The Gross–Zagier formula

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Abstract

In this note, we give a brief introduction to Arakelov theory and then describe the proof strategy in [GZ86] of the Gross–Zagier formula.

1 Arakelov theory

Let $\mathcal{X}$ be a projective scheme over $\operatorname{Spec}\mathbb{Z}$. We would like to study intersection theory on $\mathcal{X}$. However, there is one issue. By function field analogy, we think $\operatorname{Spec}\mathbb{Z}$ as analogous to $\mathbb{A}^1_k$ which is not compact. We compactify $\mathbb{A}^1_k$ to $\mathbb{P}^1_k$ by adding a point at infinity. The analogous "compactification" $\overline{\operatorname{Spec}\mathbb{Z}}$ is not readily available in the classical language of schemes. Nonetheless, from

$$\{\text{points of } \mathbb{P}^1_k\} \leftrightarrow \{\text{valuations on } k(t)\}$$

one can come up with the analogy

$$\{\text{points of } \overline{\operatorname{Spec}\mathbb{Z}}\} \leftrightarrow \{\text{valuations on } \mathbb{Q}\}$$

and think that the point at infinity in $\overline{\operatorname{Spec}\mathbb{Z}}$ corresponds to the archimedean place of $\mathbb{Q}$. Therefore, in order to obtain a satisfactory intersection theory on arithmetic schemes, one must consider contributions from the archimedean places. This is what Arakelov theory achieves.

1.1 Metrized line bundles

In classical intersection theory, one has the notion of the degree of a line bundle on a curve. One can use this to define the intersection pairing of a divisor and a curve. Let us begin by defining the Arakelov analogues of these notions. Arakelov introduced metrics on line bundles as a way to encode "orders of vanishing" at the archimedean places.

Definition 1.1. (1) Let X be a finite type scheme over $\mathbb{C}$ and let E be a vector bundle on X. A *metric* on E is a smoothly varying family of Hermitian inner products on the fibers of $E(\mathbb{C}) \to X(\mathbb{C})$.

(2) Let $\mathcal{X}$ be a finite type scheme over $\mathbb{Z}$ and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. A *metric* on $\mathcal{E}$ is a metric on $\mathcal{E}_{\mathbb{C}}$ that is invariant under the complex conjugation $F_\infty : \mathcal{E}(\mathbb{C}) \to \mathcal{E}(\mathbb{C})$.

Definition 1.2. Let $\mathcal{X}$ be a finite type scheme over $\mathbb{Z}$. The *arithmetic Picard group* $\widehat{\operatorname{Pic}}(\mathcal{X})$ is the group of isomorphism classes of metrized line bundles on $\mathcal{X}$ whose operation is given by tensor product of line bundles and fiberwise product of metrics.

Definition 1.3. Suppose $\mathcal{X}$ is a 1-dimensional integral projective scheme over $\mathbb{Z}$. Let $\mathcal{L} \in \widehat{\operatorname{Pic}}(\mathcal{X})$. The *arithmetic degree* of $\mathcal{L}$ is defined as

$$\widehat{\deg}\mathcal{L} := \sum_{x\in\mathcal{X}} \operatorname{ord}_x(s) \log|\kappa(x)| - \sum_{x\in\mathcal{X}(\mathbb{C})} \log\|s\|_x$$

for any rational section s of $\mathcal{L}$.

This definition is independent of a choice of s. If $\mathcal{X}$ is vertical (i.e. $\mathcal{X}$ is a projective curve over $\mathbb{F}_p$), then independency is classical. If $\mathcal{X}$ is horizontal (i.e. $\mathcal{X} \simeq \operatorname{Spec}\mathcal{O}$ for an order $\mathcal{O} \subseteq \mathcal{O}_K$ of a number field

K), then independency follows from the product formula.
The arithmetic degree defines a group homomorphism $\widehat{\deg} : \widehat{\mathrm{Pic}}(\mathcal{X}) \to \mathbb{R}$.

Definition 1.4. Let $\mathcal{X}$ be a projective scheme over $\mathbb{Z}$. Let $\mathcal{Y}$ be a metrized divisor of $\mathcal{X}$ (i.e. a Cartier divisor together with a metric on the line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{Y})$). Let $\mathcal{Z}$ be a 1-dimensional integral closed subscheme of $\mathcal{X}$. The *arithmetic intersection pairing* of $\mathcal{Y}$ and $\mathcal{Z}$ is defined as

$$\mathcal{Y} \cdot \mathcal{Z} := \widehat{\deg}(\mathcal{O}_{\mathcal{X}}(\mathcal{Y})|_{\mathcal{Z}}).$$

Remark 1.5. There is an asymmetry in this definition that only $\mathcal{Y}$ is enriched with some archimedean datum. We will later define the arithmetic intersection product which is symmetric in both arguments, and then use it to express $\mathcal{Y} \cdot \mathcal{Z}$.

1.2 Arithmetic Chow rings

We would like to generalize the definitions in the previous subsection to cycles of arbitrary codimension. We start with the following observation.

Definition 1.6. (1) Let X be a finite type scheme over $\mathbb{C}$ and let D be a Cartier divisor of X. A *Green function* for D is a function $g : (X - \mathrm{supp}\ D)(\mathbb{C}) \to \mathbb{R}$ that satisfies the following condition.

- If D is locally represented by a rational function f on U, then there exists a smooth function $\alpha : U(\mathbb{C}) \to \mathbb{R}$ such that $g = -\log|f|^2 + \alpha$ on $(U - \mathrm{supp}\ D)(\mathbb{C})$.

(2) Let $\mathcal{X}$ be a finite type scheme over $\mathbb{Z}$ and let $\mathcal{D}$ be a Cartier divisor of $\mathcal{X}$. A *Green function* for $\mathcal{D}$ is a Green function for $\mathcal{D}_{\mathbb{C}}$ that is invariant under the complex conjugation $F_\infty : \mathcal{X}(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$.

Proposition 1.7. Let X be a finite type scheme over $\mathbb{C}$ and let D be a Cartier divisor of X. There is a one-to-one correspondence

$$\{\text{metrics on } \mathcal{O}_X(D)\} \xrightarrow{\cong} \{\text{Green functions for } D\}$$
$$\|\cdot\| \mapsto -\log\|1\|^2.$$

So, it is enough to generalize the notion of Green functions to cycles of arbitrary codimension. To do this, we introduce currents, which generalize differential forms in the same way distributions generalize functions.

Definition 1.8. Let X be an n-dimensional complex manifold.

(1) Denote the space of compactly supported smooth complex $(n-p, n-q)$-forms on X equipped with the Schwarz topology by $\Omega_c^{n-p,n-q}(X)$. The space of (p,q)-*currents* on X is the continuous dual $D^{p,q}(X) := \Omega_c^{n-p,n-q}(X)^*$. We also set $D^m(X) := \bigoplus_{p+q=m} D^{p,q}(X)$.

(2) We define the operators $\partial : D^{p,q}(X) \to D^{p+1,q}(X)$ and $\overline{\partial} : D^{p,q}(X) \to D^{p,q+1}(X)$ by

$$\partial T(\eta) := (-1)^{p+q+1} T(\partial\eta), \quad \overline{\partial} T(\eta) := (-1)^{p+q+1} T(\overline{\partial}\eta).$$

We also define the operators $d, d^c : D^{p,q}(X) \to D^{p+q+1}(X)$ by

$$d = \partial + \overline{\partial}, \quad d^c = \frac{1}{4\pi i}(\partial - \overline{\partial}).$$

Example 1.9. (1) Let $\omega \in \Omega^{p,q}(X) \otimes_{C^\infty(X)} L^1_{\mathrm{loc}}(X)$. We obtain a current $[\omega] \in D^{p,q}(X)$ defined by

$$[\omega](\eta) := \int_X \omega \wedge \eta.$$

In particular, we obtain an embedding $[\cdot] : \Omega^{p,q}(X) \hookrightarrow D^{p,q}(X)$ that commutes with the operators ∂, $\overline{\partial}$, d, d^c due to Stokes' theorem.

(2) Suppose X is algebraic. Let $Y = \sum_i n_i Y_i \in Z^p(X)$ be a codimension p cycle. Denote the nonsingular locus of Y_i by Y_i^{ns}. We obtain a current $\delta_Y \in D^{p,p}(X)$ called the *delta current* defined by

$$\delta_Y(\eta) := \sum_i n_i \int_{Y_i^{\mathrm{ns}}} \eta,$$

which is well-defined by Hironaka's theorem on resolution of singularities.

Theorem 1.10 (Poincaré–Lelong)**.** Let X be a smooth variety over $\mathbb{C}$ and let D be a metrized divisor of X whose corresponding Green function is $g : (X-D)(\mathbb{C}) \to \mathbb{R}$. We have the equality of $(1,1)$-currents

$$dd^c[g] + \delta_D = [c_1(\mathcal{O}_X(D))].$$

On the left-hand side, $g \in L^1(X)$, so $[g] \in D^{0,0}(X)$ is well-defined. On the right-hand side, $c_1(\mathcal{O}_X(D)) \in \Omega^{1,1}(X)$ is the *first Chern form*, which is locally defined by $c_1(\mathcal{O}_X(D))|_U := -dd^c \log\|s\|^2$ for any holomorphic section s on U (which is independent of a choice of s).

Gillet and Soulé generalizd Green functions to Green currents and defined arithmetic Chow rings. From now on, $\mathcal{X}$ is an integral regular projective scheme over $\mathbb{Z}$. We use the notation where curly letters (e.g. $\mathcal{X}$) represent schemes over $\mathbb{Z}$ and usual letters represent their complex analytifications (e.g. $X = \mathcal{X}(\mathbb{C})$).

Definition 1.11. Let $\mathcal{Y} \in Z^p(\mathcal{X})$ be a codimension p cycle. A *Green current* for $\mathcal{Y}$ is a current $g \in D^{p-1,p-1}(X)$ that satisfies the following conditions.

(i) g is real, i.e. $g(\Omega_{\mathbb{R}}^{n-p+1,n-p+1}(X)) \leq \mathbb{R}$.

(ii) $F_\infty^* g = (-1)^{p-1} g$ where $F_\infty : X \to X$ is the complex conjugation.

(iii) There is $\omega \in \Omega^{p,p}(X)$ such that $dd^c g + \delta_Y = [\omega]$.

Definition 1.12. (1) We define the group

$$\widehat{Z}^p(\mathcal{X}) := \{(\mathcal{Y}, g) \mid \mathcal{Y} \in Z^p(\mathcal{X}),\ g \text{ Green current for } \mathcal{Y}\}$$

whose operation is given by $(\mathcal{Y}_1, g_1) + (\mathcal{Y}_2, g_2) := (\mathcal{Y}_1 + \mathcal{Y}_2, g_1 + g_2)$.

(2) Let $\mathcal{Y} \in Z^{p-1}(\mathcal{X})$ be irreducible and let f be a rational function on $\mathcal{Y}$. Denote the nonsingular locus of Y by Y^{ns} and consider the current $[-\log|f|^2] \in D^{p-1,p-1}(X)$ defined by

$$[-\log|f|^2](\eta) := -\int_{Y^{\mathrm{ns}}} \log|f|^2 \eta.$$

We define

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), [-\log|f|^2]) \in \widehat{Z}^p(\mathcal{X}).$$

This is well-defined since $dd^c[-\log|f|^2] + \delta_{\mathrm{div}(f)} = 0$ by the Poincaré–Lelong lemma applied to a resolution of singularities of Y.

We define $\widehat{\mathrm{Rat}}^p(X) \leq \widehat{Z}^p(X)$ to be the subgroup generated by all $\widehat{\mathrm{div}}(f)$ and all $(0, \partial u + \overline{\partial u})$ for $u \in D^{p-2,p-1}(X)$.

(3) The *arithmetic Chow group* of codimension p is $\widehat{\mathrm{CH}}^p(\mathcal{X}) := \widehat{Z}^p(\mathcal{X})/\widehat{\mathrm{Rat}}^p(\mathcal{X})$.

Remark 1.13. By 1.7, we obtain an isomorphism $\widehat{c}_1 : \widehat{\mathrm{Pic}}(\mathcal{X}) \xrightarrow{\cong} \widehat{\mathrm{CH}}^1(\mathcal{X})$.

Remark 1.14. The original theory by Arakelov and Faltings deals with a slightly different version of $\widehat{\mathrm{CH}}^1$. Suppose $\mathcal{X}$ is a 2-dimensional integral regular projective scheme over $\mathrm{Spec}\,\mathcal{O}_K$ such that $\mathcal{X}_K$ is geometrically irreducible. The original theory treats X_σ for each complex embedding $\sigma : K \hookrightarrow \mathbb{C}$ as the "vertical divisor" of $\mathcal{X}$ over σ. In the modern language, their viewpoint can be reformulated as follows. A *Néron family* is a choice of a Green function g_D for each divisor D of X_σ that satisfies the follwoing.

(i) $F_\infty^* g_{F_\infty D} = g_D$ where $F_\infty : X_\sigma \to X_{\overline{\sigma}}$ is the complex conjugation.

(ii) $g_{D_1+D_2} = g_{D_1} + g_{D_2}$ on $(X_\sigma - \operatorname{supp} D_1 - \operatorname{supp} D_2)(\mathbb{C})$.

(iii) $g_{\operatorname{div}(f)} = -\log|f|^2 + (\text{constant})$ on $(X_\sigma - \operatorname{supp} \operatorname{div}(f))(\mathbb{C})$.

(iv) (optional) $g_P(Q) = g_Q(P) =: g(P,Q)$ for $P \neq Q \in X_\sigma(\mathbb{C})$.

Existence of a Néron family can be shown by constructing a Green function for the diagonal inside each $X_\sigma \times X_\sigma$. Once we fix a Néron family λ, we can define the subgroup

$$\widehat{Z}^1(\mathcal{X}, \lambda) := \{(\mathcal{D}, g) \in \widehat{Z}^1(\mathcal{X}) \mid g - g_{D_\sigma} \text{ constant on each } (X_\sigma - D_\sigma)(\mathbb{C})\} \leq \widehat{Z}^1(\mathcal{X})$$

and the *Arakelov Chow group* $\widehat{\mathrm{CH}}^1(\mathcal{X}, \lambda) := \widehat{Z}^1(\mathcal{X}, \lambda)/\widehat{\mathrm{Rat}}^1(\mathcal{X}) \leq \widehat{\mathrm{CH}}^1(\mathcal{X})$.

Since $\mathcal{X}$ is regular, $\mathrm{CH}(\mathcal{X})_\mathbb{Q} := \bigoplus_{p\geq 0} \mathrm{CH}^p(\mathcal{X})_\mathbb{Q}$ is naturally a commutative graded $\mathbb{Q}$-algebra by the intersection product. We now define a commutative graded $\mathbb{Q}$-algebra structure on $\widehat{\mathrm{CH}}(\mathcal{X})_\mathbb{Q} := \bigoplus_{p\geq 0} \widehat{\mathrm{CH}}^p(\mathcal{X})_\mathbb{Q}$.

Lemma 1.15. Let $(\mathcal{Y}, g) \in \widehat{Z}^1(\mathcal{X})$. There exists $\widetilde{g} \in \Omega^{p-1,p-1}(X - Y)$ with logarithmic singularity along Y and $u \in D^{p-2,p-1}(X)$ such that $g = [\widetilde{g}] + \partial u + \overline{\partial u}$.

Definition 1.16. Let $(\mathcal{Y}, [\widetilde{g_1}]) \in \widehat{Z}^p(\mathcal{X})$ where $\widetilde{g_1} \in \Omega^{p-1,p-1}(X - Y)$ has logarithmic growth along Y. Let $(\mathcal{Z}, g_2) \in \widehat{Z}^q(\mathcal{X})$. Consider the currents $[\widetilde{g_1}] \wedge \delta_Z, [\omega_1] \wedge g_2 \in D^{p+q-1,p+q-1}(X)$ defined by

$$([\widetilde{g_1}] \wedge \delta_Z)(\eta) := \int_{(Z-Y)^{\mathrm{ns}}} \widetilde{g_1} \wedge \eta,$$

$$([\omega_1] \wedge g_2)(\eta) := g_2(\omega_1 \wedge \eta),$$

where $dd^c[\widetilde{g_1}] + \delta_Y = [\omega_1]$. The *star product* of $[\widetilde{g_1}]$ and g_2 is defined as

$$[\widetilde{g_1}] \star g_2 := [\widetilde{g_1}] \wedge \delta_Y + [\omega_1] \wedge g_2.$$

Theorem/Definition 1.17. Let $(\mathcal{Y}, g_1) \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $(\mathcal{Z}, g_2) \in \widehat{\mathrm{CH}}^q(\mathcal{X})$. Pick $\widetilde{g_1} \in \Omega^{p-1.p-1}(X - Y)$ that satisfies the statement of 1.15 applied to g_1. The *arithmetic intersection product* of $(\mathcal{Y}, g_1)$ and $(\mathcal{Z}, g_2)$ is defined as

$$(\mathcal{Y}, g_1) \cdot (\mathcal{Z}, g_2) := (\mathcal{Y} \cap \mathcal{Z}, [\widetilde{g_1}] \star g_2) \in \widehat{\mathrm{CH}}^{p+q}(X)_\mathbb{Q}.$$

This definition is independent of a choice of $\widetilde{g_1}$. Furthermore, this product induces a commutative graded $\mathbb{Q}$-algebra structure on $\widehat{\mathrm{CH}}(\mathcal{X})_\mathbb{Q}$. We call this ring the *arithmetic Chow ring* of $\mathcal{X}$.

1.3 Heights

Suppose $\dim \mathcal{X} = n$.

Theorem/Definition 1.18. (1) We define a group homomorphism $\int_\mathcal{X} : \widehat{\mathrm{CH}}^n(X) \to \mathbb{R}$ by

$$\int_\mathcal{X} (\mathcal{Y}, [\widetilde{g}]) := \sum_{x \in \mathcal{X}} \operatorname{ord}_x(\mathcal{Y}) \log|\kappa(x)| - \frac{1}{2} \int_X \widetilde{g},$$

where $\widetilde{g} \in \Omega^{n-1,n-1}(X)$.

(2) Let $\mathcal{Z} \in Z^p(\mathcal{X})$ with $p > 0$. Pick a Green function g_2 for $\mathcal{Z}$. We define a group homomorphism $\int_Z : \widehat{\mathrm{CH}}^{n-p}(X) \to \mathbb{R}$ by

$$\int_\mathcal{Z} (\mathcal{Y}, g_1) := \int_\mathcal{X} (\mathcal{Y}, g_1) \cdot (\mathcal{Z}, g_2) - \frac{1}{2}([\omega_1] \wedge g_2)(1),$$

where $dd^c g_1 + \delta_Y = [\omega_1]$. This definition is independent of a choice of g_2.

Definition 1.19. Let $\mathcal{L} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ and let $Y \in Z^p(\mathcal{X})$. We define the *height* of Y as

$$h_\mathcal{L}(Y) := \int_Y \widehat{c_1}(\mathcal{L})^{n-p}.$$

Remark 1.20. We now connect heights to Definition 1.4 as promised in Remark 1.5. Indeed,

$$\widehat{\deg}(\mathcal{L}|_{\mathcal{Z}}) = h_{\mathcal{L}}(\mathcal{Z}).$$

1.4 Néron pairings

Theorem 1.21 (Néron)**.** Let K be a local field of characteristic 0. Let X be a smooth projective geometrically irreducible curve over K. We also assume that $X(K)$ is Zariski dense in X. Let v be a place of K. There is a unique way of assigning a real number $\langle D, E\rangle_v$ to each $D, E \in \mathrm{Div}^0(X(K))$ with disjoint supports in a way that satisfies the following properties.

(i) $\langle D, E_1\rangle_v + \langle D, E_2\rangle_v = \langle D, E_1 + E_2\rangle_v$.

(ii) $\langle D, E\rangle_v = \langle E, D\rangle_v$.

(iii) $\langle \mathrm{div}(f), \sum_P n_P P\rangle_v = -n_P \log |f(P)|_v$.

(iv) For a fixed $P_0 \in X(F)$, $\langle D, P - P_0\rangle_v$ is continuous in P with respect to v.

The pairing descends to a well-defined symmetric $\mathbb{Z}$-bilinear pairing

$$\langle -, -\rangle_v : \mathrm{Div}^0(X(K)) \times \mathrm{Div}^0(X(K)) \to \mathbb{R}$$

called the *local Néron pairing*.

We explicitly construct a pairing that satisfies these conditions, hence showing the existence of a local Néron pairing.

- If K is archimedean, then we choose a Green function g for the diagonal $\Delta \subseteq X \times X$ and set

$$\left\langle \sum_P n_P P, \sum_Q n_Q Q \right\rangle_v := \sum_{P,Q} g(P, Q).$$

- If K is nonarchimedean, then we choose an integral regular model $\mathcal{X}$ of X over $\mathcal{O}_K$. Let $Y_1, \cdots, Y_r$ be the vertical divisors of $\mathcal{X}$. Let $\varpi \in \mathcal{O}_K$ be a uniformizer. We have $\mathrm{div}(\varpi) = n_1 Y_1 + \cdots + n_r Y_r$ for some positive integers $n_1, \cdots, n_r$. So, the symmetric matrix $(Y_i \cdot Y_j)_{i,j}$ satisfies the following properties.

  (i) $Y_i \cdot Y_j \geq 0$ for $i \neq j$.

  (ii) $(n_1, \cdots, n_r)$ is in the kernel of the matrix $(Y_i \cdot Y_j)_{i,j}$.

  By linear algebra, this implies the following proposition.

  Proposition 1.22. Let $\mathrm{Div}_{\mathrm{vert}}(\mathcal{X})$ denote the $\mathbb{Z}$-module of vertical divisors of $\mathcal{X}$. Then, the intersection pairing on $\mathrm{Div}_{\mathrm{vert}}(\mathcal{X}) \otimes \mathbb{Q} \to \mathbb{Q}$ has a 1-dimensional null space spanned by $\mathrm{div}(\varpi)$. Moreover, the intersection pairing on $(\mathrm{Div}_{\mathrm{vert}}(\mathcal{X}) \otimes \mathbb{Q})/\mathbb{Q}\mathrm{div}(\varpi)$ is negative definite.

  Given a divisor D of X, we denote its closure in $\mathcal{X}$ by $\mathbf{D}$.

  Corollary 1.23. There exists a unique $\mathbb{Z}$-linear map

$$\Phi : \mathrm{Div}^0(X(K)) \to (\mathrm{Div}_{\mathrm{vert}}(\mathcal{X}) \otimes \mathbb{Q})/\mathbb{Q}\mathrm{div}(\varpi)$$

  such that $(\mathbf{D} + \Phi(D)) \cdot Y = 0$ for all vertical divisors Y.

  We now set

$$\langle D, E\rangle_v := (\mathbf{D} + \Phi(D)) \cdot (\mathbf{E} + \Phi(E)).$$

One may check that these constructions indeed satisfy the conditions of local Néron pairings.

For later use, we fix the normalization of the absolute value on K and the local Néron pairing by

$$\begin{cases} \log |x|_v = [K_v : \mathbb{R}] \log |x|_\infty & \text{if } v \text{ archimedean,} \\ \log |x|_v = -v_p(x) \log |k(v)| & \text{if } v \text{ non-archimedean.} \end{cases}$$

Definition 1.24. Let K be a number field. Let X be a smooth projective geometrically irreducible curve over K such that $X(K)$ is Zariski dense in X. The *global Néron pairing* is the pairing on $\mathrm{Div}^0(X(K))$ defined by

$$\langle D, E\rangle := \sum_v \langle D, E\rangle_v.$$

Remark 1.25. The global Néron pairing on $\mathrm{Div}^0(X(K))$ coincides with the canonical Néron-Tate pairing on the Jacobian $J(K)$ with respect to the divisor 2Θ.

1.5 Deligne cohomology

The notion of Green currents can be interpreted using a cohomological language as in [GKK05].

Definition 1.26. (1) A *Dolbeault complex* is a complex of $\mathbb{R}$-vector spaces (A^n, d^n) together with a bigrading

$$A^n \otimes \mathbb{C} = \bigoplus_{p+q=n} A^{p,q}$$

satisfying $\overline{A^{p,q}} = A^{q,p}$ and a decomposition $d = \partial + \overline{\partial}$ into $\partial : A^{p,q} \to A^{p+1,q}$ and $\overline{\partial} : A^{p,q} \to A^{p,q+1}$.

(2) A *Dolbeault algebra* is a Dolbeault complex together with an associative graded-commutative product $\wedge : A^{p,q} \times A^{p',q'} \to A^{p+p',q+q'}$.

Definition 1.27. Let (A^n, d^n) be a Dolbeault complex.

(1) $F^pA^n := \bigoplus_{p' \geq p} A^{p',n-p'}$ and $\overline{F}^pA^n := \overline{F^pA^n}$ are the *Hodge filtrations*.

(2) $F^{p,q} : A^n \to F^pA^n \cap \overline{F}^qA^n$ is the projection map.

(3) $A^n(p) := (2\pi i)^p A^n \leq A^n \otimes \mathbb{C}$ is a $\mathbb{R}$-subspace.

(4) $\pi : A^n \otimes \mathbb{C} \to A^n(p)$ is a map defined by $\pi_p(x) := \frac{1}{2}(x + (-1)^p\overline{x})$.

Definition 1.28. Let $A = (A^n, d^n)$ be a Dolbeault complex. The *Deligne complex* associated with A is a graded complex $(\mathcal{D}^n(A,p), d^n_{\mathcal{D}})$ of $\mathbb{R}$-vector spaces defined by

$$\mathcal{D}^n(A,p) := \begin{cases} A^{n-1}(p-1) \cap F^{n-p,n-p}A^{n-1} \otimes \mathbb{C} & \text{if } n \leq 2p-1, \\ A^n(p) \cap F^{p,p}A^n \otimes \mathbb{C} & \text{if } n \geq 2p \end{cases}$$

and

$$d^n_{\mathcal{D}}x := \begin{cases} -F^{n-p+1,n-p+1}dx & \text{if } n \leq 2p-2, \\ -2\partial\overline{\partial}x & \text{if } n = 2p-1, \\ dx & \text{if } n \geq 2p. \end{cases}$$

Example 1.29. If A is the complex of real differential forms on a complex manifold X, then the cohomology of the associated Deligne complex is precisely the Deligne-Beilinson cohomology $H^n_{\mathcal{D}}(X, \mathbb{R}(p))$.

Definition 1.30. Let $f : A \to B$ be a morphism of complexes.

(1) We define the complex $\sigma_m A$ by

$$\sigma_m A^n := \begin{cases} A^n & \text{if } n \geq m, \\ 0 & \text{if } n < m. \end{cases}$$

(2) The *relative cohomology* of f is $H^n(A,B) := H^n(s(f))$ where $s(f)$ is the mapping cone of f.

(3) The *truncated relative cohomology* of f is $\widehat{H}^n(A,B) := H^n(\sigma_n A, B)$.

Tracing down the definitions, there is a concrete description

$$\widehat{H}^n(A,B) \cong \{(a,\widetilde{b}) \in (\ker d_A^n) \oplus (\operatorname{coker} d_B^{n-2}) \mid f(a) = d_B b\}.$$

Let X be a smooth complex variety and let $Z \in \mathrm{CH}^p(X)$ be a codimension p cycle. Let $\mathcal{D}(X)$ denote the Deligne complex associated with the complex of real differential forms on X. Similarly, let $\mathcal{D}(U)$ denote the Deligne complex of $U := X - Z$. We see that the truncated relative cohomology of the restriction map $\mathcal{D}(X) \to \mathcal{D}(U)$ of degree $2p$ is

$$\widehat{H}^{2p}(\mathcal{D}(X),\mathcal{D}(U)) \cong \left\{ (\omega,g) \;\middle|\; \begin{array}{c} \omega \in \Omega^{2p}(X)(p) \cap \Omega^{p,p}(X), \\ g \in \Omega^{2p-2}(U)(p-1) \cap \Omega^{p-1,p-1}(U) \\ \omega|_U = -2\partial\overline{\partial} g,\ d\omega = 0 \end{array} \right\}.$$

The condition $\omega|_U = -2\partial\overline{\partial} g$ is precisely the condition for Green currents for Z. The real subspaces that appear in $\widehat{H}^{2p}$ correspond to the fact that Green currents need to be real and satisfy $F_\infty^* g = (-1)^{p-1} g$. Therefore, the cohomology classes in $\widehat{H}^{2p}(\mathcal{D}(X),\mathcal{D}(U))$ can be interpreted as Green currents for Z.

Finally, we remark that the star product of Green currents also have a cohomological interpretation.

Definition 1.31. Let A be a Dolbeault algebra. We define a product $\bullet : \mathcal{D}^n(A,p) \times \mathcal{D}^m(A,q) \to \mathcal{D}^{n+m}(A,p+q)$ by

$$x \bullet y := \begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y) & \text{if } n < 2p, m < 2q, \\ F^{\ell-r,\ell-r}(x \wedge y) & textif n < 2p, m \geq 2g, \ell < 2r, \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r(\partial(x \wedge y)^{r-1,\ell-r}) & \text{if } n < 2p, m \geq 2q, \ell \geq 2r, \\ x \wedge y & \text{if } n \geq 2p, m \geq 2q, \end{cases}$$

where $\ell := n+m$, $r := p+q$, and $r_p(x) := 2\pi_p(F^p dx)$. We note that the product is only associative up to homotopy. We call $\mathcal{D}^n(A,p)$ the *Deligne algebra* associated with A.

Theorem 1.32. The products on Deligne algebras induce the star product of currents.

1.6 Extensions of Arakelov theory

We make short comments on how Arakelov theory can be applied to more general settings.

- One often wants to deal with Arakelov theory on Shimura varieties. However, metrics that arise naturally in the study of Shimura varieties (e.g. the Petersson metric) often have logarithmic growth along the boundary upon compactification.

  Let X be a smooth complex variety of dimension d.

  Definition 1.33. A *normal crossing compactification* of X is an open immersion $X \to \overline{X}$ such that the boundary $\partial X := \overline{X} - X$ is a normal crossing divisor of $\overline{X}$.

  Given a normal crossing compactification $\overline{X}$, let $E(\overline{X})$ denote the $\mathbb{R}$-vector space of real differential forms on X that has logarithmic growth along the boundary ∂X. If $\overline{X} \to \overline{X}'$ is a morphism between two normal crossing compactifications, then we have the pullback map $E(\overline{X}') \to E(\overline{X})$. We consider the direct limit

  $$\overline{E}(X) := \varinjlim E(\overline{X})$$

  over all normal crossing compactifications. Let $\overline{\mathcal{D}}(X)$ denote the associated Deligne algebra. If $Z \in \mathrm{CH}^p(X)$ is a p-cycle, then we can define Green currents for Z to be cohomology classes in

  $$\widehat{H}^{2p}(\overline{\mathcal{D}}(X),\overline{\mathcal{D}}(X-Z)).$$

- Another technicality is that Shimura varieties are often most naturally viewed as (Deligne-Mumford) stacks. One workaround is to introduce level structures and work with genuine schemes. But sometimes one only gets schemes over $\mathbb{Z}[1/N]$, and intersection pairings are obtained away from a finite set of primes. Considering multiple level structures may resolve the issue in some cases though. Alternatively, one may directly develop Arakelov theory on Deligne-Mumford stacks. There is no drastic change except for the fact that one has to remember to divide by the order of the stabilizer group all the time.

2 The Gross–Zagier formula

The Gross–Zagier formula relates the derivative of a Rankin L-function at the central point with the Néron-Tate height of a Heegner point. It is proved by relating both sides with certain arithmetic intersection pairings. We describe the proof strategy of [GZ86].

2.1 The statement of the Gross–Zagier formula

Lemma 2.1. Let $N \geq 1$. Let K be an imaginary quadratic field of discriminant D such that $(N, D) = 1$.

(1) There exists an ideal $\mathfrak{n} \leq \mathcal{O}_K$ satisfying $\mathcal{O}_K/\mathfrak{n} \simeq \mathbb{Z}/N$ if and only if every prime factor of N splits in K. This is also equivalent to the condition that D is a square modulo $4N$. We call any one of these conditions the *Heegner condition*.

(2) When the Heegner condition is met, there are precisely $2^{\omega(N)}$ ideals $\mathfrak{n} \leq \mathcal{O}_K$ satisfying $\mathcal{O}_K/\mathfrak{n} \simeq \mathbb{Z}/N$, where $\omega(N)$ is the number of prime factors of N.

Proof. Write $N = \prod_i p_i^{e_i}$ and $\mathfrak{n} = \prod_j \mathfrak{p}_j^{f_j}$. Then,

$$\begin{aligned}
\mathcal{O}_K/\mathfrak{n} \simeq \mathbb{Z}/N \quad &\Leftrightarrow \quad \prod_j \mathcal{O}_K/\mathfrak{p}_j^{f_j} \simeq \prod_i \mathbb{Z}/p_i^{e_i} \\
&\Leftrightarrow \quad \text{for each } i, \text{ there is only one } \mathfrak{p}_j \text{ lying over } p_i \text{ and satisfies } \mathcal{O}_K/\mathfrak{p}_j^{f_j} \simeq \mathbb{Z}/p_i^{e_i} \\
&\Leftrightarrow \quad \text{each } (p_i) = \mathfrak{p}_i\overline{\mathfrak{p}_i} \text{ splits in } K \text{ and } \mathfrak{n} = \prod_i \mathfrak{q}_i \text{ where } \mathfrak{q}_i \in \{\mathfrak{p}_i, \overline{\mathfrak{p}_i}\}.
\end{aligned}$$

□

For simplicity, we assume that $N \geq 2$ and that K is an imaginary quadratic field of discriminant $D \equiv 1$ (4) satisfying the Heegner condition. Let H_K denote the Hilbert class field of K. Consider the modular curve $X = X_0(N)$ and its Jacobian $J = J_0(N)$ defined over $\mathbb{Q}$. For a field k of characteristic 0, the points in $X_0(N)(k)$ correspond to cyclic isogenies between (generalized) complex elliptic curves of degree N.

Definition 2.2. (1) $\mathrm{CM}(\mathcal{O}_K) := \{x \in X(1)(H_K) \mid x \text{ has CM by } \mathcal{O}_K\}$.

(2) For an ideal $\mathfrak{n} \leq \mathcal{O}_K$ satisfying $\mathcal{O}_K/\mathfrak{n} \simeq \mathbb{Z}/N$, we define a map

$$\begin{aligned}
\Theta_{\mathfrak{n}} : \mathrm{Cl}(\mathcal{O}_K) &\to \{\phi \in X_0(N)(H_K) \mid \phi : E \to E' \text{ with } E, E' \in \mathrm{CM}(\mathcal{O}_K)\} \\
\mathfrak{a} &\mapsto (\mathbb{C}/\mathfrak{a} \twoheadrightarrow \mathbb{C}/\mathfrak{n}^{-1}\mathfrak{a})
\end{aligned}$$

(3) We define the set of *Heegner points* of discriminant D by $\mathrm{CM}_N(\mathcal{O}_K) := \bigcup_{\mathfrak{n}} \mathrm{im}\, \Theta_{\mathfrak{n}} \subseteq X_0(N)(H_K)$.

We recall the following standard facts.

- By the Shimura reciprocity, we have $\mathrm{Art}_K(\mathfrak{b})\Theta_{\mathfrak{n}}(\mathfrak{a}) = \Theta_{\mathfrak{n}}(\mathfrak{b}^{-1}\mathfrak{a})$ where $\mathrm{Art}_K : \mathrm{Cl}(\mathcal{O}_K) \xrightarrow{\cong} \mathrm{Gal}(H_K/K)$ is the Artin map.

- For each divisor $d \mid N$ such that $(d, N/d) = 1$, there is the *Atkin-Lehner involution* $X \to X$. These satisfy $w_d w_{d'} = w_{dd'/(d,d')^2}$ and form an abelian group W isomorphic to $(\mathbb{Z}/2)^{\omega(N)}$. If p is a prime divisor of N and $N = p^k m$ with $p \nmid m$, then we have $w_{p^k} \Theta_{\mathfrak{p}^k \mathfrak{m}}(\mathfrak{a}) = \Theta_{\overline{\mathfrak{p}}^k \mathfrak{m}}(\mathfrak{p}^{-k}\mathfrak{a})$.
- The group $\mathrm{Gal}(H_K/K) \times W$ acts simply transitively on $\mathrm{CM}_N(\mathcal{O}_K)$.
- For each $m \geq 1$, there is the *Hecke operator* $T_m : J \to J$ which is induced by an algebraic correspondence on $X \times X$ of bidegree $\sigma_1(m)$ defined over $\mathbb{Q}$. They generate the *Hecke algebra* $\mathbb{T} \subseteq \mathrm{End}_{\mathbb{Q}}(J)$.

We now define the objects that appear in the Gross–Zagier formula.

Fix $x \in \mathrm{CM}_N(\mathcal{O}_K)$ and set $c := (x) - (\infty) \in J(H_K)$. For a character $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$, define $c_\chi := \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma) c^\sigma \in J(H_K) \otimes \mathbb{C}$ in the χ-eigenspace. For $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(N))$, we also define $c_{\chi,f}$ to be the image of c_χ under the projection $J \to J/\mathrm{ann}_{\mathbb{T}}(f)J$ to the f-isotypical component.

Proposition 2.3. Let $\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_K)$ and $\sigma := \mathrm{Art}_k(\mathcal{A}) \in \mathrm{Gal}(H_K/K)$. The q-series

$$g_{\mathcal{A}} := \sum_{m \geq 1} \langle c, T_m c^\sigma \rangle q^m$$

is a cusp form of weight 2 for $\Gamma_0(N)$, where $\langle \cdot, \cdot \rangle$ denotes the global Néron pairing on $J(H_K)$.

The proposition follows from a more general lemma.

Lemma 2.4. Let $\alpha : \mathbb{T} \to \mathbb{C}$ be a $\mathbb{Q}$-linear functional. Then, $\sum_{m \geq 1} \alpha(T_m) q^m \in \mathcal{S}_2(\Gamma_0(N))$.

Proof. Let S denote the cotangent space of J at the origin. We obtain an injection $\mathbb{T} \to \mathrm{End}_{\mathbb{Q}}(S)$. Now, we identify S with the space of cusps forms of weight 2 for $\Gamma_0(N)$ with rational Fourier coefficients. This identification respects the $\mathbb{T}$-action.

Let $\alpha_m : \mathcal{S}_2(\Gamma_0(N)) \to \mathbb{C}$ denote the $\mathbb{C}$-linear functional that maps $f \in \mathcal{S}_2(\Gamma_0(N))$ to its Fourier coefficient $a_m(f)$ of degree m. We claim that

$$\begin{aligned} \mathbb{T} \times S &\to \mathbb{Q} \\ (T, f) &\mapsto \alpha_1(Tf) \end{aligned}$$

is a perfect pairing.

- Suppose $T \in \mathbb{T}$ satisfies $\alpha_1(Tf) = 0$ for all $f \in S$. Then, for any $f \in S$ and $m \geq 1$, we have $\alpha_m(Tf) = \alpha_1(T_m T f) = \alpha_1(T T_m f) = 0$. This shows that $T \in \ker(\mathbb{T} \to \mathrm{End}_{\mathbb{Q}}(S))$, so $T = 0$.
- Suppose $f \in S$ satisfies $\alpha_1(Tf) = 0$ for all $T \in \mathbb{T}$. Then, for any $m \geq 1$, we have $\alpha_m(f) = \alpha_1(T_m f) = 0$. This shows that $f = 0$.

Therefore, $\mathbb{T}$ must be finite-dimensional over $\mathbb{Q}$ and can be identified with the dual space S^* via the pairing. In particular, there exists $f \in S \otimes \mathbb{C} \leq \mathcal{S}_2(\Gamma_0(N))$ such that

$$\sum_{m \geq 1} \alpha(T_m) q^m = \sum_{m \geq 1} \alpha_1(T_m f) q^m = \sum_{m \geq 1} \alpha_m(f) q^m = f.$$

□

Definition 2.5. Let $\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_K)$. The *theta function* $\theta_{\mathcal{A}} : \mathfrak{H} \to \mathbb{C}$ is defined by

$$\theta_{\mathcal{A}}(\tau) := \frac{1}{2u_K} + \sum_{\substack{\mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{a} \in \mathcal{A}}} e^{2\pi i (N\mathfrak{a})\tau} = \sum_{n \geq 0} r_{\mathcal{A}}(n) e^{2\pi i n \tau},$$

where $u_K := |\mathcal{O}_K^\times / \{\pm 1\}|$ and

$$r_{\mathcal{A}}(n) := \begin{cases} \#\{\text{integral ideals in } \mathcal{A} \text{ of norm } n\} & \text{if } n \geq 1, \\ \frac{1}{2u_K} & \text{if } n = 0. \end{cases}$$

It is a modular form of weight 1 for $\Gamma_0(D)$ with character $\varepsilon : (\mathbb{Z}/D)^\times \to \{\pm 1\}$ associated with the quadratic extension $K/\mathbb{Q}$.

Definition 2.6. Let $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(N))$ with a Fourier expansion $f(\tau) = \sum_{n\geq 1} a_n q^n$.

(1) The L-function associated with f and $\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_K)$ is defined by

$$L_{\mathcal{A}}(f,s) := \sum_{\substack{n\geq 1 \\ (n,ND)=1}} \varepsilon(n) n^{1-2s} \cdot \sum_{n\geq 1} a_n r_{\mathcal{A}}(n) n^{-s}.$$

(2) The L-function associated with f and a character $\chi : \mathrm{Cl}(\mathcal{O}_K) \to \mathbb{C}^\times$ is defined by

$$L(f,\chi,s) := \sum_{\mathcal{A}\in \mathrm{Cl}(\mathcal{O}_K)} \chi(\mathcal{A}) L_{\mathcal{A}}(f,s).$$

Theorem 2.7. (1) $L_{\mathcal{A}}(f,s)$ converges absolutely on $\Re(s) > \frac{3}{2}$.

(2) $L_{\mathcal{A}}(f,s)$ admits a meromorphic continuation to the entire complex plane together with a functional equation relating the values at s and $2-s$.

(3) $L_{\mathcal{A}}(f,1) = 0$.

Theorem 2.8 (Gross–Zagier)**.**

$$L'_{\mathcal{A}}(f,1) = \frac{8\pi^2}{u_K^2 |D|^{1/2}} (g_{\mathcal{A}}, f),$$

$$L'(f,\chi,1) = \frac{8\pi^2 (f,f)}{h_K u_K^2 |D|_{1/2}} \widehat{h}(c_{\chi,f}).$$

Proof. (sketch) The proof consists of three main steps.

Step 1. Compute the archimedean contribution $\langle c, T_m c^\sigma \rangle_\infty$ for $(m,N)=1$ using Green functions.

Step 2. Compute the non-archimedean contribution $\langle c, T_m c^\sigma \rangle_\infty$ for $(m,N)=1$ using deformation theory of isogenies.

Step 3. Using the Rankin method and holomorphic projections, produce a modular form $\Phi_{\mathcal{A}}$ that satisfies

$$L'_{\mathcal{A}}(f,1) = \frac{8\pi^2}{u^2 |D|^{1/2}} (\Phi_{\mathcal{A}}, f)$$

for all $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(N))$.

By comparing the Fourier coefficients of $g_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$, one can show that $g_{\mathcal{A}} - \Phi_{\mathcal{A}}$ must be an oldform. This implies the first formula. The second formula follows from the first formula. □

We will see a brief overview of each step in the following subsections. But before moving on, we make a remark on how to treat the common support of c and $T_m c^\sigma$.

- By the Manin-Drinfeld theorem, $(\infty) - (0) \in J(H_K)$ is torsion. So, we have

$$\langle c, T_m c^\sigma \rangle = \langle (x) - (\infty), T_m((x) - (0))^\sigma \rangle.$$

  This is useful since ∞ is no longer in the common support of the two divisors (as long as $N \geq 2$).
- Nonetheless, (x) and $T_m(x^\sigma)$ can still have common supports.

Lemma 2.9. Let $m \geq 1$ such that $(m,N) = 1$. multiplicity of x in $T_m x^\sigma$ is $r_{\mathcal{A}}(m)$.

Proof. Write $x = \Theta_{\mathfrak{n}}(\mathfrak{a})$ and $\sigma = \mathrm{Art}_K(\mathfrak{b})$ so that $x^\sigma = \Theta_{\mathfrak{n}}(\mathfrak{b}^{-1}\mathfrak{a})$. We have

$$\begin{aligned} T_m x^\sigma &= \sum_{[\Lambda : \mathfrak{b}^{-1}\mathfrak{a}] = m} (\mathbb{C}/\Lambda \to \mathbb{C}/(\Lambda + \mathfrak{n}^{-1}\mathfrak{b}^{-1}\mathfrak{a})) \\ &= \sum_{[\mathcal{O}_K : \mathfrak{b}^{-1}\mathfrak{a}\Lambda^{-1}] = m} (\mathbb{C}/\Lambda \to \mathbb{C}/\mathfrak{n}^{-1}\Lambda). \end{aligned}$$

The multiplicity of x in $T_m x^\sigma$ is

$$\#\{\Lambda \leq K \mid \mathfrak{a}\Lambda^{-1} \text{ principal and } [\mathcal{O}_K : \mathfrak{b}^{-1}\mathfrak{a}\Lambda^{-1}] = m\} = r_{\mathfrak{b}}(m).$$

□

We will only discuss the case when $r_{\mathcal{A}}(m) = 0$. When $r_{\mathcal{A}}(m) \neq 0$, one chooses a tangent vector at x to compute the local pairings. This produces correction terms that have to be added to the result from the $r_{\mathcal{A}}(m) = 0$ case.

2.2 Archimedean local pairings

We begin by computing the local Néron pairing over the archimedean places. Fix an embedding $\sigma : K \to \mathbb{C}$ and base change X to $\mathbb{C}$.

In order to compute $\langle D, E\rangle_\sigma$ for $D, E \in \mathrm{Div}^0(X)$ with disjoint support, it is enough to compute $\langle (P) - (\infty), (Q) - (0)\rangle_\sigma$ for $P, Q \in X(\mathbb{C})$ such that P, Q, ∞, 0 are distinct. By the uniqueness of the local Néron pairing, it suffices to find a smooth function $g : \mathfrak{h} \times \mathfrak{h} - \bigcup_{\gamma\in\Gamma_0(N)}\{(\gamma\tau, \tau) \mid \tau \in \mathfrak{H}\} \to \mathbb{R}$ that satisfies the following conditions.

(i) $g(\gamma\tau, \gamma\tau') = g(\tau, \tau')$ for all $\gamma \in \Gamma_0(N)$.

(ii) $dd^c g = 0$.

(iii) For a fixed z, $g(\tau, \tau') = -|\mathrm{Stab}_{\{\pm1\}\backslash\Gamma_0(N)}(\tau)| \log|\tau - \tau'|^2 + O(1)$ as $\tau' \to \tau$.

(iv) For a fixed τ,

$$g(\tau, \tau') = \begin{cases} -4\pi\Im(\tau') + O(1) & \text{as } \tau' \to \infty, \\ O(1) & \text{as } \tau' \to \text{(any other cusp)}. \end{cases}$$

(v) For a fixed τ',

$$g(\tau, \tau') = \begin{cases} -4\pi\Im(w_N\tau) + O(1) & \text{as } \tau \to 0, \\ O(1) & \text{as } \tau \to \text{(any other cusp)}. \end{cases}$$

A natural candidate for such a function might be

$$\sum_{\gamma\in\{\pm1\}\backslash\Gamma_0(N)} \log\frac{|\tau - \gamma\tau'|^2}{|\overline{\tau} - \gamma\tau'|^2}$$

using the hyperbolic distance function. However, this sum diverges, so we need to regularize.

We introduce the geodesic polar coordinates (r, θ) centered at $i \in \mathfrak{H}$ given by

$$\tau(r, \theta) := \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} ie^r = \frac{ie^r\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{-ie^r\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} = \frac{-\sinh r\sin\theta + i}{\cosh r - \sinh r\cos\theta}.$$

We then consider the modified coordinates (u, θ) given by $u := \frac{\cosh d - 1}{2}$. In this coordinate system, the hyperbolic Laplacian becomes

$$\begin{aligned} \Delta &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &= -\frac{(1-r^2)^2}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \\ &= -\left(u(u+1)\frac{\partial^2}{\partial u^2} + (2u+1)\frac{\partial}{\partial u} + \frac{1}{4u(u+1)}\frac{\partial^2}{\partial\theta^2}\right). \end{aligned}$$

For $\Re(s) > 1$, we consider the differential equation

$$-u(u+1)Q''_{s-1}(u) - (2u+1)Q'_{s-1}(u) = s(1-s)u.$$

This is a 2^{nd} order ODE with regular singular points at $0, -1, \infty$, so it can be transformed into a hypergeometric equation. We obtain a solution

$$Q_{s-1}(u) := (u+1)^{-s}{}_2F_1\left(s, s; 2s; \frac{1}{u+1}\right)$$

on $(0,\infty)$ satisfying $G_s(u) \to 0$ as $u \to \infty$, where ${}_2F_1$ is the hypergeometric function.

Definition 2.10. For $\Re(s) > 1$, we define

$$G_s(u) := \frac{\Gamma(s)^2}{\Gamma(2s)}(u+1)^{-s}{}_2F_1\left(s, s; 2s; \frac{1}{u+1}\right) = \int_0^1 t^{s-1}(1-t)^{s-1}(u+t)^{-s}.$$

The *automorphic Green function* for $\Gamma_0(N)$ is defined by

$$G_N(\tau, \tau', s) := \sum_{\gamma \in \{\pm 1\}\backslash\Gamma_0(N)} G_s(u(\tau, \gamma\tau')) = \sum_{\gamma \in \{\pm 1\}\backslash\Gamma_0(N)} G_s\left(\frac{|z-\gamma w|^2}{4\Im(\tau)\Im(\gamma\tau')}\right)$$

Definition 2.11. The *Eisenstein series* of weight 0 for $\Gamma_0(N)$ at the cusp ∞ is defined by

$$E_N(\tau, s) := \sum_{\gamma \in \langle \pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\rangle\backslash\Gamma_0(N)} \Im(\gamma\tau)^s$$

Using the standard techniques (similar to that in Section 2), one can prove the following results.

Theorem 2.12. $G_N(\tau, \tau', s)$ converges when $\Re(s) > 1$ and admits a meromorphic continuation to a neighborhood of $s = 1$ with a simple pole of residue $\kappa_N/2$, where

$$\kappa_N := \frac{12}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} = \frac{12}{N}\prod_{p|N}\frac{p}{p+1}$$

Theorem 2.13. $E_N(\tau, s)$ converges when $\Re(s) > 1$ and admits a meromorphic continuation to a neighborhood of $s = 1$ with a simple pole of residue $\kappa_N/4\pi$.

Definition 2.14. The *regularized automorphic Green function* for $\Gamma_0(N)$ is defined by

$$g_N(\tau, \tau') := \lim_{s\to 1}\left(G_N(\tau, \tau', s) - 2\pi E_N(w_N\tau, s) - 2\pi E_N(\tau', s) + \frac{\kappa_N}{2(s-1)}\right) + C_N,$$

where C_N is a constant that makes $g_N(z, w) \to 0$ as $z \to 0$. One can show that

$$C_N = \frac{\kappa_N}{2}\left(2 - \log N - 2\log 2 + 2\gamma - 2\frac{\zeta'}{\zeta}(2) - 2\sum_{p|N}\frac{p\log p}{p^2-1}\right).$$

Theorem 2.15. The archimedean local Néron pairing on $\mathrm{Div}^0(X)$ at $\sigma : K \to \mathbb{C}$ is given by

$$\langle (\tau) - (\infty), (\tau') - (0)\rangle_\sigma = g_N(\tau, \tau').$$

for $\tau, \tau' \in \mathfrak{H}$ satisfying $\tau \notin \Gamma_0(N)\tau'$.

Proof. Check that g_N satisfy the conditions (i)-(v). □

Theorem 2.16. Assume $(m, N) = 1$.

$$\begin{aligned}\langle c, T_m c^\sigma\rangle_\infty = \lim_{s\to 1}\Bigg(&-2u_K^2\sum_{n\geq 1}\sigma_{\mathcal{A}}(n)r_{\mathcal{A}}(m|D| + nN)Q_{s-1}\left(\frac{nN}{m|D|}\right) - \frac{h_k\kappa_N\sigma_1(m)}{s-1}\Bigg)\\ &+ h_k\kappa_N\left(\sigma_1(m)\left(\log\frac{N}{|D|} + 2\sum_{p|N}\frac{\log p}{p^2-1} + 2 + 2\frac{\zeta'}{\zeta}(2) - 2\frac{L'}{L}(1,\varepsilon)\right) + \sum_{d|m} d\log\frac{m}{d^2}\right)\\ &+ h_k u_k r_{\mathcal{A}}(m)\left(2\frac{L'}{L}(1,\varepsilon) - 2\gamma - 2\log 2\pi + \log|D|\right).\end{aligned}$$

2.3 Non-archimedean local pairings

We next compute the local Néron pairing over the non-archimedean places. Fix a prime $\mathfrak{P}$ of H_K. We must compute $\langle (x)-(\infty), T_m((x)-(0))^\sigma\rangle_{\mathfrak{P}}$.

Consider the functor that assigns to a base scheme S over $\mathbb{Z}$ the set of isogenies $\phi : E \to E'$ of degree N such that $\ker f$ admits a generator fppf-locally on S (which is called the *Drinfeld* $\Gamma_0(N)$*-structure*). It has a coarse moduli space $\mathcal{Y}_0(N)$. Moreover, $\mathcal{Y}_0(N)$ admits a natural compactification $\mathcal{X}_0(N)$. Set $\mathbf{x}$ to be the closure of x in $\mathcal{X} := \mathcal{X}_0(N)$.

Proposition 2.17. Let $p \mid N$ be a prime and write $N = p^k M$ with $p \nmid M$.
(1) $\mathcal{X}_{\mathbb{F}_p}$ consists of $k+1$ irreducible components $\mathcal{X}_{a,b}$ indexed by $0 \le a, b \le k$ with $a+b=k$.
(2) Each component $\mathcal{X}_{a,b}$ is isomorphic to $\mathcal{X}_0(M)_{\mathbb{F}_p}$.
(3) Ordinary points of $\mathcal{X}_{a,b}$ parametrize isogenies of generalized elliptic curves over $\mathbb{F}_p$ whose kernal is fppf-locally $\mu_{p^a} \times \mathbb{Z}/p^b$.
(4) At supersingular points, all of $\mathcal{X}_{a,b}$ intersect. These are the only crossings of $\mathcal{X}_{a,b}$.

Lemma 2.18. One of the two divisors $\mathbf{x}-(\infty)$ and $\mathbf{x}-(0)$ intersects all vertical divisors of $\mathcal{X}_0(N)$ over $\mathfrak{P}$ trivially.

Proof. Let p be the prime lying over $\mathfrak{P}$. We only need to consider the case when $\mathcal{X}_0(N)$ has a bad reduction at p, which is equivalent to $p \mid N$.
Write $x = \Theta_{\mathfrak{n}}(\mathfrak{a})$. By the Heegner condition, $\mathfrak{P}$ divides exactly one of $\mathfrak{n}$ or $\overline{\mathfrak{n}}$. Let ϕ be the isogeny corresponding to $\mathbf{x}$. If $\mathfrak{P} \mid \mathfrak{n}$, then ϕ is étale over $\mathcal{O}_{\mathfrak{P}}$, so $\mathbf{x}$ reduces to an ordinary point in $\mathcal{X}_{0,n}$. Since (0) also reduces to an ordinary point on $\mathcal{X}_{0,n}$, we see that $\mathbf{x}-(0)$ intersects all $\mathcal{X}_{a,b}$ trivially. Similarly, if $\mathfrak{P} \mid \overline{\mathfrak{n}}$, then $\widehat{\phi}$ is étale over $\mathcal{O}_{\mathfrak{P}}$, and $\mathbf{x}-(\infty)$ intersects all $\mathcal{X}_{a,b}$ trivially. □

Therefore, we obtain

$$\begin{aligned}\langle (x)-(\infty), T_m((x)-(0))^\sigma\rangle_{\mathfrak{P}} &= i_{\mathfrak{P}}(\mathbf{x}-(\infty), T_m(\mathbf{x}^\sigma-(0)))\log N\mathfrak{P}\\ &= i_{\mathfrak{P}}(\mathbf{x}, T_m\mathbf{x}^\sigma)\log N\mathfrak{P}\end{aligned}$$

It suffices to compute $i_{\mathfrak{P}}(\mathbf{x}, T_m\mathbf{x}^\sigma)$. This is done by using the Serre–Tate deformation theory.

Definition 2.19. Let R be a complete noetherian local domain with algebraically closed residue field k. We define the category $\mathrm{Def}(R)$ consisting of triples (E, Γ, ε) satisfying the following.
(i) E is an elliptic curve over k.
(ii) Γ is a p-divisible group over R.
(iii) $\varepsilon : \Gamma_k \to E(p^\infty)$ is an isomorphism of p-divisoble groups over k.

Theorem 2.20 (Serre–Tate)**.** Let $\mathrm{Ell}(R)$ denote the category of elliptic curves over R. Then, the functor

$$\begin{aligned}\mathrm{Ell}(R) &\to \mathrm{Def}(R)\\ \mathcal{E} &\mapsto (\mathcal{E}_k, \mathcal{E}(p^\infty), \varepsilon)\end{aligned}$$

is a natural equivalence.

As an application of the Serre–Tate theory, we obtain the intersection degree formula.

Theorem 2.21. Let $m \ge 1$. We have

$$i_{\mathfrak{P}}(\mathbf{x}, T_m\mathbf{x}^\sigma) = \frac{1}{2}\sum_{n\ge 1} \#\mathrm{Hom}_{W_{\mathfrak{P}}/\varpi^n}(\mathbf{x}^\sigma, \mathbf{x})_{\deg=m},$$

where $W_{\mathfrak{P}} := W(\overline{k_{\mathfrak{P}}})$ is the ring of Witt vectors and $\varpi \in W_{\mathfrak{P}}$ is a uniformizer.

Proposition 2.22. Assume $(m,N)=1$ and $r_{\mathcal{A}}(m)=0$. Let p be a prime that splits in K. Then,

$$i_{\mathfrak{P}}(\mathbf{x}, T_m\mathbf{x}^\sigma)=0.$$

Proof. (idea) It is enough to show $\mathrm{Hom}_{k_{\mathfrak{P}}}(\mathbf{x}^\sigma,\mathbf{x})_{\deg=m}=0$. By Serre–Tate canonical lifts, we have

$$\mathrm{Hom}_{k_{\mathfrak{P}}}(\mathbf{x}^\sigma,\mathbf{x})_{\deg=m}=\mathrm{Hom}_{W_{\mathfrak{P}}}(\mathbf{x}^\sigma,\mathbf{x})_{\deg=m}.$$

However, the right-hand side is 0 since $r_{\mathcal{A}}(m)=0$. □

Proposition 2.23. Assume $(m,N)=1$ and $r_{\mathcal{A}}(m)=0$. Let p be a prime that does not split in K.

(1) $\mathrm{End}_{k_{\mathfrak{P}}}(\mathbf{x})$ is an order in a quaternion algebra B over $\mathbb{Q}$ that is ramified exactly at p and ∞.

(2) There is an identification

$$\mathrm{Hom}_{W_{\mathfrak{P}}/\varpi^n}(\mathbf{x}^\sigma,\mathbf{x})_{\deg=m}\to \mathrm{End}_{W_{\mathfrak{P}}/\varpi^n}(\mathbf{x})\mathfrak{a}\subseteq B$$
$$\phi\mapsto b(\phi)$$

(3) We have

$$\deg\phi=\frac{N(b(\phi))}{N\mathfrak{a}}.$$

(4) Since $\mathbf{x}$ has CM by $\mathcal{O}_K$, we have an embedding $K\to B$. Let $j\in B$ be an element that satisfies $jaj^{-1}=\overline{a}$ for all $a\in K$. Then, we have

$$i_{\mathfrak{P}}(\mathbf{x},T_m\mathbf{x}^\sigma)=\sum_{\substack{b\in R\mathfrak{a}/\{\pm1\}\\ N(b)/N\mathfrak{a}=m}}\begin{cases}\frac{1}{2}(v_p(N(b^-))+1) & \text{if } p \text{ is inert in } K,\\ v_p(N(b^-)) & \text{if } p \text{ is ramified in } K,\end{cases}$$

where $b=b^+ + b^-$ is the decomposition into j-eigenvectors of eigenvalue ±1.

Theorem 2.24. Assume $(m,N)=1$.

$$\langle c, T_m c^\sigma\rangle_f=-u_K^2\sum_{1\le n\le m|D|/N}\sigma'_{\mathcal{A}}(n)r_{\mathcal{A}}(m|D|-nN)+h_k u_K r_{\mathcal{A}}(m)\log\frac{N}{m}.$$

2.4 Central value of the L-function

Recall

$$L_{\mathcal{A}}(f,s+1)=\sum_{\substack{n\ge1\\(n,ND)=1}}\frac{\varepsilon(n)}{n^{2s+1}}\sum_{n\ge1}\frac{a_n r_{\mathcal{A}}(n)}{n^{s+1}}.$$

By Parseval's theorem, we have

$$\begin{aligned}
&\frac{\Gamma(s+1)}{(4\pi)^{s+1}}\sum_{n\ge1}\frac{a_n r_{\mathcal{A}}(n)}{n^{s+1}}\\
&=\int_0^\infty\left(\sum_{n\ge1}a_n r_{\mathcal{A}}(n)e^{-4\pi ny}\right)y^s dy\\
&=\int_0^\infty\left(\int_0^1\overline{\theta_{\mathcal{A}}(x+iy)}f(x+iy)dx\right)y^s dy\\
&=\int_{\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle\backslash\mathfrak{H}}\overline{\theta_{\mathcal{A}}(\tau)}f(\tau)y^{s+2}\frac{dxdy}{y^2}\\
&=\frac{1}{2}\int_{\Gamma_0(ND)\backslash\mathfrak{H}}\sum_{\gamma\in\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle\backslash\Gamma_0(ND)}\overline{\theta_{\mathcal{A}}(\gamma\tau)}f(\gamma\tau)\Im(\gamma\tau)^{s+2}\frac{dxdy}{y^2}\\
&=\frac{1}{2}\int_{\Gamma_0(ND)\backslash\mathfrak{H}}\sum_{\gamma\in\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle\backslash\Gamma_0(ND)}\overline{\theta_{\mathcal{A}}(\tau)}f(\tau)\frac{\varepsilon(d)(c\overline{\tau}+d)(c\tau+d)^2}{|c\tau+d|^{2s+4}}y^{s+2}\frac{dxdy}{y^2}
\end{aligned}$$

$$= \frac{1}{2}\int_{\Gamma_0(ND)\backslash\mathfrak{H}} \overline{\theta_{\mathcal{A}}(\tau)} f(\tau) \left(\sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \Gamma_0(ND)} \frac{\varepsilon(d) y^s}{(c\overline{\tau}+d)|c\tau+d|^{2s}} \right) y^2 \frac{dxdy}{y^2}$$

We also have

$$\begin{aligned}
&\frac{1}{2} \sum_{\substack{n \geq 1 \\ (n, ND)=1}} \frac{\varepsilon(n)}{n^{2s+1}} \sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \Gamma_0(ND)} \frac{\varepsilon(d) y^s}{(c\overline{\tau}+d)|c\tau+d|^{2s}} \\
&= \frac{1}{2} \sum_{\substack{n \geq 1 \\ (n, ND)=1}} \sum_{\substack{c \in \mathbb{Z} \\ ND | c}} \sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} \frac{\varepsilon(n)}{n^{2s+1}} \frac{\varepsilon(d) y^s}{(c\overline{\tau}+d)|c\tau+d|^{2s}} \\
&= \frac{1}{2} \sum_{ND|c} \sum_{(n,ND)=1} \sum_{(c,d)=1} \frac{\varepsilon(nd) y^s}{(nc\overline{\tau}+nd)|nc\tau+nd|^{2s}} \\
&= \frac{1}{2} \sum_{ND|c'} \sum_{(d',ND)=1} \frac{\varepsilon(d') y^s}{(c'\overline{\tau}+d')|c'\tau+d'|^{2s}}.
\end{aligned}$$

We have thus proved the following.

Definition 2.25. The *non-homolorphic Eisenstein series* of weight 1 for $\Gamma_0(N|D|)$ with character ε is defined by

$$E_1(\tau, s) := \frac{1}{2} \sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \Gamma_0(D)} \frac{\varepsilon(d)}{c\tau+d} \frac{y^s}{|c\tau+d|^{2s}}$$

is a non-homolorphic Eisenstein series.

Theorem 2.26. Let Φ be the non-holomorphic modular form of weight 2 for $\Gamma_0(N)$ with character ε defined by

$$\Phi(\tau, s) := \sum_{\gamma \in \Gamma_0(N|D|)\backslash\Gamma_0(N)} (\theta_{\mathcal{A}}(\tau) E_1(\tau, s))|_2 \gamma.$$

Then, we have

$$\frac{\Gamma(s+1)}{(4\pi)^{s+1}} L_{\mathcal{A}}(f, s+1) = (\Phi, f)$$

for all $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(N))$.

Proof. We have already showed that the left-hand side is equal to $(\theta_{\mathcal{A}} E_1, f)_{\Gamma_0(N|D|)}$. But we have

$$(\theta_{\mathcal{A}} E_1, f)_{\Gamma_0(N|D|)} = (\Phi, f)_{\Gamma_0(N)}.$$

□

We have a $\mathbb{C}$-linear functional $(\Phi, -) : \mathcal{S}_2(\Gamma_0(N)) \to \mathbb{C}$. There exists a unique $\Psi \in \mathcal{S}_2(\Gamma_0(N))$ that satisfies $(\Psi, f) = (\Phi, f)$ for all $f \in \mathcal{S}_2(\Gamma_0(N))$. Moreover, the Fourier coefficients of Ψ are completely determined by the condition

$$a_m(\Psi) = 2\pi m (P_m, \Psi) = 2\pi m (P_m, \Phi),$$

where P_n is the Poincaré series of weight m.

Theorem 2.27. There exists a cusp form $\Psi_A = \sum_{n \geq 1} a_{\mathcal{A},m} q^m \in \mathcal{S}_2(\Gamma_0(N))$ that satisfies

$$L'_{\mathcal{A}}(f, 1) = \frac{8\pi^2}{u_K^2 |D|^{1/2}} (\Psi_{\mathcal{A}}, f)$$

for all $f \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(N))$. The m^{th} Fourier coefficients for $(m, N) = 1$ is given by

$$a_{\mathcal{A},m} = -\sum_{1 \leq n \leq m|D|/N} \sigma'_{\mathcal{A}}(n) r_{\mathcal{A}}(m|D| - nN)$$

$$+\frac{h_K}{u_K}r_{\mathcal{A}}(m)\left(\log\frac{N|D|}{4\pi^2 m}-2\gamma+2\frac{L'}{L}(\varepsilon,1)\right)$$
$$+\lim_{s\to 1}\left(-2u_K^2\sum_{n\geq 1}\sigma_{\mathcal{A}}(n)r_{\mathcal{A}}(m|D|+nN)Q_{s-1}\left(\frac{nN}{m|D|}\right)-\frac{h_k\kappa_N\sigma_1(m)}{s-1}\right)$$
$$+h_k\kappa_N\left(\sigma_1(m)\left(\log\frac{N}{|D|}+2\sum_{p|N}\frac{\log p}{p^2-1}+2+2\frac{\zeta'}{\zeta}(2)-2\frac{L'}{L}(1,\varepsilon)\right)+\sum_{d|m}d\log\frac{m}{d^2}\right).$$

Notice that the m^{th} Fourier coefficients of $g_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$ match completely for $(m,N)=1$. In particular, $g_{\mathcal{A}}-\Phi_{\mathcal{A}}\in\mathcal{S}_2^{\text{old}}(\Gamma_0(N))$, and the Gross–Zagier formula follows.

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