Rigid meromorphic cocycles of weight 2

Bomsok Kwon

Abstract

These are notes for my seminar talk at the University of Michigan RTG Number Theory Seminar on rigid meromorphic cocycles of weight 2. It is a more complete version of the talk. The presentation follows Chapter 1 of [DV21]. The proof of the classification theorem of rational period functions of weight 2 is taken from [CZ92].

1 Parabolic cohomology

Set $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$. Set Γ_∞ to be the subgroup of upper triangular matrices in Γ.

Definition 1.1. For a (left) $\mathrm{GL}_2(\mathbb{Q}_p)$-module Ω, we define the *parabolic cohomology* as

$$H^1_{\mathrm{par}}(\Gamma, \Omega) := \ker\left(H^1(\Gamma, \Omega) \to H^1(\Gamma_\infty, \Omega)\right)$$

Rigid meromorphic cocycles are defined as parabolic cohomology classes in the additive case, or as quasi-parabolic cohomology classes in the multiplicative case. Let $\mathcal{M}$ be the $\mathbb{C}_p$-algebra of rigid meromorphic functions on the Drinfeld p-adic upper half-plane $\mathfrak{H}_p$. Let $\mathcal{O} \subseteq \mathcal{M}$ be the subalgebra of rigid analytic functions on $\mathfrak{H}_p$. There is the weight k action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ on $\mathcal{M}$ and $\mathcal{O}$ given by

$$(f|_k\gamma)(z) := (\det\gamma)^{k/2}(cz+d)^{-k}f(\gamma z).$$

This is a right action, but we can turn it into a left action by setting $\gamma \cdot_k f := f|_k\gamma^{-1}$. We denote these $\mathrm{PGL}_2(\mathbb{Q}_p)$-modules as $\mathcal{M}_k$ and $\mathcal{O}_k$ respectively. When $k = 0$, we may drop the subscript and simply write $\mathcal{M}$ and $\mathcal{O}$ instead of $\mathcal{M}_0$ and $\mathcal{O}_0$.

Definition 1.2. (1) A *multiplicative rigid meromorphic cocycle* is a class in $H^1_f(\Gamma, \mathcal{M}^\times)$, which we define as the preimage of $H^1(\Gamma_\infty, \mathbb{C}_p^\times)$ under the map $H^1(\Gamma, \mathcal{M}^\times) \to H^1(\Gamma_\infty, \mathcal{M}^\times)$.

(2) An *additive rigid meromorphic (resp. analytic) cocycle of weight k* is a class in $H^1_{\mathrm{par}}(\Gamma, \mathcal{M}_k)$ (resp. $H^1_{\mathrm{par}}(\Gamma, \mathcal{O}_k)$).

Multiplicative meromorphic cocycles are the main objects of interest. They are related to additive meromorphic cocycles of weight 2 by the map

$$H^1_f(\Gamma, \mathcal{M}^\times) \to H^1_{\mathrm{par}}(\Gamma, \mathcal{M}_2)$$

which is induced by the map $\mathrm{dlog} : \mathcal{M}^\times \to \mathcal{M}_2$ given by $g \mapsto g'/g$.

2 Modular symbols

Instead of thinking directly about H^1, we will introduce two alternative notions that are equivalent to parabolic cohomology (under a certain hypothesis that is satisfied in the situations we care) and are easier to deal with than H^1.

Definition 2.1. A Ω-*valued modular symbol* is a function $m : \mathbb{P}^1(\mathbb{Q})^2 \to \Omega$ that satisfies

$$m\{r,s\} + m\{s,t\} = m\{r,t\} \quad \forall\, r,s,t \in \mathbb{P}^1(\mathbb{Q}).$$

Remark 2.2. This condition automatically implies $m\{r,r\} = 0$ and $m\{r,s\} = -m\{s,r\}$.

Let $\mathrm{MS}(\Omega)$ be the group of Ω-valued modular symbols. There is an action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ on $\mathrm{MS}(\Omega)$ defined by

$$(\gamma \cdot m)\{r,s\} := \gamma \cdot (m\{\gamma^{-1}r, \gamma^{-1}s\}).$$

Remark 2.3. If $m \in \mathrm{MS}(\Omega)$ is fixed by $\gamma \in \mathrm{PGL}_2(\mathbb{Q}_p)$, then we have $\gamma(m\{r,s\}) = m\{\gamma r, \gamma s\}$.

Lemma 2.4. There is an exact sequence

$$0 \to \Omega^\Gamma \to \Omega^{\Gamma_\infty} \to \mathrm{MS}(\Omega)^\Gamma \xrightarrow{\delta} H^1(\Gamma,\Omega) \to H^1(\Gamma_\infty,\Omega).$$

Proof. There is a short exact sequence of Γ-modules

$$0 \to \Omega \to \mathrm{Func}(\mathbb{P}^1(\mathbb{Q}),\Omega) \xrightarrow{d} \mathrm{MS}(\Omega) \to 0$$

where Γ acts on $\mathrm{Func}(\mathbb{P}^1(\mathbb{Q}),\Omega)$ by $(\gamma \cdot m)(r) := \gamma \cdot (m(\gamma^{-1}r))$ and d is defined by

$$(df)\{r,s\} := f(r) - f(s).$$

We obtain the long exact sequence of cohomology groups

$$0 \to H^0(\Gamma,\Omega) \to H^0(\Gamma,\mathrm{Func}(\mathbb{P}^1(\mathbb{Q}),\Omega)) \to H^0(\Gamma,\mathrm{MS}(\Omega)) \to H^1(\Gamma,\Omega) \to H^1(\Gamma,\mathrm{Func}(\mathbb{P}^1(\mathbb{Q}),\Omega)).$$

Observe that Γ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ and that the stabilizer of ∞ is Γ_∞. So, we can identify $\mathbb{P}^1(\mathbb{Q}) \cong \Gamma/\Gamma_\infty$ and $\mathrm{Func}(\mathbb{P}^1(\mathbb{Q}),\Omega) \cong \mathrm{Ind}_{\Gamma_\infty}^{\Gamma}\Omega$. Then,

$$H^i(\Gamma,\mathrm{Func}(\mathbb{P}^1(\mathbb{Q}),\Omega)) \cong H^i(\Gamma,\mathrm{Ind}_{\Gamma_\infty}^{\Gamma}\Omega) \cong H^i(\Gamma_\infty,\Omega)$$

by Shapiro's lemma. This concludes the proof. □

Corollary 2.5. There is a natural surjection $\delta : \mathrm{MS}(\Omega)^\Gamma \twoheadrightarrow H^1_{\mathrm{par}}(\Gamma,\Omega)$, which is an isomorphism precisely when $\Omega^\Gamma = \Omega^{\Gamma_\infty}$.

Remark 2.6. If $m \in \mathrm{MS}(\Omega)^\Gamma$, then $\delta(m)$ is the class represented by a 1-cocycle $\Phi(\gamma) := m\{\gamma\infty, \infty\}$.

The condition $\Omega^\Gamma = \Omega^{\Gamma_\infty}$ in 2.5 is true when Ω is one of $\mathcal{M}^\times$, $\mathcal{M}_k$, or $\mathcal{O}_k$. Indeed, one can use the Weierstrass preparation lemma to show the following.

Lemma 2.7.

$$H^0(\Gamma_\infty,\mathcal{M}^\times) = \mathbb{C}_p^\times, \quad H^0(\Gamma_\infty,\mathcal{M}_k) = \begin{cases} \mathbb{C}_p & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

3 Period functions

Consider the matrices

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad D := \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}$$

Definition 3.1. We define

$$\Omega^\dagger := \{\omega \in \Omega \mid \omega + S\omega = \omega + U\omega + U^2\omega = 0\}.$$

Lemma 3.2. There is a natural isomorphism

$$\mathrm{MS}(\Omega)^{\mathrm{SL}_2(\mathbb{Z})} \to \Omega^\dagger$$
$$m \mapsto m\{0,\infty\}.$$

Proof. We first check that if $m \in \mathrm{MS}(\Omega)^{\mathrm{SL}_2(\mathbb{Z})}$ then $m\{0,\infty\} \in \Omega^\dagger$. Note that S swaps 0 and ∞ while U cyclically permutes $0, 1, \infty$. By 2.3, we have

$$m\{0,\infty\} + S(m\{0,\infty\}) = m\{0,\infty\} + m\{\infty,0\} = 0,$$
$$m\{0,\infty\} + U(m\{0,\infty\}) + U^2(m\{0,\infty\}) = m\{0,\infty\} + m\{1,0\} + m\{\infty,1\} = 0.$$

To show bijectivity, we state a few facts about unimodular pairs in $\mathbb{P}^1(\mathbb{Q})$. A pair $(r,s) \in \mathbb{P}^1(\mathbb{Q})^2$ is called *unimodular* if $ad - bc = \pm 1$ where $r = \frac{a}{b}$ and $s = \frac{c}{d}$ are the reduced fractions. So, $0 = \frac{0}{1}$ and $\infty = \frac{1}{0}$ form a unimodular pair.

(a) $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on unimodular pairs.

(b) For any pair $(r,s) \in \mathbb{P}^1(\mathbb{Q})^2$, there exists a unimodular sequence joining r and s, i.e. a finite sequence $r = \alpha_0, \alpha_1, \cdots, \alpha_{r-1}, \alpha_r = s$ for which each (α_i, α_{i+1}) is unimodular.

(c) Any two unimodular sequences with the same endpoints admit a common refinement which are formed by applying a finite number of replacements of the form $\frac{a}{b}, \frac{c}{d} \rightsquigarrow \frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}$ and $\frac{a}{b}, \frac{c}{d}, \frac{-a}{-b} \rightsquigarrow \frac{a}{b}$.

Injectivity follows from (a) and (b), while surjectivity follows from (c). □

Definition 3.3. We define $\Omega^\ddagger$ as the image of $\mathrm{MS}(\Omega)^\Gamma$ in $\Omega^\dagger$.

Remark 3.4. If $m \in \mathrm{MS}(\Omega)^\Gamma$ then $D(m\{0,\infty\}) = m\{0,\infty\}$ as D fixes 0 and ∞. So, we have an inclusion

$$\Omega^\ddagger \subseteq \{\omega \in \Omega^\dagger \mid D\omega = \omega\}.$$

The containment is strict in general.

Provided that $\Omega^\Gamma = \Omega^{\Gamma_\infty}$, we obtain a commutative diagram of isomorphisms

$$\begin{array}{ccc} \mathrm{MS}(\Omega)^\Gamma & \longrightarrow & \Omega^\ddagger \\ & \searrow \quad \nearrow & \\ & H^1_{\mathrm{par}}(\Gamma,\Omega) & \end{array}$$

Recall from 2.3 that $\delta(m)$ is represented by $\Phi(\gamma) := m\{\gamma\infty,\infty\}$. This is the *parabolic representative* in the sense that $\Phi(\Gamma_\infty) = \{0\}$. Moreover, $\Phi(S) = m\{0,\infty\}$ recovers the image in $\Omega^\ddagger$.

Definition 3.5. (1) A *multiplicative rigid meromorphic period function* is a function in $\mathcal{M}^\times$ of the form $\Phi(S)$ where Φ is the quasi-parabolic representative (i.e. $\Phi(\Gamma_\infty) \subseteq \mathbb{C}_p^\times$) of a meromorphic cocycle.

(2) An *additive rigid meromorphic (resp. analytic) period function of weight k* is a function in $\mathcal{M}_k$ (resp. $\mathcal{O}_k$) of the form $\Phi(S)$ where Φ is the parabolic representative (i.e. $\Phi(\Gamma_\infty) = \{0\}$) of a meromorphic (resp. analytic) cocycle of weight k. Equivalently, it is just an element of $\mathcal{M}_k^\ddagger$ (resp. $\mathcal{O}_k^\ddagger$).

We denote the group of meromorphic period functions by $\mathcal{R}^\times$. We also denote the $\mathbb{C}_p$-vector space of rigid meromorphic period functions of weight k by $\mathcal{R}_k$, which is just $\mathcal{M}_k^\ddagger$.

4 Rational period functions of weight 2

Our goal is to classify functions in $\mathcal{R}_2$. But before doing so, let us classify so-called rational period functions in $\mathcal{R}_2^{\mathrm{rat}}$. The end result is almost identical to that for $\mathcal{R}_2$, but $\mathcal{R}_2^{\mathrm{rat}}$ involves less technicalities.

Let $\mathcal{M}_k^{\mathrm{rat}}$ denote the $\mathbb{C}$-vector space of rational functions on $\mathbb{P}^1_{\mathbb{C}}$ endowed with the weight k action by $\mathrm{PGL}_2(\mathbb{Q})$. By what we have discussed so far (but with Γ replaced by $\mathrm{SL}_2(\mathbb{Z})$ every time), we obtain isomorphisms

$$H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_k^{\mathrm{rat}}) \cong \mathrm{MS}(\mathcal{M}_k^{\mathrm{rat}})^{\mathrm{SL}_2(\mathbb{Z})} \cong (\mathcal{M}_k^{\mathrm{rat}})^\dagger = \mathcal{R}_k^{\mathrm{rat}}.$$

Rational period functions of weight k in $\mathcal{R}_k^{\mathrm{rat}}$ are precisely the functions $\phi \in \mathcal{M}_k^{\mathrm{rat}}$ that satisfy

$$\phi\left(-\frac{1}{z}\right) = -z^k \phi(z), \tag{1}$$

$$\phi(z) + z^{-k}\phi\left(\frac{z-1}{z}\right) + (z-1)^{-k}\phi\left(\frac{-1}{z-1}\right) = 0. \tag{2}$$

Let us construct examples of functions in $\mathcal{R}_2^{\mathrm{rat}}$.

Example 4.1. $\phi^\circ_\infty(z) := \frac{1}{z}$ is a rational period function of weight 2. While one can quickly verify (1) and (2), there is also a conceptual explanation. Let Ω^1_{rat} be the $\mathbb{C}$-vector space of rational 1-forms on $\mathbb{P}^1_{\mathbb{C}}$. For $r, s \in \mathbb{P}^1(\mathbb{Q})$, set $\omega^\circ_\infty\{r, s\} \in \Omega^1_{\mathrm{rat}}$ to be the unique 1-form whose residual divisor is $[r] - [s]$, i.e. whose poles are all simple and concentrated at r and s with residue 1 and -1 respectively. This defines a function

$$\omega^\circ_\infty : \mathbb{P}^1(\mathbb{Q})^2 \to \Omega^1_{\mathrm{rat}},$$

which is a modular symbol since $([r] - [s]) + ([s] - [t]) = [r] - [t]$. It is clearly invariant under $\mathrm{SL}_2(\mathbb{Z})$. Passing over the isomorphism $\mathrm{MS}(\Omega^1_{\mathrm{rat}})^{\mathrm{SL}_2(\mathbb{Z})} \cong (\Omega^1_{\mathrm{rat}})^\dagger$, we obtain

$$\omega^\circ_\infty\{0, \infty\} = \frac{dz}{z} \in (\Omega^1_{\mathrm{rat}})^\dagger.$$

By the isomorphism

$$\begin{aligned} \mathcal{M}_2^{\mathrm{rat}} &\xrightarrow{\cong} \Omega^1_{\mathrm{rat}} \\ g &\mapsto g dz, \end{aligned}$$

we see $\frac{1}{z} \in \mathcal{R}_2^{\mathrm{rat}}$.

Example 4.2. Let $\mathbb{C}^{\mathrm{RM}}$ denote the set of RM points in $\mathbb{C}$. Fix an orbit $\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{C}^{\mathrm{RM}}$. We will construct a rational period function ϕ°_τ of weight 2 using the same idea of specifying residual divisors. For $r, s \in \mathbb{P}^1(\mathbb{Q})$, the upper half-plane $\overline{\mathfrak{H}} := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{R})$ gets disconnected by the geodesic $\gamma(r, s)$ joining r to s into two connected components

$$\overline{\mathfrak{H}} - \gamma(r, s) = \mathfrak{H}^+(r, s) \amalg \mathfrak{H}^-(r, s).$$

Here, $\mathfrak{H}^+(r, s)$ (resp. $\mathfrak{H}^-(r, s)$) is the component that is on the right (resp. left) side of the geodesic as one travels from r to s. We define the set

$$\Sigma^\circ_\tau(r, s) := \{w \in \mathrm{SL}_2(\mathbb{Z})\tau \mid w \text{ and } w' \text{ lie in different components of } \overline{\mathfrak{H}} - \gamma(r, s)\}$$

where w' is the Galois conjugate of w. We also define

$$\Sigma^{\circ,+}_\tau(r, s) := \Sigma^\circ_\tau(r, s) \cap \mathfrak{H}^+(r, s), \quad \Sigma^{\circ,-}_\tau(r, s) := \Sigma^\circ_\tau(r, s) \cap \mathfrak{H}^-(r, s).$$

Claim. $|\Sigma^\circ_\tau(r, s)| < \infty$ and $|\Sigma^{\circ,+}_\tau(r, s)| = |\Sigma^{\circ,-}_\tau(r, s)|$.

Proof. For simplicity, we only prove this when $(r, s) = (0, \infty)$. In this case,

$$\Sigma^\circ_\tau(r, s) = \{w \in \mathrm{SL}_2(\mathbb{Z})\tau \mid ww' < 0\}.$$

For each $w \in \mathrm{SL}_2(\mathbb{Z})\tau$, there is a unique (up to sign) primitive integral quadratic form $F_w(x, y) = Ax^2 + Bxy + Cy^2$ such that $F_w(w, 1) = 0$. The discriminant $D := \mathrm{disc}(F_w) > 0$ is independent of w and

depends only on τ. Moreover, $ww' < 0$ holds if and only if $AC < 0$. The first part of the claim follows since there are only finitely many integral quadratic forms of fixed discriminant with $AC < 0$.
The second part follows from the bijection $w \mapsto -1/w$ between $\Sigma_\tau^{\circ,+}(0,\infty)$ and $\Sigma_\tau^{\circ,-}(0,\infty)$. □

We define a function $\delta_{r,s} : \mathrm{SL}_2(\mathbb{Z})\tau \to \{1, 0, -1\}$ by

$$\delta_{r,s}(w) := \begin{cases} 1 & \text{if } w \in \Sigma_\tau^{\circ,+}(r,s), \\ -1 & \text{if } w \in \Sigma_\tau^{\circ,-}(r,s), \\ 0 & \text{otherwise.} \end{cases}$$

Set $\omega_\tau^\circ\{r,s\} \in \Omega^1_{\mathrm{rat}}$ to be the unique 1-form whose residual divisor is

$$\Delta_\tau^\circ\{r,s\} := \sum_{w \in \mathrm{SL}_2(\mathbb{Z})\tau} \delta_{r,s}(w)[w] \in \mathrm{Div}^0(\mathbb{P}^1_{\mathbb{C}}).$$

This defines $\omega_\tau^\circ \in \mathrm{MS}(\Omega^1_{\mathrm{rat}})^{\mathrm{SL}_2(\mathbb{Z})}$ as one can check $\delta_{r,s} + \delta_{s,t} = \delta_{r,t}$. This modular symbol induces

$$\phi_\tau^\circ(z) := \sum_{w \in \Sigma_\tau^\circ(0,\infty)} \delta_{0,\infty}(w) \frac{1}{z-w} \in \mathcal{R}_2^{\mathrm{rat}}.$$

These two examples generate all rational period functions of weight 2.

Theorem 4.3 (Knopp, Ash, Choie–Zagier)**.** $\mathcal{R}_2^{\mathrm{rat}}$ has a $\mathbb{C}$-basis $\{\phi_\infty^\circ\} \cup \{\phi_\tau^\circ \mid \tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{C}^{\mathrm{RM}}\}$.

Proof. Linear independency is clear as the set of poles of ϕ_∞° and ϕ_τ° are $\{0\}$ and $\Sigma_\tau^\circ(0,\infty)$, which are pairwise disjoint. It remains to show that any $\phi \in \mathcal{R}_2^{\mathrm{rat}}$ is spanned by ϕ_∞° and ϕ_τ°. Let Σ_ϕ be the set of poles of ϕ, which must be finite. By (1) and (2), it has the following properties.

$$0, \infty \neq w \in \Sigma_\phi \quad \Rightarrow \quad Sw = -1/w \in \Sigma_\phi,$$

$$0, 1, \infty \neq w \in \Sigma_\phi \quad \Rightarrow \quad Uw = (w-1)/w \in \Sigma_\phi \text{ or } U^2 w = -1/(w-1) \in \Sigma_\phi.$$

In particular, if $0, -1, \infty \neq w \in \Sigma_\phi$, then either $USw = w + 1 \in \Sigma_\phi$ or $U^2 Sw = w/(w+1) \in \Sigma_\phi$.
Step 1. $\Sigma_\phi \subseteq \mathbb{P}^1(\mathbb{R})$.
Assume $w_0 \in \Sigma_\phi - \mathbb{P}^1(\mathbb{R})$. Let w_1 be $w_0 + 1$ or $w_0/(w_0+1)$, whichever is contained in Σ_ϕ. In either case, $w_1 \notin \mathbb{P}^1(\mathbb{R})$ and $|\arg(w_0)| > |\arg(w_1)|$. Repeating this process yields an infinite sequence $w_0, w_1, w_2, \cdots$ in Σ_ϕ, which is a contradiction.
Step 2. $\Sigma_\phi - \{0, \infty\}$ is a disjoint union of $\Sigma_\tau^\circ(0,\infty)$.
Suppose $0, \infty \neq w_0 \in \Sigma_\phi$. We must show that $w_0 \in \Sigma_\tau^\circ(0,\infty) \subseteq \Sigma_\phi$ for some τ. We may assume $w_0 > 0$ by replacing w_0 with $-1/w_0$ if necessary. Construct a sequence $w_0, w_1, w_2, \cdots$ in $\Sigma_\phi \cap (0,\infty)$ where w_{i+1} is either $w_i + 1$ or $w_i/(w_i+1)$. If we define a function $T : [0,\infty) \to [0,\infty)$ by

$$T(w) := \begin{cases} w - 1 & \text{if } w \geq 1, \\ w/(1-w) & \text{if } 0 \leq w < 1, \end{cases}$$

then $w_i = T(w_{i+1})$ always holds. By finiteness of Σ_ϕ, the T-orbit $\mathrm{Orb}_T(w_0)$ is finite and is contained in Σ_ϕ. One can show using the theory of continued fractions that $\{0\}$ and $\Sigma_\tau^{\circ,+}(0,\infty)$ are all finite T-orbits. The claim follows since $\Sigma_\tau^\circ(0,\infty) = \Sigma_\tau^{\circ,+}(0,\infty) \cup S(\Sigma_\tau^{\circ,+}(0,\infty))$.
Step 3. Every $w_0 \in \Sigma_\phi - \{0,\infty\}$ is a simple pole.
Consider the T-orbit $w_0, w_1, w_2, \cdots, w_r = w_0$ in Σ_ϕ with $w_{i+1} = \gamma_i w_i$ where γ_i is either SU or SU^2. Let $p_i(z)$ bs the principal part of the Laurent series of $\phi(z)$ at w_i. Then, (1) and (2) imply $p_{i+1} = p_i|_2\gamma_i$. So, $p_0 = p_0|_2\gamma$, where $\gamma := \gamma_0\gamma_1\cdots\gamma_{r-1}$ is a generator of $\mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{Z})}(w_0)$. If $c(z-w_0)^{-m}$ is the lowest

degree term of $p_0(z)$, then writing $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ gives

$$p_0(z) = c(z-w_0)^{-m} + O((z-w_0)^{-m+1}),$$
$$(p_0|_2\gamma)(z) = c(rw_0+s)^{2-2m}(z-w_0)^{-m} + O((z-w_0)^{-m+1}),$$

hence $(rw_0+s)^{2-2m} = 1$. But rw_0+s is a fundamental unit of an order in $\mathbb{Q}(w_0)$, so $m = 1$.

Step 4. ϕ is spanned by ϕ°_∞ and ϕ°_τ.

Set $\lambda_\infty := \mathrm{res}_0\phi$ as well as $\lambda_\tau := \mathrm{res}_w\phi$ for any $w \in \Sigma^{\circ,+}_\tau$. This is well-defined: if we write $\Sigma^{\circ,+}_\tau$ as a T-orbit $w_0, w_1, w_2, \cdots$, then (1) and (2) imply $\mathrm{res}_{w_{i+1}}\phi = \mathrm{res}_{w_i}\phi$. We also have $-\lambda_\tau = \mathrm{res}_w\phi$ for any $w \in \Sigma^{\circ,-}_\tau$ since (1) implies $\mathrm{res}_{Sw_i}\phi = -\mathrm{res}_{w_i}\phi$. Therefore,

$$\phi - \lambda_\infty\phi^\circ_\infty - \sum_\tau \lambda_\tau\phi^\circ_\tau$$

is a rational period function of weight 2 whose pole can only occur at ∞. This has to be 0. □

5 Meromorphic period functions of weight 2

We now return to the task of describing $\mathcal{R}_2$.

Let $\mathrm{red} : \mathfrak{H}_p(\mathbb{C}_p) \to \mathcal{T}$ denote the reduction map to the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Let v_0 denote the standard vertex of $\mathcal{T}$. For $r \in \mathbb{R}_{\geq 0}$, we have an affinoid subset

$$\mathfrak{H}_p^{\leq r} := \mathrm{red}^{-1}(\{x \in \mathcal{T} \mid \mathrm{dist}(x, v_0) \leq r\}),$$

which has an explicit description

$$\mathfrak{H}_p^{\leq r} = \mathbb{P}^1(\mathbb{C}_p) - \bigcup_{x \in \mathbb{P}^1(\mathbb{Q}_p)} \mathcal{B}(x, p^{-r})$$

where

$$\mathcal{B}([x_0 : x_1], d) := \{[y_0 : y_1] \in \mathbb{P}^1(\mathbb{C}_p) \mid |x_0y_1 - x_1y_0| < d\}.$$

Here, we are using coordinates such that $x_0, x_1 \in \mathcal{O}_{\mathbb{C}_p}$ while at least one of x_0, x_1 is in $\mathcal{O}^\times_{\mathbb{C}_p}$ (and similarly for y_0, y_1). We form an admissible cover $\{\mathfrak{H}_p^{\leq n} \mid n \in \mathbb{N}\}$ of $\mathfrak{H}_p$.

Let $\mathfrak{H}_p^{\mathrm{RM}}$ denote the set of RM points in $\mathfrak{H}_p(\mathbb{C}_p)$. By the *discriminant* of $w \in \mathfrak{H}_p^{\mathrm{RM}}$, we mean the discriminant of the unique (up to sign) primitive integral quadratic form $F_w(x, y)$ such that $F_w(w, 1) = 0$.

Lemma 5.1. For $w \in \mathfrak{H}_p^{\mathrm{RM}}$, we have $\mathrm{dist}(\mathrm{red}(w), v_0) = v_p(\mathrm{disc}(w))/2$.

Proof. For simplicity, we only prove this when $p \neq 2$.

Let $D := disc(w)$. Write $F_w(x, y) = Ax^2 + Bxy + Cy^2$ so that $w = \frac{-B \pm \sqrt{D}}{2A}$. We may assume $p \nmid A$ by replecing w with γw for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ if necessary. Then, we have $|w| \leq 1$. It follows from the description of $\mathfrak{H}^{\leq r}$ that

$$\mathrm{dist}(\mathrm{red}(w), v_0) = \max_{n \in \mathbb{Z}} v_p(2Aw - n).$$

Write $D = D_0p^m$ with $p \nmid D_0$. Since $w \notin \mathbb{Q}_p$, either m is odd or D_0 is a quadratic non-residue mod p. We obtain

$$\max_{n \in \mathbb{Z}} v_p(2Aw - n) = \frac{1}{2}\max_{n \in \mathbb{Z}}(n^2 - D) = \frac{1}{2}m.$$

□

After choosing an isomorphism $\mathbb{C}_p \simeq \mathbb{C}$, we can embed $\mathfrak{H}_p^{\mathrm{RM}}$ into $\mathbb{R}$. For $\tau \in \Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}$, we consider

$$\Sigma_\tau(r, s) := \{w \in \Gamma\tau \mid w \text{ and } w' \text{ lie in different components of } \overline{\mathfrak{H}} - \gamma(r, s)\}.$$

We also define $\Sigma_\tau^+(r,s)$, $\Sigma_\tau^-(r,s)$, and $\delta_{r,s}$ in the analogous way. The set $\Sigma_\tau(r,s)$ is no longer finite (unless it's empty), but it is discrete instead.

Claim. $\Sigma_\tau(r,s)$ is *discrete* in the sense that the intersection with any affinoid subset is finite. Moreover, $|\Sigma_\tau^+(r,s)\cap\mathcal{A}_v| = |\Sigma_\tau^-(r,s)\cap\mathcal{A}_v|$ and $|\Sigma_\tau^+(r,s)\cap\mathcal{W}_v| = |\Sigma_\tau^-(r,s)\cap\mathcal{W}_v|$ for every vertex v of $\mathcal{T}$, where $\mathcal{A}_v$ (resp. $\mathcal{W}_v$) is the preimage of v (resp. the open cone of v) under the reduction map.

Proof. For simplicity, we only prove this when $(r,s) = (0,\infty)$.
For the first part of the claim, we must show $|\Sigma_\tau(0,\infty)\cap\mathfrak{H}_p^{\le n}| < \infty$ for each $n \in \mathbb{N}$. This follows from 5.1 together with the fact that there are only finitely many integral quadratic forms of bounded discriminant with $AC < 0$.
For the second part of the claim, consider the $\Gamma_v := \mathrm{Stab}_\Gamma(v)$-equivariant map

$$\begin{aligned}\deg_{\mathcal{A}_v} : \mathrm{Div}_{\mathrm{dis}}(\mathcal{H}_p) &\to \mathbb{Z}\\ \Delta &\mapsto \deg(\Delta|_{\mathcal{A}_v}),\end{aligned}$$

where $\mathrm{Div}_{\mathrm{dis}}(\mathcal{H}_p)$ is the group of divisors with discrete support. The induced map

$$\mathrm{MS}(\mathrm{Div}_{\mathrm{dis}}(\mathcal{H}_p))^\Gamma \to \mathrm{MS}(\mathbb{Z})^{\Gamma_v} = 0$$

has to be the zero map. Applying this map to $\Delta_\tau \in \mathrm{MS}(\mathrm{Div}_{\mathrm{dis}}(\mathcal{H}_p))^\Gamma$ defined by

$$\Delta_\tau\{r,s\} := \sum_{w\in\Gamma\tau} \delta_{r,s}(w)[w] \in \mathrm{Div}_{\mathrm{dis}}(\mathcal{H}_p)$$

gives the desired result. The statement about $\mathcal{W}_v$ is proved in the same way. □

Lemma 5.2.

$$\phi_\tau(z) := \sum_{w\in\Sigma_\tau(0,\infty)} \delta_{0,\infty}(w)\frac{1}{z-w}$$

converges to a function in $\mathcal{R}_2$ in the rigid analytic topology.

Proof. Once we show convergence, the proof of $\phi_\tau \in \mathcal{R}_2$ is identical to the argument of 4.2.
Fix $n \ge 0$. We must show that

$$\sum_{w\in\Sigma_\tau(0,\infty)\cap\mathcal{H}_p^{\le N}} \delta_{0,\infty}(w)\frac{1}{z-w}$$

converges as $N\to\infty$ on $\mathfrak{H}_p^{\le n}$ with respect to the supremum norm.
Case 1. $v_p(\mathrm{disc}(\tau))$ is even.
In this case, 5.1 tells us that every $w \in \Gamma\tau$ reduces to a vertex of $\mathcal{T}$. It suffices to show that

$$\left\| \sum_{w\in\Sigma_\tau(0,\infty)\cap\mathcal{A}_v} \delta_{0,\infty}(w)\frac{1}{z-w} \right\| \to 0 \text{ as } N := \mathrm{dist}(v,v_0)\to\infty$$

where v ranges over vertices of $\mathcal{T}$. This follows from the estimate

$$\left|\frac{1}{z-w_1} - \frac{1}{z-w_2}\right| = \left|\frac{w_1-w_2}{(z-w_1)(z-w_2)}\right| \le \frac{p^{-N}}{p^{-n}\cdot p^{-n}} = p^{2n-N}$$

for $z \in \mathfrak{H}^{\le n}$ and $w_1, w_2 \in \mathcal{A}_v$.
Case 2. $v_p(\mathrm{disc}(\tau))$ is odd.
In this case, 5.1 tells us that every $w \in \Gamma\tau$ reduces to the midpoint of an edge of $\mathcal{T}$. Instead of $\mathcal{A}_v$, we use $\mathcal{W}_v$ where v ranges over vertices of $\mathcal{T}$ for which $\mathrm{dist}(v,v_0)$ is even. The rest is the same. □

Here is the main theorem of this section.

Theorem 5.3. Any $\phi \in \mathcal{R}_2$ is a finite linear combination of ϕ_τ for $\tau \in \Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}$ and an analytic period function of weight 2.

Proof. Note that $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is 2-torsion in $\mathrm{PGL}_2(\mathbb{Z}[1/p])/\mathrm{PSL}_2(\mathbb{Z}[1/p])$. It induces an involution ϖ_p on $\mathcal{R}_2$ given by

$$(\varpi_p\phi)(z) := -(\phi|_2 P)(z) = -p\phi(pz).$$

We say that $\phi \in \mathcal{R}_2$ is *p-even* (resp. *p-odd*) if $\varpi_p\phi = \phi$ (resp. $\varpi_p\phi = -\phi$). Any $\phi \in \mathcal{R}_2$ can be broken down into a p-even part and a p-odd part. So, we may assume that ϕ is either p-even or p-odd.
Let Σ_ϕ be the set of poles of ϕ. Then, each $\Sigma_\phi^{\leq n} := \Sigma_\phi \cap \mathfrak{H}_p^{\leq n}$ must be a finite set. Running the proof of 4.3 again shows that $\Sigma_\phi^{\leq n}$ is a disjoint union of $\Sigma_\tau^\circ(0,\infty)$ with $\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash(\mathfrak{H}_p^{\mathrm{RM}} \cap \mathfrak{H}^{\leq n})$, and that we can take a finite linear combination of ϕ_τ such that the difference

$$\phi' := \phi - \sum_\tau \lambda_\tau \phi_\tau$$

is a meromorphic period function of weight 2 that is still p-even or p-odd and has no poles in $\mathfrak{H}_p^{\leq 1}$.
It remains to show that ϕ' is analytic. Assume that there is a pole w of ϕ'. We already showed $w \in \mathfrak{H}^{\mathrm{RM}}$. Since ϕ' is either p-even or p-odd, w/p^n is also a pole of ϕ' for all $n \in \mathbb{Z}$. By 5.1, we have $w/p^n \in \mathfrak{H}_p^{\leq 1}$ for sufficiently large n, which is a contradiction. □

There is also a classification theorem for analytic cocycles of weight 2, which appears in Chapter 2.5 of [DV21].

Theorem 5.4. There is an isomorphism

$$\mathrm{MS}(\mathcal{O}_2)^\Gamma \cong \mathrm{MS}(\mathbb{C}_p)^{\Gamma_0(p)}.$$

References

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