A Geometric Theory of Quantisation

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1 Introduction

Quantum mechanics has been perceived as a beautiful yet mysterious description of our world. Its implications, such as the probabilistic nature of wavefunctions, or the wave-particle duality, have baffled many physicists and led to a century of great philosophical debates. Niels Bohr once said 'anyone who is not shocked by quantum mechanics has not understood it'.

Mathematics behind quantum mechanics, however, seems to be less mysterious. In classical mechanics, an observable is defined as a function of position x and momentum p. One example is the energy of a particle, which is given by

$$H = H(x, p) = \frac{p^2}{2m} + V(x).$$

In quantum mechanics, on the other hand, observables are Hermitian operators. For example, position and momentum have quantum counterparts

$$\hat{x} = x, \quad \hat{p} = -i\hbar\nabla,$$

and the Hamiltonian of a particle is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = -\frac{\hbar^2}{2m}\nabla^2 + V(x).$$

So, while it is true that the consequences of quantum theory are weird and counterintuitive, the mathematical foundation is really akin to that of classical theory, at least on a formal level.

Thus it is natural to ask if there is a correspondence between classical and quantum systems. In particular, we are interested in promoting functions f to operators $\hat{f}$ which obey certain compatibility conditions. One naive approach would be to replace every occurrence of x and p in f by $\hat{x}$ and $\hat{p}$, so that

$$\hat{f} = f\left(x, -i\hbar\nabla\right).$$

But this isn't very satisfactory. It does not specify the space on which $\hat{f}$ acts on; it is ill-defined since $\hat{x}$ and $\hat{p}$ do not commute while x and p do; and the construction might be dependent on a choice of coordinates.

Geometric quantisation is a quantisation scheme developed by Bertram Kostant [5] and Jean-Marie Souriau [7] in the 1970s, which attempts to construct quantum operators in an unambiguous manner. This is usually done in four stages: prequantisation, polarisation, metaplectic

correction, and quantisation. In order to yield a coordinate-free theory of quantisation, it is essential to work with abstract differential geometry, in which one can make geometrically intrinsic constructions.

We will start by having a brief review of the rigorous foundations of classical and quantum mechanics. Then we will see that there is an analogy between the symmetries of classical and quantum systems, enabling a meaningful discussion on quantisation. Having done preliminary work, we study how the full quantisation scheme runs.

2 Geometrising classical mechanics

Newton's idea that all natural phenomena can be described by a small set of mathematical axioms is probably the most thought-provoking concept in the history of physics. In *Principia*, Newton gave the three laws of motion, encapsulating all motions in the universe in a single equation

$$F = ma.$$

Although the Newtonian framework is conceptually straightforward, it isn't the best setting to perform quantisation.

Lagrangian formalism

Lagrange was the first to reformulate classical mechanics. One of his motivations came from constrained motions. Imagine a simple pendulum whose motion is restricted to a circle due to the tension force. In a Newtonian setting, we decompose the gravity and tension into their horizontal and vertical components, then apply Newton's second law to predict the motion. However, the motion has only one degree of freedom namely the angle with the vertical. Thus we are complicating the problem by introducing unnecessary variables. The problem with Newtonian mechanics is that it only works in Cartesian coordinates.

Lagrangian mechanics is rather based on the *principle of least action*, whose historical origin goes up to the principle of least time. Fermat suggested that light always travels a path which takes the shortest time. Light follows a straight line in a vacuum, whereas it gets refracted under a change in a medium as a straight line is no longer the fastest path.
In a Lagrangian setting, every path has a property called *action*. The principle of least action states that the actual path taken by a physical system is the one making the action stationary. Hence dynamics is rephrased as an optimisation problem. We are now free to choose arbitrary non-Cartesian coordinates, since the action functional is an intrinsic property of paths.

Let's get into some actual maths. Consider a single particle in a Euclidean space subject to a conservative force with potential V. The *Lagrangian* of the particle is a function of position x and velocity v defined by

$$L = L(x, v) := \frac{1}{2}mv^2 - V(x).$$

Each path $x = x(t)$ $(0 \leq t \leq T)$ is associated with the *action* defined by

$$S = S[x] := \int_0^T L(x, \dot{x})dt.$$

Theorem. The actual path taken by a physical system is a stationary point of S.

Proof. The Euler-Lagrange equations for S are

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = -\frac{\partial V}{\partial x^a} - m\ddot{x}_a = 0.$$

These are equivalent to

$$m\ddot{x} = -\nabla V$$

which is precisely Newton's second law. □

Returning back to constrained motions, suppose that a particle has Lagrangian L but is restricted to an n-dimensional submanifold $Q \subseteq \mathbb{R}^m$. Locally, there exist functions $f^1, \cdots, f^{m-n}$ for which Q is the locus $f^1 = \cdots = f^{m-n} = 0$. If $\{q^a\}$ is a local coordinate system of Q, then $\{q^a, f^b\}$ is a local coordinate system of $\mathbb{R}^m$.
We consider a new Lagrangian $L' = L + \lambda_b f^b$ where λ_b are Lagrange multipliers. As mentioned earlier, Lagrange's equations hold for any local coordinates, so we have

$$\frac{\partial L'}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}^a}\right) = 0$$

or equivalently

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \lambda_b \frac{\partial f^b}{\partial q^a}.$$

But the right-hand side vanishes as $f^b = 0$ on Q, so we obtain

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0.$$

These are Lagrange's equations on Q.

It is possible to axiomatise Lagrangian mechanics in a completely abstract setting. The space of position is represented by a smooth manifold Q called the *configuration space*. If a particle is moving on Q following a path $q = q(t)$, then the *generalised velocity* $\dot{q} \in T_qQ$ is a tangent vector based at q. The *velocity phase space* is defined as the tangent bundle TQ and represents all possible combinations of position and velocity.
A *Lagrangian* can be any smooth function on TQ. The dynamics of a system is determined by specifying $L \in C^\infty(TQ)$, and the equations of motion are the corresponding Lagrange's equations

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0. \tag{1}$$

Hamiltonian formalism

Lagrange's equations for an n-dimensional system give n second-order differential equations. One may want to decouple these into $2n$ first-order differential equations. An obvious thing to try is to substitute $v^a = \dot{q}^a$, but this doesn't help too much. A better substitution is to use $p_a = \partial L/\partial \dot{q}^a$, the *generalised momenta*.

Mathematics behind this is the *Legendre transformation*. For a convex function $f(x)$ on a convex subdomain of $\mathbb{R}^n$, its Legendre transform is defined by

$$f^*(y) := x \cdot y - f(x) \quad \text{where} \quad \nabla f(x) = y.$$

One can show that $\nabla f(x) = y$ implies $\nabla f^*(y) = x$. In particular, it is always true that $f^{**} = f$. So there is a nice duality between x and y, and between f and f^*.

The *Hamiltonian* is defined as the Legendre transform of the Lagrangian

$$H = H(q,p) := p \cdot \dot{q} - L(q,\dot{q}) \quad \text{where} \quad \frac{\partial L}{\partial \dot{q}^a} = p_a.$$

Substituting $p_a = \partial L/\partial \dot{q}^a$ into Lagrange's equations (1) gives

$$\dot{p}_a = \frac{\partial L}{\partial q^a} = -\frac{\partial H}{\partial q^a}. \tag{2}$$

Moreover, Legendre duality implies

$$\dot{q}^a = \frac{\partial H}{\partial p_a}. \tag{3}$$

The combination of (2) and (3) is a beautiful set of anti-symmetrical equations, which are known as *Hamilton's equations*.

The substitution $p_a = \partial L/\partial \dot{q}^a$ involves a choice of local coordinates, so one may hope to have an intrinsic construction of p. We start from an observation that

$$dL = \frac{\partial L}{\partial q^a} dq^a + \frac{\partial L}{\partial \dot{q}^a} d\dot{q}^a \in \Omega^1(TQ)$$

is a 1-form on TQ. Under the natural isomorphism

$$T^*_{(q,v)}(TQ) \simeq T^*_q Q \times T^*_q Q,$$

$dL_{(q,v)}$ is mapped to

$$\left(\frac{\partial L}{\partial q^a} dq^a,\ \frac{\partial L}{\partial \dot{q}^a} dq^a \right) \in T^*_q Q \times T^*_q Q.$$

The second component provides an intrinsic definition of $p \in T^*_q Q$ as a cotangent vector. Hence in a geometric perspective, the Legendre transformation is a smooth map $\varphi : TQ \to T^*Q$.

The cotangent bundle $M = T^*Q$ thus represents all possible combinations of position and momentum. It is called the *momentum phase space* or simply *phase space*. Assume, for now, that the Legendre transformation $\varphi : TQ \to T^*Q$ is a diffeomorphism. Then the Hamiltonian can be rewritten as

$$H = H(q,p) = p(q) - L(\varphi^{-1}(q,p)).$$

In particular, $H \in C^\infty(T^*Q)$ is a smooth function on T^*Q.

We define an *observable* as a smooth function on T^*Q. If $f \in C^\infty(T^*Q)$ is an observable and $(q,p) = (q(t),p(t))$ is a solution to Hamilton's equations, then

$$\frac{df}{dt} = \frac{\partial f}{\partial q^a}\dot{q}^a + \frac{\partial f}{\partial p_a}\dot{p}_a = \frac{\partial f}{\partial q^a}\frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H}{\partial q^a}.$$

This motivates us to introduce the *Poisson bracket*

$$\{f, g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}$$

so that one may simply write

$$\frac{df}{dt} = \{f, H\}.$$

Note that replacing f by q^a or p_a recovers Hamilton's equations.

It is again possible to axiomatise Hamiltonian mechanics abstractly. Let Q be a configuration space and $M = T^*Q$ be the phase space. A *Hamiltonian* can be any smooth function on M. The dynamics is determined by specifying $H \in C^\infty(M)$, and the equation of motion is

$$\frac{df}{dt} = \{f, H\}. \tag{4}$$

The Poisson bracket satisfies four important properties: for $f, g, h \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$,

(i) $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$, (linearity)

(ii) $\{f, g\} = -\{g, f\}$, (anti-symmetry)

(iii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, (Jacobi identity)

(iv) $\{fg, h\} = \{f, h\}g + f\{g, h\}$, (Leibniz rule)

In particular, $\{\cdot, \cdot\}$ is a Lie bracket on $C^\infty(M)$ which satisfies the Leibniz rule. It is reasonable to make the following definition:

Definition. A *Poisson manifold* is a manifold M with a *Poisson bracket*

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying (i)-(iv).

Symplectic Mechanics

Hamilton's equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}. \tag{5}$$

involve a choice of local coordinates. We desire to have coordinate-free mechanics, so we hope to find an intrinsic way of describing (5). To do this, we regard the solutions to (5) as integral curves of a certain vector field. Such a vector field must be given by

$$X_H = \dot{q}^a\frac{\partial}{\partial q^a} + \dot{p}^a\frac{\partial}{\partial p^a} = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p^a} \in V(M).$$

Observe that

$$dH = \frac{\partial H}{\partial q^a}dq^a + \frac{\partial H}{\partial p_a}dp^a \in \Omega^1(M)$$

is a 1-form on M. To transform dH to X_H, what we need is a linear map $T_mM \to T_m^*M$, or equivalently a bilinear map $T_mM \times T_mM \to \mathbb{R}$.

Fortunately, the tangent bundle $M = T^*Q$ admits the *canonical* 2*-form* defined by

$$\omega := dq^a \wedge dp_a.$$

Now we have

$$\omega\left(\frac{\partial}{\partial q^a}\right) = dp_a, \quad \omega\left(\frac{\partial}{\partial p_a}\right) = -dq^a$$

so that

$$\omega(X_H) = dH.$$

Moreover, X_H is the unique vector field with this property since ω is non-degenerate. We call X_H the *Hamiltonian vector field* generated by H. This is the intrinsic description of Hamilton's equations.

The canonical 2-form ω is clearly non-degenerate. It is also exact since $M = T^*Q$ admits the *canonical* 1*-form* defined by

$$\theta := -p_a dq^a$$

for which $\omega = d\theta$. In particular, ω is closed. Now we make the following definition:

Definition. A *symplectic manifold* is a manifold M with a closed non-degenerate 2-form.

Lemma. Symplectic manifolds are naturally Poisson manifolds.

Proof. Let (M, ω) be a symplectic manifold. According to Darboux's theorem, every point of M has a local coordinate system $\{q^a, p_b\}$ for which $\omega = dq^a \wedge dp_a$. We have

$$\omega(X_f, X_g) = \omega\left(\frac{\partial f}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial f}{\partial q^a}\frac{\partial}{\partial p^a}, \frac{\partial g}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial g}{\partial q^a}\frac{\partial}{\partial p^a}\right) = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}.$$

Hence defining

$$\{f, g\} := \omega(X_f, X_g)$$

gives a natural Poisson structure on M. □

Why do we generalise cotangent bundles and care about abstract symplectic manifolds? The reason is to take account of more mechanical problems, which do appear in reality. We previously assumed that the Legendre transformation $\varphi : TQ \to T^*Q$ is a diffeomorphism. But in general, φ need not be a diffeomorphism nor even a local one. Thus we may lose some information when transporting from Lagrangian to Hamiltonian mechanics.

It turns out that the Lagrangian approach directly fits in a symplectic framework. Given a Lagrangian $L \in C^\infty(TQ)$,

$$\omega_L = \frac{\partial^2 L}{\partial q^a \partial v^b} dq^a \wedge dq^b - \frac{\partial^2 L}{\partial v^a \partial v^b} dq^a \wedge dv^b$$

defines a 2-form on TQ. It is exact since TQ admits a 1-form

$$\theta_L = \frac{\partial L}{\partial v^a} dq^a.$$

One may show that the solutions to (1) are generated by a unique vector field $X \in V(TQ)$ satisfying

$$\omega_L(X) = d\left(\frac{\partial L}{\partial v^a} dv^a - L\right).$$

The only remaining issue is that ω_L might be degenerate. If

$$\det\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right) \neq 0,$$

then L is said to be *regular*, and ω_L gives a symplectic structure on TQ. Otherwise there is no guarantee of non-degeneracy, but provided that the *characteristic foliation* has constant rank, (TQ, ω_L) is a *presymplectic manifold* and its *reduction* produces a symplectic manifold, although information might be lost.

Hamilton-Jacobi theory

Hamilton-Jacobi theory provides yet another formulation of classical mechanics. Recall that the Lagrangian approach is based on position and velocity, whereas the Hamiltonian approach is based on position and momentum. Hamilton-Jacobi theory will reformulate mechanics in terms of position and action.

Let $L \in C^\infty(TQ)$ be a Lagrangian and fix an initial point $q_0 \in Q$. Under moderate assumptions, there is a unique solution $q = q(t)$ $(0 \leq t \leq T)$ to Lagrange's equations joining $q(0) = q_0$ and $q(T) = q_1$. We define

$$S = S(q_1, T) := \int_0^T L(q, \dot{q}) dt$$

where the integral is along that unique path. The variation of S is given by

$$\delta S = \int_0^T \left(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\right) \delta q^a dt + \left|\frac{\partial L}{\partial \dot{q}^a} \delta q^a\right|_0^T = \left.\frac{\partial L}{\partial \dot{q}^a}\right|_T \delta q^a,$$

so Hamiltonian's equations imply

$$\frac{\partial S}{\partial q^a} = p_a. \tag{6}$$

In particular, S contains all information about momenta. Thus it is reasonable to rewrite Hamilton's equations in terms of q and S. From (6), we have

$$\frac{dS}{dT} = \frac{\partial S}{\partial T} + \frac{\partial S}{\partial q^a}\dot{q}^a = \frac{\partial S}{\partial T} + p_a \dot{q}^a.$$

At the same time, the fundamental theorem of calculus implies

$$\frac{dS}{dT} = L.$$

Combined together, we get

$$\frac{\partial S}{\partial T} = L - p_a \dot{q}^a = -H. \tag{7}$$

If we substitute (6) into (7), then we obtain the *Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial T} = -H\left(q, \nabla S\right). \tag{8}$$

Conversely, suppose that S is a solution to (6) and (8). Then

$$\begin{aligned}\dot{p}_a = \frac{d}{dt}\left(\frac{\partial S}{\partial q^a}\right) &= \frac{\partial^2 S}{\partial t \partial q^a} + \frac{\partial^2 S}{\partial q^b \partial q^a}\dot{q}^b \\ &= -\frac{\partial H}{\partial q^a} - \frac{\partial H}{\partial p_b}\frac{\partial p_b}{\partial q^a} + \frac{\partial^2 S}{\partial q^b \partial q^a}\dot{q}^b \\ &= -\frac{\partial H}{\partial q^a} - \frac{\partial^2 S}{\partial q^b \partial q^a}\dot{q}^b + \frac{\partial^2 S}{\partial q^b \partial q^a}\dot{q}^b \\ &= -\frac{\partial H}{\partial q^a},\end{aligned}$$

so the generalised momenta $p_a = \partial S/\partial q^a$ satisfy Hamilton's equations.

3 Axiomatising quantum mechanics

In the 1920s, Heisenberg invented *matrix mechanics* which was the first successful mathematical foundation of quantum mechanics. However, physicists at the time were not familiar with Heisenberg's idea of using linear algebra as a foundation of physics. They often preferred *wave mechanics* subsequently introduced by Schrödinger, which uses differential equations to formulate quantum mechanics.
It was Paul Dirac who brought reconciliation between the two theories. In *Principles of Quantum Mechanics* [4], Dirac gave a new foundation of quantum mechanics based on Hilbert spaces and showed that Heisenberg and Schrödinger pictures are just two different perspectives on the time evolution of Hilbert spaces.

Consider a particle on a configuration space Q subject to a potential V. Suppose that Q has a non-vanishing volume form ε. In quantum mechanics, the particle is no longer represented by definite position and momentum. Its state is rather given by a *wavefunction* $\psi \in L^2(Q)$ which is a square-integrable complex function on Q.
The Hilbert space $\mathcal{H} = L^2(Q)$ has a Hermitian inner product defined by

$$\langle \psi_1, \psi_2 \rangle := \int_Q \overline{\psi}_1 \psi_2 \varepsilon.$$

We say two wavefunctions $\psi_1, \psi_2 \in L^2(Q)$ are *physically equivalent* if there is a nonzero constant $c \in \mathbb{C} \setminus \{0\}$ such that $\psi_1 = c\psi_2$. Hence one can always rescale a nonzero ψ and *normalise* it to make it have the unit norm

$$\|\psi\| = \left(\int_Q |\psi|^2 \varepsilon\right)^{1/2} = 1.$$

Under the Copenhagen interpretation, the squared modulus $|\psi|^2$ of a normalised wavefunction ψ represents the probability density of finding the particle at position x.

We define an *observable* as a Hermitian operator on $L^2(Q)$. The position operator $\hat{q}$ and the momentum operator $\hat{p}$ act on a wavefunction ψ via

$$(\hat{q}^a \psi)(q) := q^a \psi(q), \quad (\hat{p}_a \psi)(x) := -i\hbar \frac{\partial \psi}{\partial q^a}(q)$$

where $\hbar$ is the Planck constant. There is also the Hamiltonian operator defined by

$$\hat{H} := \frac{\hat{p}^2}{2m} + V(\hat{x})$$

which acts on ψ via

$$(\hat{H}\psi)(q) = -\frac{\hbar^2}{2m}\nabla^2\psi(q) + V(q)\psi(q).$$

Eigenvalues of Hermitian operators are always real, and eigenvectors associated to distinct eigenvalues are always orthogonal.

The dynamics of a system is given by the *Schrödinger's equation*

$$i\hbar\dot{\psi} = \hat{H}\psi. \tag{9}$$

If $\psi = \psi(t)$ is a solution to the Schrödinger's equation, then

$$\begin{aligned}\frac{\partial}{\partial t}\langle\psi,\psi\rangle &= \langle\psi,\dot{\psi}\rangle + \langle\dot{\psi},\psi\rangle \\ &= -\frac{i}{\hbar}\langle\psi,\hat{H}\psi\rangle + \frac{i}{\hbar}\langle\hat{H}\psi,\psi\rangle = 0\end{aligned}$$

since $\hat{H}$ is Hermitian. Hence normalised wavefunctions stay normalised, which is consistent with the Copenhagen interpretation.
If modulus represents probability amplitude, what does phase represents? Write $\psi = R\exp(iS/\hbar)$, then the Schrödinger's equation reads

$$i\hbar\left(\frac{\partial R}{\partial t} + \frac{iR}{\hbar}\frac{\partial S}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(\nabla^2 R + \frac{2i}{\hbar}\nabla R\cdot\nabla S + \frac{iR}{\hbar}\nabla^2 S - \frac{R}{\hbar^2}\nabla S\cdot\nabla S\right) + VR.$$

Neglecting all the higher order terms, we obtain

$$\frac{\partial S}{\partial t} = -\left(\frac{(\nabla S)^2}{2m} + V\right) + O(\hbar).$$

In the classical limit $\hbar \to 0$, this coincides with the Hamilton-Jacobi equation (8). Thus phase can be thought of as the quantum counterpart of classical action.

There are even more cases where quantum mechanics comes close to classical mechanics. Given an observable $\hat{A}$ and a normalised wavefunction ψ, we define its *expected value* by

$$\langle\hat{A}\rangle_\psi := \langle\psi,\hat{A}\psi\rangle = \int_Q \overline{\psi}\hat{A}\psi\varepsilon.$$

Then we have the *Ehrenfest's theorem*

$$\begin{aligned}\frac{d}{dt}\langle\hat{A}\rangle_\psi &= \langle\psi,\hat{A}\dot{\psi}\rangle + \langle\dot{\psi},\hat{A}\psi\rangle \\ &= -\frac{i}{\hbar}\langle\psi,\hat{A}\hat{H}\psi\rangle + \frac{i}{\hbar}\langle\hat{H}\psi,\hat{A}\psi\rangle \\ &= -\frac{i}{\hbar}\langle[\hat{A},\hat{H}]\rangle_\psi\end{aligned}$$

where $[\hat{A},\hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A}$ is the commutator. This is very reminiscent of (4). Indeed, the commutator satisfies the same four conditions for a Poisson bracket:

(i) $[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h]$, (linearity)

(ii) $[f, g] = -[g, f]$, (anti-symmetry)

(iii) $[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$, (Jacobi identity)

(iv) $[fg, h] = [f, h]g + f[g, h]$. (Leibniz rule)

Thus we see that the commutator is analogous to a Poisson bracket.

4 Symmetries

Symmetry is one of the most important concepts in physics. In 1915, Noether proved the celebrated *Noether's theorem*, stating that every continuous symmetry of a system induces a corresponding conservation law. In Newtonian mechanics, for example, the laws of physics are invariant under Galilean transformations; spacial translations correspond to the conservation of momentum; and time translations correspond to the conservation of energy.
Henceforth we will study the symmetries of classical and quantum systems. The symmetries of a symplectic manifold are Hamiltonian actions, whereas the symmetries of a Hilbert space are unitary representations. We will see that there are formal similarities between the two theories.

Symmetries of a classical system

Let (M, ω) be a connected symplectic manifold. We consider smooth symmetries of M which preserve the symplectic structure.

Definition. A left action ρ of a Lie group G on M is *symplectic* if $\rho(g) : M \to M$ is symplectomorphism for each $g \in G$.

Consider an element $g = g(t)$ of G changing in time, starting from the identity element $g(0) = 1$. The local behaviour of an action ρ is determined by the difference between $g \cdot m$ and m as $t \to 0$, which can be approximated by a tangent vector in $T_m M$.
Let $\mathfrak{g}$ be the Lie algebra associated with G. Formally, ρ induces a Lie algebra anti-homomorphism

$$\phi : \mathfrak{g} \to V(M)$$

defined by

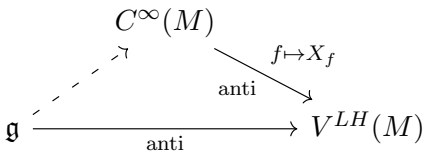
$$\phi(A) := \left(m \mapsto \left. \frac{d}{dt} \right|_0 (\exp(tA) \cdot m) \right)$$

where $\exp : \mathfrak{g} \to G$ is the exponential map. Write $X_A \in V(M)$ for the image of $A \in \mathfrak{g}$.

If ρ is required to be symplectic, then each X_A is *locally Hamiltonian*. That is, there locally exists a function f with $\omega(X_A) = df$. Let $V^{LH}(M)$ and $V^H(M)$ be the set of locally and globally Hamiltonian vector fields respectively. One may show that these are subalgebras of $V(M)$. In particular, we obtain an anti-homomorphism $\mathfrak{g} \to V^{LH}(M)$.

There is another anti-homomorphism $C^\infty(M) \to V^H(M)$ given by $f \mapsto X_f$, where X_f is the Hamiltonian vector field generated by f. The kernel consists of constants, so $C^\infty(M)/\mathbb{R}$ and $V^H(M)$ are anti-isomorphic.

Now given a symplectic action, it is natural to ask whether the action is *Hamiltonian* or not. In other words, we want to determine if there is a factorisation:



There is a beautiful answer in terms of Lie algebra cohomology. Consider an anti-symmetric bilinear form $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. We say α is a 2-*cocycle* if

$$\alpha([A, B], C) + \alpha([B, C], A) + \alpha([C, A], B) = 0$$

and a 2-*coboundary* if there is a linear map $\beta : \mathfrak{g} \to \mathbb{R}$ with

$$\alpha(A, B) = \beta([A, B]).$$

The definitions of cocycles and coboundaries are analogous to that of Čech cohomology. Let $Z^2(\mathfrak{g})$ and $B^2(\mathfrak{g})$ denote the sets of 2-cocycles and 2-coboundaries respectively. It is easy to check $B^2(\mathfrak{g}) \leq Z^2(\mathfrak{g})$, and the second cohomology group is defined as the quotient

$$H^2(\mathfrak{g}) := Z^2(\mathfrak{g})/B^2(\mathfrak{g}).$$

Theorem. Let $\phi : \mathfrak{g} \to V^{LH}(M)$ be a symplectic action. Fix any $m \in M$, then $\Omega(A, B) := \omega_m(X_A, X_B)$ defines a well-defined cohomology class $[\Omega] \in H^2(\mathfrak{g})$ independent of m. Furthermore the action is Hamiltonian if and only if $(\operatorname{im} \phi) \leq V^H(M)$ and $[\Omega] = 0$.

Proof. See [9], p 40. □

A rather special situation occurs when the second cohomology group vanishes. If $H^2(\mathfrak{g}) = 0$ then $[\Omega] = 0$ automatically, so it only remains to check $(\operatorname{im} \phi) \leq V^H(M)$.

Symmetries of a quantum system

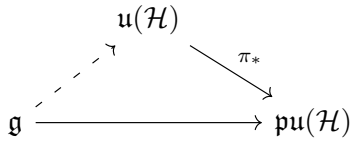
Let $\mathcal{H}$ be a Hilbert space. We want to consider smooth symmetries of $\mathcal{H}$ which preserve inner products, but this is too strict. Recall that two wavefunctions are equivalent if they differ by a scalar multiple. Thus we should consider the modified group action law which only requires

$$\rho(gh) = c(g, h)\rho(g)\rho(h) \quad \forall\, g, h \in G$$

for a constant $c(g, h) \in \mathbb{C} \setminus \{0\}$ dependent on g and h. Together with the requirement that ρ should preserve the inner product, what we get is a *projective unitary representation.*

Definition. A *projective unitary representatioin* of a Lie group G on $\mathcal{H}$ is a Lie group homomorphism $\rho : G \to PU(\mathcal{H})$.

Its pushforward induces a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{pu}(\mathcal{H})$ which determines the local behaviour of ρ. As in the classical case, we come up with a natural question asking whether the action can be lifted to an actual unitary representation or not. In other words, we want to determine if there is a factorisation:



Here, the homomorphism $\pi_* : \mathfrak{u}(\mathcal{H}) \to \mathfrak{pu}(\mathcal{H})$ is the pushforward of the projection map $\pi : PU(\mathcal{H}) \to U(\mathcal{H})$. The answer is again pretty much related to the Lie algebra cohomology of $\mathfrak{g}$.

Theorem. Let $\phi : \mathfrak{g} \to \mathfrak{pu}(\mathcal{H})$ be a projective unitary representation. If $H^2(\mathfrak{g}) = 0$ then ϕ lifts to a unitary representation.

Proof. See [2]. □

Finally, note that $\mathfrak{u}(\mathcal{H})$ consists of skew-Hermitian operators on $\mathcal{H}$. These are the same as Hermitian operators multiplied by i. So one may write $\mathfrak{u}(\mathcal{H}) = -(i/\hbar)O(\mathcal{H})$ where $O(\mathcal{H})$ is the set of Hermitian operators on $\mathcal{H}$.

5 Quantisation viewed from Representation Theory

We have seen that there is a formal relationship between Hamiltonian actions and unitary representations. If there ever exists a theory of quantisation, it should be the case that a classical system and its quantum counterpart share the same symmetries.

A natural thing to do is to construct a Lie algebra homomorphism $C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ so that any Hamiltonian action induces a unitary representation via composition. Let's say $f \in C^\infty(M)$ is mapped to $-(i/\hbar)\hat{f} \in \mathfrak{u}(\mathcal{H})$ where $\hat{f} \in O(\mathcal{H})$. The homomorphism condition reads

$$\left[-\frac{i}{\hbar}\hat{f}, -\frac{i}{\hbar}\hat{g}\right] = -\frac{i}{\hbar}\widehat{\{f, g\}},$$

or equivalently

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}. \tag{10}$$

Historically, the analogy between Poisson bracket and commutator is found by Dirac [4]. On a symplectic manifold, one has the Poisson bracket relation

$$\{q^a, q^b\} = 0, \quad \{q^a, p_b\} = \delta^a_b, \quad \{p_a, p_b\} = 0.$$

On a Hilbert space, on the other hand, one has the canonical commutation relation

$$[\hat{q}^a, \hat{q}^b] = 0, \quad [\hat{q}^a, \hat{p}_b] = i\hbar\delta^a_b, \quad [\hat{p}_a, \hat{p}_b] = 0.$$

Dirac's observation motivated himself to suggest the *canonical quantisation rule*. Whenever we promote functions f to operators $\hat{f}$, the Poisson bracket and the commutator should be compatible in the sense that (10) holds.

We therefore arrive at the mathematical definition of quantisation. A *quantisation* of a symplectic manifold M is a Hilbert space $\mathcal{H}$ together with a map

$$Q : C^\infty(M) \to O(\mathcal{H})$$

which satisfies:

(Q1) $f \mapsto \hat{f}$ is $\mathbb{R}$-linear, (linearity)

(Q2) $[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}$ holds, (homomorphism)

(Q3) the constant 1 is mapped to $\mathrm{id}_{\mathcal{H}}$. (commutation relation)

However, such maps are not unique as one can make $\mathcal{H}$ as huge as we want. To make a quantisation physically reasonable, one further imposes a condition on elementary systems namely that:

(Q4) any transitive Hamiltonian action induces an irreducible unitary representation.

The *prequantisation scheme* we will meet soon produces a Hilbert space and quantum observables satisfying (Q1)-(Q3). Unfortunately, the resulting Hilbert space is too big, and the irreducibility condition (Q4) fails. Indeed, mathematicians and physicists found obstructions, such as Groenewold's theorem, claiming (Q1)-(Q4) cannot be satisfied at the same time.
Geometric quantisation avoids this problem by restricting $C^\infty(M)$ to a certain subalgebra. However, this process involves an arbitrary choice of a *polarisation* of M. Even then, the result is still unsatisfactory because the construction might fail or give physically incorrect answers. This is fixed by an arbitrary choice of a metaplectic correction on M. Equipped with these additional choices and structures, one may able to produce a quantisation satisfying (Q1)-(Q4).

6 Prequantisation

Let (M, ω) be a symplectic manifold. It has a natural non-vanishing volume form $\varepsilon = (\omega/2\pi\hbar)^n$ called the *Liouville form*. We can easily imagine a Hilbert space $\mathcal{H} = L^2(M)$ of square-integrable complex functions on M with the Hermitian inner product

$$\langle \psi_1, \psi_2 \rangle := \int_M \overline{\psi_1}\psi_2\varepsilon.$$

One may attempt to construct a map $C^\infty(M) \to O(\mathcal{H})$ by taking

$$\hat{f}\psi := -i\hbar X_f(\psi).$$

This satisfies (Q1) and (Q2) but violates (Q3). So we make a modification

$$\hat{f}\psi := -i\hbar X_f(\psi) + f\psi.$$

This satisfies (Q1) and (Q3) but (Q2) now fails. Adding yet another compensation term

$$\hat{f}\psi := -i\hbar X_f(\psi) + \theta(X_f)\psi + f\psi = -i\hbar\left(X_f(\psi) + \frac{i}{\hbar}\theta(X_f)\psi\right) + f\psi, \tag{11}$$

where θ is a local potential for which $\omega = d\theta$, reconcile all the three conditions. But in doing so, we have introduced an ambiguity θ which is non-unique and defined only locally. Therefore the construction fails unless ω is exact.
A rescue comes from an observation that the expression $X_f(\psi) + (i/\hbar)\theta(X_f)\psi$ is something we encounter in the theory of vector bundles. We have a brief review on connected Hermitian line bundles.

Complex line bundles

In differential geometry, connections on vector bundles are used to transport a vector from one fibre to another.

Definition. Let $\pi : B \to M$ be a complex vector bundle. A *connection* on B is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(B) \to \Gamma(T^*_{\mathbb{C}}M \otimes B)$$

which satisfies the Leibniz rule

$$\nabla(fs) = f\nabla s + df \otimes s \quad \forall\, f \in C^\infty(M) \quad \forall\, s \in \Gamma(B). \tag{12}$$

If ∇ is a connection and X is a vector field, then they induce the *covariant derivative* along X

$$\nabla_X : \Gamma(B) \to \Gamma(B)$$

defined by

$$\nabla_X s := (\nabla s)(X).$$

Intuitively, if $s \in \Gamma(B)$ and $X \in V(M)$, then $\nabla_X s \in \Gamma(B)$ measures the failure of s being parallel in the direction of X.

Suppose that B is a line bundle. Fix a local trivialisation $\tau : \pi^{-1}(U) \to U \times \mathbb{C}$ on some open set U. Then τ determines a local section

$$\sigma = \tau^{-1}(\cdot, 1) \in \Gamma(B|_U).$$

Moreover, there is a unique 1-form $\Theta \in \Omega^1_{\mathbb{C}}(U)$ such that

$$\nabla\sigma = \Theta \otimes \sigma. \tag{13}$$

Any local section $s \in \Gamma(B|_U)$ can be written uniquely as $s = \psi\sigma$ for some $\psi \in C^\infty(U)$. Then (12) and (13) together imply

$$\begin{aligned} \nabla s &= d\psi \otimes \sigma + \psi\nabla\sigma \\ &= (d\psi + \Theta\psi) \otimes \sigma \end{aligned} \tag{14}$$

and

$$\begin{aligned} \nabla_X(\psi s) &= (d\psi(X) + \Theta(X)\psi)s \\ &= (X(\psi) + \Theta(X)\psi)s, \end{aligned} \tag{15}$$

which looks suspiciously similar to (11).

Let $\gamma = \gamma(t)$ be a closed loop on M based at m. A section s on γ is said *parallel* if $\nabla_{\dot\gamma} s = 0$. Writing $s = \psi\sigma$, (15) shows that the parallelity condition reads

$$\dot\gamma(\psi) + \Theta(\dot\gamma)\psi = 0$$

or equivalently

$$\dot\psi = -\Theta(\dot\gamma)\psi.$$

This is a first-order linear equation and can be solved via integrating factor. If γ encloses a surface Σ, then Stokes' theorem implies

$$\psi(1) = \psi(0)\exp\left(-\int_\gamma \Theta\right) = \psi(0)\exp\left(-\int_\Sigma \Omega\right)$$

where $\Omega = d\Theta$. One can check that Ω is independent of choice of τ and Θ. This is called the *curvature* of ∇ and represents local holonomy.

A *Hermitian structure* on B is a Hermitian inner product $(\cdot,\cdot)_m : B_m \times B_m \to \mathbb{C}$ varying smoothly in m. We say such a structure is *compatible* with ∇ if

$$i_X(d(s_1, s_2)) = (\nabla_X s_1, s_2) + (s_1, \nabla_X s_2).$$

For sections $s_1, s_2 \in \Gamma(B)$, we define

$$\langle s_1, s_2\rangle := \int_M (s_1, s_2)\varepsilon$$

whenever the integral exists.

Prequantum line bundles

Observe that (11) and (15) share the same expression provided that $\Theta = i\theta/\hbar$ and $X = X_f$.

> **Definition.** A *prequantum line bundle* on M is a connected Hermitian line bundle (B, ∇) on M whose curvature is $i\omega/\hbar$.

Such a bundle need not exist in general. Imagine a closed orientable surface Σ in M, partitioned into Σ_1 and Σ_2 by a path γ. The curvature provides two different ways to calculate the holonomy of γ. They are

$$\exp\left(-\frac{i}{\hbar}\int_{\Sigma_1}\omega\right) = \exp\left(\frac{i}{\hbar}\int_{\Sigma_2}\omega\right),$$

where Σ_1 and Σ_2 have the same orientation as Σ. This implies

$$\exp\left(\frac{i}{\hbar}\int_\Sigma \omega\right) = 1$$

or equivalently

$$\frac{1}{2\pi\hbar}\int_\Sigma \omega \in \mathbb{Z}.$$

As the choice of Σ can be made arbitrary, de Rham isomorphism implies

$$\frac{\omega}{2\pi\hbar} \in H^2(M, \mathbb{Z}). \tag{16}$$

This is the *Bohr-Sommerfeld condition*, whose origin goes up to the old quantum theory. It is a necessary for the existence of a prequantum line bundle; in fact it is also a sufficient condition.

Theorem. A prequantum line bundle exists if and only if the Bohr-Sommerfeld condition is true. Once (16) is satisfied, the equivalence classes of prequantum line bundles are in bijection with $H^1(M, \mathbb{S}^1)$.

Proof. It suffices to prove that (16) implies the existence of a prequantum line bundle. Using the isomorphism between de Rham and Čech cohomology, there exists a contractible open cover $\mathcal{U} = \{U_i\}$ together with $\theta_i \in \Omega^1(U_i)$ and $u_{ij} \in C^\infty(U_i \cap U_j)$ satisfying

(i) $d\theta_i = \omega$,

(ii) $du_{ij} = \theta_i - \theta_j$,

(iii) $u_{ij} + u_{jk} + u_{ki} \in 2\pi\hbar\mathbb{Z}$.

Define $c_{ij} = \exp(iu_{ij}/\hbar)$, then c_{ij} satisfies the cocycle condition $c_{ij}c_{jk}c_{ki} = 1$. So there exists a connected line bundle with transition functions c_{ij} and curvature $i\omega/\hbar$.
Now fix a prequantum line bundle B_0. By taking a tensor product with B_0^{-1}, the prequantum line bundles are in one-to-one correspondence with the flat connected line bundles up to equivalence. A similar argument as above shows that this bijects to $H^1(M, \mathbb{S}^1)$. □

Prequantisatoin

Let (M, ω) be a symplectic manifold and (B, ∇) be a prequantum line bundle. We construct the *prequantum Hilbert space* as

$$\mathcal{H} := L^2(\Gamma(B)),$$

the space of square-integrable sections of B. If $f \in C^\infty(M)$ is a classical observable, then it promotes to a *prequantum operator* $\hat{f} \in O(\mathcal{H})$ defined by

$$\hat{f} := -i\hbar\nabla_{X_f} + f. \tag{17}$$

This geometrically intrinsic construction satisfies (Q1)-(Q3) and we are done.

Admittedly the formula (17) looks quite random, but there is a much more natural interpretaion. Fix $f \in C^\infty(M)$ and consider the Hamiltonian vector field X_f. Choose a local trivialisation $\tau : \pi^{-1}(U) \to U \times \mathbb{C}$ and its corresponding 1-form $i\theta/\hbar$. In local canonical coordinates,

$$\theta(X_f) = (-p_a da^a)\left(\frac{\partial f}{\partial p_b}\frac{\partial}{\partial q^b} - \frac{\partial f}{\partial q^b}\frac{\partial}{\partial p_b}\right) = -p_a\frac{\partial f}{\partial p_a}.$$

The quantity

$$L_f := -\theta(X_f) - f = p_a\frac{\partial f}{\partial p_a} - f$$

can be thought of as the analogue of *Lagrangian* in symplectic geometry. Now $\hat{f}$ becomes

$$\hat{f} = -i\hbar X_f + \theta(X_f) + f = -i\hbar X_f - L_f. \tag{18}$$

Let $\varphi_t : M \to M$ be the flow of X_f. It lifts to a unitary operator $\Phi_t : \mathcal{H} \to \mathcal{H}$ defined by

$$(\Phi_t \psi)(m) := \psi(\varphi_t(m)) \exp\left(-\frac{i}{\hbar}\int_0^t L_f dt\right). \tag{19}$$

The term $\psi(\varphi_t(m))$ contributes as a pushforward of the Hamiltonian flow, whereas the term $\exp(-iS/\hbar)$ reflects the fact that phase corresponds to action. Combining (18) and (19), one may show that

$$\hat{f}\psi = -i\hbar \frac{d}{dt}\bigg|_0 (\Phi_t \psi) = i\hbar \frac{d}{dt}\bigg|_0 (\Phi_{-t}\psi). \tag{20}$$

If we think that $\psi = \psi(t)$ is a wavefunction evolving through $\psi(t) := \Phi_{-t}\psi(0)$, then (20) reads

$$i\hbar\dot{\psi} = \hat{f}\psi$$

which is the Schrödinger's equation.

Example. Let Q be a configuration space and $M = T^*Q$ be the phase space. Consider the position and momentum observables $q^a, p_b \in C^\infty(M)$. If we choose a local potential to be the canonical 1-form $\theta = -p_a dq^a$, then we have

$$X_{q^a} = -\frac{\partial}{\partial p_a}, \quad X_{p_b} = \frac{\partial}{\partial q^b}$$

and

$$L_{q^a} = -q^a, \quad L_{p_b} = 0.$$

Plugging these into (18) yields the prequantum operators

$$\hat{q}^a = i\hbar\frac{\partial}{\partial p_a} + q^a, \quad \hat{p}_b = -i\hbar\frac{\partial}{\partial q^b}. \tag{21}$$

7 Geometry of polarisations

Mathematics has been elegant so far. Nevertheless, prequantisation has some problems. The prequantum Hilbert space is too huge to satisfy the irreducibility condition; and it does not match our physical theory as seen in (21) which contains an unnecessary $-i\hbar\partial/\partial p_a$ term.
These issues are leading us in the right direction. If Q is a configuration space, then the standard phase space is $M = T^*Q$ while the standard Hilbert space is $\mathcal{H} = L^2(Q)$. But prequantisation produces a Hilbert space $\mathcal{H}' = L^2(\Gamma(B))$ which is comparable to $L^2(T^*Q)$. Thus $\mathcal{H}'$ is much larger in dimension than expected. If we are able to recover Q from T^*Q and cut down the dimension into half, this will allow us to reduce the size of $\mathcal{H}$.

A cotangent bundle $M = T^*Q$ is naturally partitioned into its vertical subspaces $\Lambda_q = T^*_q Q$. One may recover Q from M by taking the quotient space. Moreover, if $\psi \in L^2(M)$ is constant on each leaf, i.e. independent of momentum p, then the prequantum operator $\hat{q}^a$ given in (21) acts on ψ by

$$\hat{q}^a\psi = q^a\psi$$

and the term $-i\hbar\partial/\partial p_a$ disappears as desired.
The vertical subspaces Λ_q are spanned by the vertical tangent vectors $\partial/\partial p_a$. In general, we will talk about *distributions*, i.e. subbundles of TM. An integrable distribution is called a *foliation*, and its maximal integral submanifolds are called the *leaves*. According to The vertical foliation of M has a special feature that each $T_m\Lambda_q \leq T_mM$ is a Lagrangian subspace, which we will define soon.

Real polarisation

Let (M, ω) be a symplectic manifold. The symplectic form ω induces a non-degenerate anti-symmetric bilinear form $\omega_m : T_mM \times T_mM \to \mathbb{R}$ for each $m \in M$.

Definition. A *symplectic vector space* is a vector space V with a non-degenerate anti-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. For a subspace $U \leq V$, define the *symplectic complement* by

$$U^\perp = \{v \in V \,|\, \omega(v, u) = 0 \;\forall\, u \in U\}.$$

We say U is

(i) *isotropic* if $W \leq W^\perp$,

(ii) *coisotropic* if $W^\perp \leq W$,

(iii) *Lagrangian* if $W = W^\perp$.

We hope to generalise vertical foliations of cotangent bundles. It is easy to check that the subspace spanned by the vertical tangent vectors $\partial/\partial p_a$ is a Lagrangian submanifold of $T_m(T^*Q)$.

Definition. A distribution P on M is *Lagrangian* (resp. *isotropic*, *coisotropic*) if P_m is a Lagrangian (resp. isotropic, coisotropic) subspace of T_mM for each $m \in M$. A *real polarisation* is a different name for Lagrangian foliation.

Thus cotangent bundles admit natural real polarisations. These are, in fact, locally generic examples in the sense that every real polarisation on a symplectic manifold is locally equivalent to some vertical polarisation.

Lemma. A real polarisation P on M induces a flat torsion-free partial affine connection

$$\nabla : V_P(M) \times V_P(M) \to V_P(M),$$

called the *Weinstein connection*, defined by

$$\omega(\nabla_X Y) = i_X(d(\omega(Y))).$$

In particular, each leaf admits a flat torsion-free affine connection.

Proof. See [9], p 68. □

Theorem. Let P be a real polarisation whose leaves are simply-connected and geodesically-complete. Suppose that there exists a Lagrangian submanifold Q which intersects every leaf exactly once. Then there is a symplectomorphism $M \to T^*Q$, under which P is mapped to the vertical foliation and Q is mapped to the zero section.

Proof. According to affine geometry, any manifold admitting a flat torsion-free affine connection is an affine manifold. So by the previous lemma, each leaf of P is an affine manifold. Let Λ_q be the leaf which intersect Q at q. Simply-connectedness and geodesically-completeness imply that Λ_q is an affine space. Taking q as the origin makes it a vector space.
For each $q \in Q$, the bilinear map

$$\omega_q : P_q \times T_qQ \to \mathbb{R},$$

is non-degenerate as P_q and T_qQ are complementary Lagrangian subspaces of T_qM. This induces isomorphisms $\Lambda_q \simeq P_q \simeq T_q^*Q$. Varying $q \in Q$, we get an identification $M \simeq T^*Q$ as desired. □

Removing the topological requirements on leaves gives a local version of the theorem.

Theorem. Let P be a real polarisation. Each point $m \in M$ has a neighbourhood U which is symplectomorphic to an open subset of a cotangent bundle with the polarisation preserved.

Complex polarisation

Real polarisations may not exist in general. For example, $M = \mathbb{S}^2$ does not admit a real polarisation; it doesn't even have a non-vanishing vector field at all. Reminding that prequantisation was all about complex line bundles, it is reasonable to consider *complex polarisations* as well.

Let (M, ω) be a symplectic manifold. Each tangent and cotangent space can be complexified to obtain complex vector spaces

$$(T_mM)_{\mathbb{C}} := T_mM \otimes_{\mathbb{R}} \mathbb{C}, \quad (T_m^*M)_{\mathbb{C}} := T_m^*M \otimes_{\mathbb{R}} \mathbb{C}.$$

Varying $m \in M$ yields the complexified tangent and cotangent bundles $T_{\mathbb{C}}M$ and $T_{\mathbb{C}}^*M$. Now we can talk about *complex distributions* which are subbundles of $T_{\mathbb{C}}M$.
The symplectic form ω is complexified to $\omega_{\mathbb{C}} \in \Omega_{\mathbb{C}}^2(M)$, which in turn induces a non-degenerate anti-symmetric bilinear form

$$(\omega_{\mathbb{C}})_m : (T_mM)_{\mathbb{C}} \times (T_mM)_{\mathbb{C}} \to \mathbb{C}$$

for each $m \in M$. One may discuss about complex symplectic vector spaces whose definition is straightforward.

Let P be a complex Lagrangian foliation. For each $m \in M$, the Lagrangian subspace $P_m \leq (T_m M)_{\mathbb{C}}$ induces the conjugate subspace $\overline{P}_m \leq (T_m M)_{\mathbb{C}}$ which is also Lagrangian. One may define

$$D_m := (P_m \cap \overline{P}_m) \cap T_m M, \quad E_m := (P_m + \overline{P}_m) \cap T_m M.$$

But there is no guarantee of constant rank or integrability of D_m and E_m, so we put them into definition:

Definition. A *complex polarisation* is a complex Lagrangian foliation P for which D has constant rank. We say P is *strongly integrable* if E is integrable.

If P is a complex polarisation, then the rank of $D = P \cap \overline{P} \cap TM$ can be thought of as the real part of P. At one extreme, we have (rank D) $= 0$ and P arises as a complexification of a real polarisation. At the other extreme, we have (rank D) $= n$.

Definition. A complex polarisation P is *real* (resp. *Kähler*) if the rank of D is n (resp. 0).

We have already studied real polarisations, so we look at Kähler polarisations. The middle case where the rank is between 0 and n is a mixture of the two theories.
Let P be a Kähler polarisation, then $P_m \cap \overline{P}_m = 0$ for each $m \in M$. It is a known fact that for a real symplectic vector space (V, ω), the subspaces $P \leq V_{\mathbb{C}}$ satisfying $P \cap \overline{P} = 0$ are in bijection with the *complex structures* J on (V, ω). Hence we obtain a complex structure J_m on each $T_m M$ varying smoothly in m, which is known as an *almost complex structure* on M. Now the integrability of P promotes M to a complex manifold compatible with ω. This is called a *pseudo-Kähler manifold.*

Definition. A *pseudo-Kähler manifold* is a complex manifold M with compatible symplectic and pseudo-Riemannian structures ω and g.

Conversely, every pseudo-Kähler manifolds admit natural Kähler polarisations. One is the *holomorphic polarisation* spanned by $\partial/\partial\overline{z}^a$, and the other is the *anti-holomorphic polarisation* spanned by $\partial/\partial z^a$.

8 Quantisation and metaplectic correction

Now we come to the heart of geometric quantisation. Given a symplectic manifold (M, ω) satisfying the Bohr-Sommerfeld condition, the prequantisation scheme yields a prequantum Hilbert space, which satisfies all the quantum conditions except the irreducibility one. To cut the dimension of M into half, we introduced a polarisation P on M.

The first step is the construction of a *quantum Hilbert space* $\mathcal{H}_P$. We expect that taking a cotangent bundle $M = T^*Q$ and its vertical polarisation P constructs a Hilbert space which consists of wavefunctions depending only on position q but not on momentum p. In general, a

quantum Hilbert space will consist of *polarised* wavefunctions.
The construction of $\mathcal{H}_P$ works the best when P is a *positive* Kähler polarisation. But otherwise, we meet a serious problem that polarised wavefunctions may not be square-integrable and $\mathcal{H}_P$ is empty. To fix this issue, we need to choose a metaplectic correction.
But even in the positive Kähler case, a choice of metaplectic correction is desirable. Without such, mathematics will provide us with a well-defined quantisation, but it just gives physically incorrect answers. For example, running a quantisation scheme for a harmonic oscillator naively without a correction gives the energy spectrum $n\hbar$ rather than $(n+1/2)\hbar$.

Having constructed a Hilbert space, the second step is the construction of *quantum operators*. As mentioned earlier, there are obstructions which make it impossible to associate $\hat{f} \in O(\mathcal{H}_P)$ to every $f \in C^\infty(M)$ without making $\mathcal{H}_P$ too large. Hence we should restrict ourselves only to a subalgebra of $C^\infty(M)$.
The easiest case occurs when f preserves the polarisation P. In such case, f is said to be *polarised*, and the prequantum operator $\hat{f}$ immediately promotes to a quantum operator. However, polarised functions cover only a small portion of $C^\infty(M)$ and they miss several important classical observables, including energy.
Therefore we should try to construct quantum operators for non-polarised functions, and this is the point where the quantisation scheme gets rather ad hoc. One possible approach is the *Blattner-Kostant-Sternberg (BKS) method* we will see later. The BKS method may not work depending on what functions f we put. But whenever it works, it exhibits how geometric quantisation is deeply interrelated to a profound mathematical ideas such as Fourier transform and Feynman path integrals.

Quantum Hilbert space

Let (M,ω) be a symplectic manifold, (B,∇^B) be a prequantum line bundle, and P be a complex polarisation.

Definition. A section $s \in \Gamma(B)$ is *polarised* if it is covariantly constant along the leaves of P. That is,

$$\nabla^B_X s = 0 \quad \forall\, X \in V_P(M). \tag{22}$$

The set of polarised sections is denoted by $\Gamma_P(B)$.

Our first attempt is to define the *quantum Hilbert space* for the polarisation P as

$$\mathcal{H}_P := L^2(\Gamma_P(B)),$$

the space of polarised square-integrable sections. This does not quite work, however, because $\mathcal{H}_P$ may be empty. A typical example is a cotangent bundle $M = T^*Q$ with its vertical polarisation P. For a polarised section $s \in \Gamma_P(B)$, (22) implies

$$i_X(d(s,s)) = (\nabla^B_X s, s) + (s, \nabla^B_X s) = 0.$$

so (s,s) is constant on each vertical subspace. Thus the integral

$$\langle s,s\rangle = \int_M (s,s)\varepsilon = \left(\frac{1}{2\pi\hbar}\right)^n \int_Q (s,s)d^nq \int_{\mathbb{R}^n} d^np = \infty \tag{23}$$

diverges, and $\mathcal{H}_P = 0$.

We see from (23) that integrals along leaves are responsible for blowups. Thus a natural solution is to take the integral of (s,s) over the quotient space M/P rather than M. But M/P does not have a canonical volume element, so we need to make an arbitrary choice. This is called a *metaplectic correction*.

Define the quotient space $\pi : M \to M/P =: Q$. Assume that Q is orientable and let

$$\Delta_P := \Lambda^n T^*_{\mathbb{C}}(Q)$$

be the complex line bundle of n-forms on Q. Its pullback

$$K_P := \pi^*(\Delta_P) \leq \Lambda^n T^*_{\mathbb{C}}(M)$$

is a line bundle on M. One can show that, for each $\alpha \in \Gamma(K_P)$, there exists $\beta \in \Omega^n_{\mathbb{C}}(Q)$ satisfying $\alpha = \pi^*(\beta)$ if and only if $d\alpha(X) = 0$ for all $X \in V_P(M)$. Now we define a *metaplectic correction* as a square root bundle of K_P.

Definition. A *metaplectic correction* is a line bundle δ_P on M such that $\delta_P \otimes \delta_P \simeq K_P$.

Lemma. The line bundle K_P has a flat partial connection

$$\nabla^{K_P} : V_P(M) \times \Gamma(K_P) \to \Gamma(K_P)$$

defined by

$$\nabla^{K_P}_X \alpha := d\alpha(X).$$

If δ_P is a metaplectic correction, then this induces a flat partial connection

$$\nabla^{\delta_P} : V_P(M) \times \Gamma(\delta_P) \to \Gamma(\delta_P)$$

defined by

$$2\nu \otimes \nabla^{\delta_P}_X \nu = \nabla^{K_P}_X (\nu \otimes \nu).$$

Since Q was assumed orientable, Δ_P is a trivial line bundle, and so is K_P. This guarantees the existence of a metaplectic correction, but such a correction is also trivial. This does not make our construction meaningless, because the connection ∇^{δ_P} might be inequivalent for different choices, depending on how we identify $\delta_P \otimes \delta_P$ with K_P. On the other hand, if Q is not orientable then we should use *densities* instead of differential forms.

Now we make a modification to our prequantum line bundle by defining

$$B_P := B \otimes \delta_P.$$

Since ∇^{δ_P} is flat, δ_P has a vanishing curvature. In particular, $B \otimes \delta_P$ also has the curvature $i\omega/\hbar$ and satisfies all the conditions for prequantum line bundle. We call this corrected line bundle the *quantum line bundle*. Every $\tilde{s} \in \Gamma(B_P)$ can be written non-uniquely as $\tilde{s} = s \otimes \nu$ for some $s \in \Gamma(B)$ and $\nu \in \Gamma(\delta_P)$.

Lemma. The quantum line bundle B_P has a partial connection

$$\nabla^{B_P} : V_P(M) \times \Gamma(B_P) \to \Gamma(B_P)$$

defined by

$$\nabla^{B_P}_X(s \otimes \nu) := (\nabla^B_X s) \otimes \nu + s \otimes (\nabla^{\delta_P}_X \nu).$$

and a Hermitian structure $(\cdot,\cdot)_m : (B_P)_m \times (B_P)_m \to (K_P)_m$ defined by

$$(s_1 \otimes \nu_1, s_2 \otimes \nu_2) := (s_1, s_2)(\nu_1 \otimes \nu_2)$$

which are compatible in the sense that

$$\nabla^{K_P}_X(\tilde{s}_1, \tilde{s}_2) = (\nabla^{B_P}_X \tilde{s}_1, \tilde{s}_2) + (\tilde{s}_1, \nabla^{B_P}_X \tilde{s}_2).$$

Now for sections $\tilde{s}_1$ and $\tilde{s}_2$, we define

$$\langle \tilde{s}_1, \tilde{s}_2 \rangle := \int_Q (\tilde{s}_1, \tilde{s}_2)$$

whenever the integral exists.

Definition. A section $\tilde{s} \in \Gamma(B_P)$ is *polarised* if

$$\nabla^{B_P}_X \tilde{s} = 0 \quad \forall\, X \in V_P(M).$$

The set of polarised sections is denoted by $\Gamma_P(B_P)$.

Then we may construct the *quantum Hilbert space* as

$$\mathcal{H}_P := L^2(\Gamma_P(B_P)),$$

the space of polarised square-integrable sections of the quantum line bundle B_P.

Quantum operators for polarised observables

Henceforth we drop B_P and write ∇ instead of ∇^{B_P}.
Recall from (17) that $f \in C^\infty(M)$ corresponds to the prequantum operator

$$\hat{f}s := -i\hbar \nabla_{X_f} s + fs.$$

In general, $\hat{f}$ sends polarised sections to non-polarised sections. However, there are special situations. If the Hamiltonian flow generated by f fixes the leaves, i.e. X_f is tangent to P, then

$$s \in \Gamma_P(B_P) \quad \Rightarrow \quad \nabla_{X_f} s = 0 \quad \Rightarrow \quad \hat{f}s = fs \quad \Rightarrow \quad \hat{f}s \in \Gamma_P(B_P).$$

So f immediately promotes to a quantum operator $\hat{f} : \mathcal{H}_P \to \mathcal{H}_P$ defined by

$$\hat{f}s := fs.$$

One may hope to find a general criterion for when $\hat{f}$ maps porised sections to themselves.

Lemma. The prequantum operator $\hat{f}$ restricts to $\Gamma_P(B_P) \to \Gamma_P(B_P)$ if and only if

$$[X, X_f] = 0 \quad \forall\, X \in V_P(M). \tag{24}$$

Proof. Suppose that $\psi \in \Gamma_P(B_P)$ so that $\nabla_X \psi = 0$ for all $X \in V_P(M)$. Then

$$\begin{aligned}
\nabla_X(\hat{f}s) &= -i\hbar \nabla_X \nabla_{X_f} s + \nabla_X(fs) \\
&= -i\hbar \left(\nabla_{[X,X_f]} s + \nabla_{X_f} \nabla_X s + \frac{i}{\hbar} \omega(X, X_f) s \right) + (f \nabla_X s + X(f) s) \\
&= -i\hbar \nabla_{[X,X_f]} s.
\end{aligned}$$

As ψ can be chosen arbitrarily, $\hat{f}$ restricts to $\Gamma_P(B_P) \to \Gamma_P(B_P)$ if and only if (24) holds. □

Definition. A function $f \in C^\infty(M)$ is *polarised* if it satisfies (24). The set of polarised functions is denoted by $C^\infty_P(M)$.

Therefore if $f \in C^\infty_P(M)$ then f induces a well-defined quantum operator $\hat{f} : \mathcal{H}_P \to \mathcal{H}_P$ via

$$\hat{f} := -i\hbar \nabla_{X_f} + f.$$

Example. Let $Q = \mathbb{R}^n$ be a configuration space and $M = T^*Q$ be the phase space. The criterion (24) shows that $f \in C^\infty(M)$ is polarised if and only if there exist functions

$$f^0, \cdots, f^n \in C^\infty(Q)$$

such that

$$f(q, p) = f^0(q) + f^a(q) p_a, \tag{25}$$

i.e. f is linear in momentum p. In such case, f induces the quantum operator

$$\hat{f} = f^0(q) - i\hbar f^a(q) \frac{\partial}{\partial q^a}.$$

In particular, this yields the quantum operators

$$\hat{q}^a = q^a, \quad \hat{p}_b = -i\hbar \frac{\partial}{\partial q^b}$$

as expected.

Quantum operators for non-polarised observables

Most functions, nonetheless, are not polarised. The traditional Hamiltonian $H = p^2/2m$, for example, is a non-polarised function since it is quadratic in p and cannot be expressed in the form of (25). But in order to determine the time evolution and the dynamics of the quantum theory, one must find a quantum counterpart of the Hamiltonian. The Blattner-Kostant-Sternberg (BKS) kernel provides one method to construct such quantum operators.

Fix $f \in C^\infty(M)$ and its Hamiltonian flow φ_t. We saw from (19) and (20) that φ_t induces the unitary operator $\Phi_t : \mathcal{H}_P \to \mathcal{H}_P$, which is essentially the pushforward with phase shifts. Again, one may think that the wavefunction $s = s(t)$ is evolving through

$$s(t) := \Phi_{-t} s(0).$$

The issue is that $s(t)$ is not polarised with respect to P. But we can at least say that $s(t)$ is polarised with respect to the pullback polarisation

$$P_t := \Phi_{-t}^*(P)$$

so that $s(t) \in \mathcal{H}_{P_t}$. The *BKS kernel*, which we will see in a moment, provides a linear map

$$\pi_t : \mathcal{H}_{P_t} \to \mathcal{H}_P.$$

If π_t is unitary in addition, then the composition $\pi_t \circ \Phi_{-t} : \mathcal{H}_P \to \mathcal{H}_P$ is a unitary operator. Now we may define a quantum operator $\hat{f}$ by

$$\hat{f} s := i\hbar \frac{d}{dt}\bigg|_0 (\pi_t(\Phi_{-t} s))$$

which is indeed Hermitian.

It remains to provide a linear map $\pi : \mathcal{H}_{P'} \to \mathcal{H}_P$ for a polarisation P'. Consider

$$(\cdot,\cdot)_m : (K_P)_m \times (K_{P'})_m \to \mathbb{C}$$

defined by

$$(\alpha, \alpha')\varepsilon = \alpha \wedge \alpha'$$

which is a non-degenerate pairing varying smoothly in m. If δ_P and $\delta_{P'}$ are metaplectic corrections, then we also have a pairing

$$(\cdot,\cdot)_m : (\delta_P)_m \times (\delta_{P'})_m \to \mathbb{C}$$

defined by

$$(\nu, \nu') := \sqrt{(\nu \otimes \nu, \nu' \otimes \nu')}.$$

For $\tilde{s} = s \otimes \nu \in \Gamma_P(B_P)$ and $\tilde{s}' = s' \otimes \nu' \in \Gamma_{P'}(B_{P'})$, We define a pairing

$$\langle \cdot, \cdot \rangle : \mathcal{H}_P \times \mathcal{H}_{P'} \to \mathbb{C}$$

by

$$\langle \tilde{s}, \tilde{s}' \rangle := \int_M (s, s')(\nu, \nu')\varepsilon.$$

Combining with the Riesz representation $\mathcal{H}_P \simeq \mathcal{H}_P^*$, we obtain a linear map $\pi : \mathcal{H}_{P'} \to \mathcal{H}_P$.

Mathematical issues are still remaining. The BKS map $\pi : \mathcal{H}_{P'} \to \mathcal{H}_P$ need not be unitary, in which case $\hat{f}$ may fail to be Hermitian. It is a very difficult question to determine whether $\hat{f}$ is a well-defined quantum operator or not, and is not well understood even today.
However, there are some instances where the map are easily shown to be unitary. Let $M = T^*\mathbb{R}^n$ be the phase space of a Euclidean space. Let P and P' be the vertical and horizontal polarisations respectively. Then $\mathcal{H}_P$ and $\mathcal{H}_{P'}$ represent the space of position and momentum wavefunctions. The BKS map $\pi : \mathcal{H}_{P'} \to \mathcal{H}_P$ now becomes

$$\psi(q) = (\pi\psi')(q) = \left(\frac{1}{2\pi\hbar}\right)^n \int \psi'(p) e^{ip_a q^a/\hbar} d^n p$$

which is precisely the Fourier inverse transform.

The BKS construction also shows a deep relationship between geometric quantisation and *Feynman path integral*. The time evolution of a wavefunction can be interpreted in a following way. Fix an initial state $\psi(0) = \psi$. Divide the time interval $[0, T]$ into N subintervals with $\delta t = T/N$. We set $\psi_0 = \psi(0)$ and

$$\psi_{i+1} = \pi_{\delta t}(\Phi_{-\delta t}\psi_i),$$

where $\pi_{\delta t}$ is the BKS map. Hence ψ is experiencing a time evolution, except that we are projecting ψ back to $\mathcal{H}_P$ and mixing position and momentum every time step. The upshot is that even non-classical paths contribute to the wavefunction ψ. Taking limit $N \to \infty$, one concludes that

$$\psi(q, t) = \int_\Gamma \psi(q') \exp\left(\frac{i}{\hbar}\int_\gamma L\right) d\gamma,$$

where the integral is over the space Γ of all paths γ joining $\gamma(0) = q'$ and $\gamma(1) = q$. This leads to a beautiful path integral formulation of quantum mechanics.

9 Discussion

We originally had a very ambitious goal at the beginning: the quantisation of all symplectic manifolds and classical systems existing in the universe of mathematics and physics. In the end, however, we are left with a huge amount of arbitrary choices and ad hoc theories. We had to choose a prequantum line bundle, a complex polarisation, and a metaplectic correction; even then, quantum operators are not clearly understood, and different situations require different prescriptions. This looks like a retreat.

One optimistic solution is to put more things into consideration. Starting from Dirac and Weyl, physicists have developed many alternative theories of quantisation. One of them is *deformation quantisation*, which concentrates more on the relationship between classical and quantum observables.
The clash between commutativity and non-commutativity is a huge obstruction when quantising systems. Hence one may introduce some non-commutativity to the classical realm. Let f and g be classical observables and $\hat{f}$ and $\hat{g}$ be quantum operators. In the 1940s, Moyal and Groenewold

independently considered the problem of dequantising the product $\hat{f}\hat{g}$. Such dequantisation can be expanded in a Taylor series to give an expression

$$f \star g = \sum_{n=0}^{\infty} B_n(f,g)\hbar^n.$$

There are quantum conditions which are expected to be obeyed:

(i) $\star$ is bilinear and associative,

(ii) $\star$ has the constant 1 as the identity,

(iii) B_n is bidifferential,

(iv) $B_0(f,g) = fg$ and $B_1(f,g) - B_1(g,f) = \{f,g\}$ hold.

Kontsevich proved that every Poisson manifolds M admits a *Moyal product*

$$\star : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\hbar]]$$

satisfying (i)-(iv), where $C^\infty(M)[[\hbar]]$ is the algebra of formal power series. Then one may define the *Moyal bracket* by

$$\{\{f,g\}\} := f \star g - g \star f.$$

If one replaces the quantum condition (Q2) by $[\hat{f},\hat{g}] = i\hbar\{\{f,g\}\}$, then one only gets

$$[\hat{f},\hat{g}] = i\hbar\{\{f,g\}\} + O(\hbar^3),$$

so the original condition (Q2) is true only under the classical limit $\hbar \to 0$.

Compared to geometric quantisation, one may regard deformation quantisation as a formal or algebraic theory of quantisation. Admittedly it does not construct any Hilbert space so is not a physical quantisation in a strict sense. Nevertheless, once allowing higher-order terms of $\hbar$ to contribute, more interesting thing can be said.

But there is yet another exciting possibility. Probably it was just the case that our grand goal was impossible in the first place. Under the correspondence principle, classical results are recovered from quantum theory in the limit of $\hbar \to 0$. In doing so, we lose information about the quantum system. Hence quantum systems have essentially more data than classical systems. This viewpoint suggests that all the additional structures introduced in geometric quantisation are just ingredients to overcome this deficiency.

Hence it might be interesting to consider the problem of *dequantisation*. Let $\mathcal{H}$ be a Hilbert space, then a physical state is represented by a ray in $\mathcal{H}$. This is the same as an element of the *projective Hilbert space* $\mathbb{P}\mathcal{H}$. Now $\mathbb{P}\mathcal{H}$ is naturally a Kähler manifold with the Fubini-Study symplectic form. Moreover, $\mathcal{H}$ gives the tautological Hermitian line bundle on $\mathcal{H}$, which is connected by a unique Chern connection. These are precisely the conditions required for a prequantum line bundle.

The projective space $\mathbb{P}\mathcal{H}$ is not a physical dequantisation, and it should only be thought of as a heuristic to explaining why geometric quantisation is a natural theory of quantisation. But still, it is enlightening to see how mathematics behind symplectic manifolds are Hilbert spaces are so much related.

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