Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let (X, d) be a metric space. Verify that the collection $\mathcal{T}_d$ of open sets in this metric space does indeed form a topology on the set X.
2. Find all possible topologies on the set $X = \{0, 1\}$.
3. Let $X = \{0, 1, 2\}$. Show that the collection of subsets $\Big\{\varnothing, X, \{0, 1\}, \{1, 2\}\Big\}$ is **not** a topology on X.
4. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show by induction that the intersection of any **finite** collection of open sets is open.
   (b) Explain why this argument does not apply to an infinite collection of open sets.
5. Consider the following collection of subsets of $\mathbb{R}$. Note that $\varnothing$ is considered a finite set.

$$\mathcal{R} = \{A \mid A \text{ is finite}\} \cup \{\mathbb{R}\}.$$

   (a) Show that $\varnothing, \mathbb{R} \in \mathcal{R}$.
   (b) Show that if $U, V \in \mathcal{R}$, then so is $U \cap V$.
   (c) Show that if $U, V \in \mathcal{R}$, then so is $U \cup V$. Show, in fact, that the union of any finite number of elements in $\mathcal{R}$ is an element of $\mathcal{R}$.
   (d) Show that arbitrary unions of elements in $\mathcal{R}$ may not be in $\mathcal{R}$. Conclude that $\mathcal{R}$ is not a topology on $\mathbb{R}$.
   (e) Conclude that the second axiom of a topology—that it is closed under arbitrary unions—is truly stronger than the condition that it be closed under finite unions.
6. Let X be a set. Show that the discrete topology on X is induced by the discrete metric on X.
7. Let X be a topological space consisting of either 0 or 1 points. Explain why X satisfies both the T_1 and T_2 properties.
8. Let X be a set, and $A \subseteq X$ a proper subset. What are the interior and closure (Assignment Problems 2 and 3) of A if X is given
   (a) the discrete topology? (b) the indiscrete topology?
9. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

   Find the interior, closures, boundaries, and accumulation points of of the subsets
   (a) $\{a, b, c\}$ (b) $\{a, c, d\}$ (c) $\{a, b, d\}$ (d) $\{b\}$ (e) $\{d\}$ (f) $\{b, d\}$
10. Consider the following subsets of $\mathbb{R}$

- $\mathbb{R}$
- $\varnothing$
- $\{0, 1\}$
- $[0, \infty)$
- $(1, 2)$
- $[1, 2] \cup [3, \infty)$
- $(-\infty, 0)$
- $(-\infty, 0]$
- $\mathbb{N}$
- $\{-n \mid n \in \mathbb{N}\}$

Find the interior, closures, boundaries, and accumulation points of these subsets ...

(a) ... when $\mathbb{R}$ has the discrete topology.

(b) ... when $\mathbb{R}$ has the indiscrete topology.

(c) ... when $\mathbb{R}$ has the Euclidean topology.

(d) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

(e) ... when $\mathbb{R}$ has the $\mathcal{T} = \{A \mid \mathbb{R} \setminus A \text{ is finite}\} \cup \{\varnothing\}$.

(f) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.

(g) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

Worksheet problems

(Hand these questions in!)

- Worksheet #9 Problems 3.

Assignment questions

(Hand these questions in!)

1. (a) **Definition (T_1-space).** Let $(X, \mathcal{T})$ be a topological space. Then we say that X has the T_1 *property*, or call X a T_1*-space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists some neighbourhood U_x of x that does not contain y, and there exists some neighbourhood U_y of y that does not contain x.

   Prove that $(X, \mathcal{T})$ is a T_1-space if and only if, for every point $x \in X$, the singleton set $\{x\}$ is closed. (Mathematicians often refer to this property by the slogan "points are closed").

   (b) **Definition (Hausdorff property; T_2-space).** Let $(X, \mathcal{T})$ be a topological space. Then we say that X has the T_2 *property*, or call X a *Hausdorff space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists open neighbourhoods U_x of x and U_y of y such that U_x and U_y are disjoint.

   Show that every space satisfying the T_2 property must satisfy the T_1 property. Conclude in particular that, in a Hausdorff space, singleton sets $\{x\}$ are closed.

   (c) For each of the following topologies on $\mathbb{R}$, state whether the topology has the T_1 property, the T_1 and T_2 properties, or neither. No justification needed.

   (i) The topology induced by the Euclidean metric

   (ii) $\mathcal{T} = \{\mathbb{R}, \varnothing\}$

   (iii) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$

   (iv) $\mathcal{T} = \{\mathbb{R}, (0, 1), \varnothing\}$

   (v) $\mathcal{T} = \{\mathbb{R}, \{0, 1\}, \{0\}, \{1\}, \varnothing\}$

   (vi) $\mathcal{T} = \{A \mid \mathbb{R} \setminus A \text{ is finite}\} \cup \{\varnothing\}$

   (vii) $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$

   (viii) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$

   (ix) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$

(x) $\mathcal{T} = \{A \mid A \text{ is a union of intervals of the form } [a, b) \text{ for } a, b \in \mathbb{R}\ \} \cup \{\varnothing\}$

(d) Explain why all the topoloigical spaces on this list that fail to have the Hausdorff property cannot be metrizable.

In the following questions, we will generalize the notions of interior, closure, and boundary to general topological spaces—and verify that many of the results we proved for metric spaces hold in general. This is an opportunity to revisit their proofs!

2. **Definition (Interior of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. A point $a \in A$ is an *interior point* of A if there exists an open subset $U \subseteq X$ such that $a \in U \subseteq A$. Define the *interior* of A to be the set of interior points of A. Concretely,

$$\text{Int}(A) = \{\ a \in A \mid \text{there is some open neighbourhood } U \text{ of } a \text{ such that } U \subseteq A\ \}.$$

Observe that, by definition, $\text{Int}(A) \subseteq A$. Prove the following.

(a) A is open if and only if $A = \text{Int}(A)$.

(b) $\text{Int}(A)$ is an open set.

(c) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.

(d) Suppose that $A \subseteq X$ is any subset, and $U \subseteq A$ is an open subset of X. Prove that $U \subseteq \text{Int}(A)$.

(e) $\text{Int}(A) = \bigcup_{\substack{U \subseteq A, \\ U \text{ open in } X}} U.$

3. **Definition (Closure of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *closure* of A to be the set

$$\overline{A} = \{\ x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A\ \}.$$

Prove the following.

(a) $A \subseteq \overline{A}$.

(b) A is closed if and only if $A = \overline{A}$.

(c) $\overline{\overline{A}} = \overline{A}$.

(d) $\overline{A}$ is a closed set.

(e) Suppose that $A \subseteq X$ is any subset, and C is a closed set containing A. Then $\overline{A} \subseteq C$.

(f) $\overline{A} = \bigcap_{\substack{C \text{ closed in } X, \\ A \subseteq C}} C.$

4. **Definition (Boundary of a set A).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \text{Int}(A)$.

(a) Prove that $\partial A = \overline{A} \cap (\overline{X \setminus A})$.

(b) Use this result to conclude that (*i*) ∂A is closed, and (*ii*) $\partial A = \partial(X \setminus A)$.

(c) Prove the following.

Theorem (An equivalent definition of ∂A). Let X be a topological space, and let $A \subseteq X$. Then

$$\partial A = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{every open neighbourhood } U \text{ of } x \text{ contains at least one point of } A, \\ \text{and at least one point of } X \setminus A \end{array} \right\}$$

(d) Prove that every point of X falls into one of the following three categories of points, and that the three categories are mutually exclusive:

(i) interior points of A;

(ii) interior points of $X \setminus A$;

(iii) points in the (common) boundary of A and $X \setminus A$.