dense** subset of a metric space X. Show that, for any **open** subset $U \subseteq X$, there is equality

$$\overline{U \cap A} = \overline{U}.$$

Solution. We first recall from Worksheet #5 Theorem 1.7 that, for any subsets $B \subseteq C$ in a metric space X, we have $\overline{B} \subseteq \overline{C}$. Thus the containment $U \cap A \subseteq U$ implies that we have containment $\overline{U \cap A} \subseteq \overline{U}$. To prove the result, then, we must show containment $\overline{U} \subseteq \overline{U \cap A}$.

Let $u \in \overline{U}$. To prove that $u \in \overline{U \cap A}$ we must show (by definition of closure, Worksheet #5 Theorem 1.5) that every open neighbourhood of u contains a point of $U \cap A$. Let V be an open neighbourhood of u.

Since $u \in \overline{U}$, by definition of closure, the set V must contain a point of U. Thus the intersection $U \cap V$ is nonempty, and because U and V are open it follows (by Worksheet #2 Problem 2(c)) that $U \cap V$ is open. We saw on Homework #3 Problem 4 that one definition of a dense subset of X is that it intersects every nonempty open subset of X. Thus $U \cap V$ contains a point a of A.

But then the point a is contained in U, V, and A. In particular, it is a point of $U \cap A$ contained in V. Since V was an arbitrary neighbourhood of u, this concludes our argument that u is contained in $\overline{U \cap A}$.

We have shown that $\overline{U \cap A} = \overline{U}$, as claimed.