The Colmez Conjecture

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Abstract

In this note, we motivate the Colmez conjecture and give its proof for abelian CM fields. This note is essentially a summary and an English translation of the original paper [Col93].

0.1 Colmez's period conjecture

In [Col93], Pierre Colmez asked the question of whether periods have a product formula like algebraic numbers too or not.

Denote the set of places of $\mathbb{Q}$ by $\mathcal{P}$. We fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ for each $p \in \mathcal{P}$. Let K be a number field and let X be a variety over K. By p-adic Hodge theory, there are comparison isomorphisms

$$H^1_{\mathrm{dR}}(X) \otimes_K B_p \simeq H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_p$$

where $B_p := B^+_{\mathrm{dR}}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is one of Fontaine's *period rings*. Therefore, we obtain a pairing

$$\langle -, - \rangle_p : H^1_{\mathrm{dR}}(X) \times H_1(X^\sigma(\mathbb{C}), \mathbb{Z}) \to B_p$$

for each $p \in \mathcal{P}$ and an embedding $\sigma : K \to \overline{\mathbb{Q}}$, where we interpret $B_\infty := \mathbb{C}$. The ring B_p is equipped with the p-adic norm. Let $\omega \in H^1_{\mathrm{dR}}(X)$ be given. Choose $u_\sigma \in H_1(X^\sigma(\mathbb{C}), \mathbb{Z})$ for each $\sigma : K \to \overline{\mathbb{Q}}$. We may ask if the product

$$\prod_{p \in \mathcal{P}} \prod_{\sigma : K \to \overline{\mathbb{Q}}} |\langle \omega^\sigma, u_\sigma \rangle_p|_p$$

is well-defined and equal to 1.

There are several obvious problems to this naive question. Firstly, the norm $|\langle \omega^\sigma, u_\sigma \rangle_p|_p$ depends on the choice of the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ that we started with. Secondly, the term $\langle \omega^\sigma, u_\sigma \rangle_p$ depends on the choices of u_σ. Lastly, the product diverges in general. Nonetheless, each of these difficulties have a workaround.

- When X is a CM abelian variety, it turns out that $|\langle \omega^\sigma, u_\sigma \rangle_p|_p$ does not depend on the choice of the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$.
- To address the third difficulty, we investigate the case when $X = \mathbb{G}_m$. Set $\omega := \frac{dx}{x} \in H^1_{\mathrm{dR}}(\mathbb{G}_m)$. We also set $u \in H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Z})$ to be the generator that rotates around the origin once counterclockwise. We define $t_p := \langle \omega, u \rangle_p \in B_p$. Then, the (log of the) product we are interested is given by

  $$\sum_{p \in \mathcal{P}} \log |t_p|_p = \log 2\pi - \sum_{p < \infty} \frac{\log p}{p - 1}$$

  which diverges. Nonetheless, we can formally interpret the sum as $\zeta'(1)/\zeta(1)$. Now, it is possible to regularize and assign a meaningful value. That is, we can use the constant term of the Laurent expansion at $s = 1$ to assign a meaningful value to the sum. However, Colmez chose an alternative but equivalent method of using the functional equation to transform the sum into a special value at $s = 0$. This makes the presentation of the formula cleaner.

Definition 1. We define $\mathbb{Q}$-vector spaces

$$\mathcal{C} := \{\text{locally constant functions } G_{\mathbb{Q}} \to \mathbb{Q}\},$$

$$\mathcal{C}^0 := \{a \in \mathcal{C} \mid a \text{ constant on each conjugacy class}\},$$

$$\mathcal{C}^{\mathrm{CM}} := \{a \in \mathcal{C} \mid a \text{ factors through } \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/\mathbb{Q}) \text{ and } a(g) + a(cg) \text{ constant}\},$$

$$\mathcal{C}^{\mathrm{CM},0} := \mathcal{C}^{\mathrm{CM}} \cap \mathcal{C}^0.$$

Here, $c \in G_{\mathbb{Q}}$ denotes the complex conjugate.

Note that $\mathcal{C}^0$ has a $\mathbb{Q}$-basis consisting of all Artin characters. Similarly, $\mathcal{C}^{\mathrm{CM},0}$ has a $\mathbb{Q}$-basis consisting of all Artin characters χ of CM fields satisfying $L(\chi, 0) \neq 0$. We define $\mathbb{Q}$-linear maps $\mu_{\mathrm{Art}}(-)$ and $Z(-, s)$ on $\mathcal{C}^0$ by

$$\mu_{\mathrm{Art}}(\chi) := \log f_\chi, \quad Z(\chi, s) := L'(\chi, s)/L(\chi, s)$$

for Artin characters χ, where f_χ is the Artin conductor of χ.

Definition 2. We define $\mathbb{Q}$-vector spaces

$$\mathcal{C}_p := \{\text{locally constant functions } G_{\mathbb{Q}_p} \to \mathbb{Q}\},$$

$$\mathcal{C}_p^0 := \{a \in \mathcal{C}_p \mid a \text{ constant on each conjugacy class}\}.$$

We have the restriction map $\mathcal{C}^0 \to \mathcal{C}_p^0$. We obtain decompositions

$$\mu_{\mathrm{Art}}(-) = \sum_{p<\infty} \mu_{\mathrm{Art},p}(-) \log p, \quad Z(-, s) = -\sum_{p<\infty} Z_p(-, s) \log p.$$

Definition 3. For $a \in \mathcal{C}^{\mathrm{CM},0}$, we assign a value to the divergent series

$$\sum_{p<\infty} Z_p(a, 1) \log p := Z(a^*, 0) + \mu_{\mathrm{Art}}(a),$$

where $a^* \in \mathcal{C}^{\mathrm{CM},0}$ is an element defined by $a^*(g) := a(g^{-1})$. We also assign a value to a divergent series if the series differs from $\sum_{p<\infty} Z_p(a, 1) \log p$ at only finitely many terms.

Based on this convention, we conclude that

$$\log 2\pi - \sum_{p<\infty} \frac{\log p}{p-1} = \log 2\pi - \sum_{p<\infty} Z_p(1, 1) \log p = \log 2\pi - \frac{\zeta'(0)}{\zeta(0)} = 0.$$

In this sense, the period $2\pi i$ has a product formula.

- We come back to the second difficulty. Let X be a CM abelian variety over a number field K that has a CM by $\mathcal{O}_E$. Then, there is a decomposition $H^1_{\mathrm{dR}}(X) = \oplus_{\tau: E \to \overline{\mathbb{Q}}} H^1_\tau(X)$ that satisfies $\alpha^* \omega = \tau(\alpha)\omega$ for all $\alpha \in E$ and $\omega \in H^1_\tau(X)$. Let $\omega_\tau \in H^1_\tau(X)$ be given for each $\tau : E \to \overline{\mathbb{Q}}$. Choose u_σ for each σ such that $u_{c\sigma} = c_* u_\sigma$.

  Definition 4. We define

  $$\langle \omega_\tau^\sigma, \omega_{c\tau}^\sigma, u_\sigma \rangle_p := \left(\langle \omega_\tau^\sigma, u_\sigma \rangle \frac{t_p}{\langle \omega_{c\tau}^\sigma, u_\sigma \rangle} \right)^{1/2},$$

  where $t_p \in B_p$ is the element obtained from the pairing for $\mathbb{G}_m$.

  It turns out that $\langle \omega_\tau^\sigma, \omega_{c\tau}^\sigma, u_\sigma \rangle_p^2 / \langle \omega_\tau^\sigma, u_\sigma \rangle_p^2$ is an element of $K^\times$ that is independent of $p \in \mathcal{P}$. Furthermore, $\langle \omega_\tau^\sigma, \omega_{c\tau}^\sigma, u_\sigma \rangle_p \cdot \langle \omega_\tau^{c\sigma}, \omega_{c\tau}^{c\sigma}, u_{c\sigma} \rangle_p$ is independent of the choices of u_σ.

Based on these observations, Colmez conjectured a product formula for CM periods.

Conjecture 5 (Colmez)**.** Let X be a CM abelian variety over a number field K. Then,

$$\prod_{p \in \mathcal{P}} \prod_{\sigma: K \to \overline{\mathbb{Q}}} |\langle \omega_\tau^\sigma, \omega_{c\tau}^\sigma, u_\sigma \rangle_p|_p = 1,$$

where the divergent product is assigned a value using the convention that we introduced.

0.2 The statement of the Colmez conjecture

Let us now reformulate the period conjecture using a more down-to-earth language. Let E be a CM field of degree $2d$ and Φ let be a CM type of E. Let X be a CM abelian variety over a number field K whose CM type is (E, Φ).

Theorem 6. The quantity

$$\frac{1}{[K:\mathbb{Q}]}\sum_{p\in\mathcal{P}}\sum_{\sigma:K\to\overline{\mathbb{Q}}}\log|\langle\omega_\tau^\sigma,\omega_{c\tau}^\sigma,u_\sigma\rangle_p|_p$$

is indenepdent of X, K, ω_τ, u_σ.

Definition 7. Let $\mathcal{X}$ be a model of X^σ over $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ with good reduction. Set $H^1_\tau(\mathcal{X}) := H^1_{\mathrm{dR}}(\mathcal{X}) \cap H^1_\tau(\mathcal{X})$. Pick an $\mathcal{O}_{\overline{\mathbb{Q}_p}}$-module generator ω_0 of $H^1_\tau(\mathcal{X})$ (resp. $H^0(\mathcal{X}, \Omega^d_{\mathcal{X}})$). We define the *p-adic valuation* on $H^1_{\mathrm{dR}}(X^\sigma)$ (resp. $H^0(X^\sigma, \Omega^d_{X^\sigma})$) by $v_p(a\omega_0) := v_p(a)$ for $a \in \overline{\mathbb{Q}_p}$.

Definition 8. (1) For $\tau : E \to \overline{\mathbb{Q}}$, we define $a_{\Phi,\tau} \in \mathcal{C}^{\mathrm{CM}}$ by

$$a_{E,\Phi,\tau}(g) := \begin{cases} 1 & \text{if } g\tau \in \Phi, \\ 0 & \text{if } g\tau \notin \Phi. \end{cases}$$

Then, we define $a_\Phi \in \mathcal{C}^{\mathrm{CM}}$ by

$$a_{E,\Phi} := \sum_{\tau\in\Phi} a_{E,\Phi,\tau}.$$

(2) For $\tau : E \to \overline{\mathbb{Q}}$, we define $a^0_{E,\Phi,\tau} \in \mathcal{C}^{\mathrm{CM},0}$ by

$$a^0_{E,\Phi,\tau}(g) := \frac{1}{[E':E]}\sum_{g\in\mathrm{Gal}(E'/E)} a_{gE,g\Phi,g\tau},$$

where E' is the Galois closure of E, while gE, $g\Phi$, $g\tau$ are understood as post-compositions by a lift of g in $G_{\mathbb{Q}}$. Then, we define $a_\Phi \in \mathcal{C}^{\mathrm{CM},0}$ by

$$a^0_{E,\Phi} := \sum_{\tau\in\Phi} a^0_{E,\Phi,\tau}.$$

Theorem 9. Let u be an $\mathcal{O}_E \otimes \mathbb{Z}_p$-module generator of the Tate module T_pX.
(1) $v_p(\langle\omega_\tau^\sigma, u\rangle) = v_p(\omega_\tau^\sigma) + Z_p(a_{E,\Phi,\tau}, 1) - \mu_{\mathrm{Art},p}(a_{E,\Phi,\tau})$.
(2) $v_p(\bigwedge_{\tau\in\Phi}\omega_\tau^\sigma) = \sum_{\tau\in\Phi} v_p(\omega_\tau^\sigma) - \frac{1}{2}\mu_{\mathrm{Art},p}(a_{\sigma E,\sigma\Phi}) + \frac{1}{4}v_p(D_E)$.

Corollary 10. The quantity

$$h(E,\Phi,\tau) := \frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\to\overline{\mathbb{Q}}}\left(\log|\langle\omega_\tau^\sigma,\omega_{c\tau}^\sigma,u_\sigma\rangle_\infty|_\infty - \frac{1}{2}\sum_{p<\infty}(v_p(\omega_\tau^\sigma) - v_p(\omega_{c\tau}^\sigma))\log p\right)$$

is independent of X, K, ω_τ, u_σ.

Proof. By 9(1), this quantity is equal to

$$\frac{1}{[K:\mathbb{Q}]}\sum_{p\in\mathcal{P}}\sum_{\sigma:K\to\overline{\mathbb{Q}}}\log|\langle\omega_\tau^\sigma,\omega_{c\tau}^\sigma,u_\sigma\rangle_p|_p + \sum_{p<\infty}(Z_p(a^0_{E,\Phi,\tau},1) - \mu_{\mathrm{Art},p}(a^0_{E,\Phi,\tau}))\log p$$

is independent of X, K, ω_τ, u_σ. Then by the convention, this evaluates to

$$\frac{1}{[K:\mathbb{Q}]}\sum_{p\in\mathcal{P}}\sum_{\sigma:K\to\overline{\mathbb{Q}}}\log|\langle\omega_\tau^\sigma,\omega_{c\tau}^\sigma,u_\sigma\rangle_\infty|_\infty + Z((a^0_{E,\Phi,\tau})^*,0)$$

which is independent of X, K, ω_τ, u_σ by 6. □

Theorem 11. There is a unique $\mathbb{Q}$-linear map $h : \mathcal{C}^{\mathrm{CM},0} \to \mathbb{R}$ that satisfies $h(a^0_{E,\Phi,\tau}) = h(E,\Phi,\tau)$.

Proof. (idea) Show that the relations among $a^0_{E,\Phi,\tau}$ are satisfied by $h(E,\Phi,\tau)$ as well. □

Based on the computation done in the proof of 10, we arrive at the standard formulation of the Colmez conjecture.

Conjecture 12 (Colmez)**.** $h(a) = Z(a^*, 0)$ holds for all $a \in \mathcal{C}^{\mathrm{CM},0}$.

0.3 Relation to Faltings heights

Definition 13. The *Faltings height* of an abelian variety X over K is defined by

$$h_{\mathrm{Fal}}(X) := \frac{1}{[K:\mathbb{Q}]}\left(-\frac{1}{2}\sum_{\sigma:K\to\overline{\mathbb{Q}}}\log\left|\int_{X^\sigma(\mathbb{C})}\omega^\sigma\wedge\overline{\omega^\sigma}\right| + \sum_{p<\infty}\sum_{\sigma:K\to\overline{\mathbb{Q}}} v_p(\omega^\sigma)\log p\right)$$

for any $0 \neq \omega \in H^0(X, \Omega_X^d)$. This definition is independent of a choice of ω by the product formula.

Proposition 14. Suppose that X has CM and has a good reduction everywhere. Then,

$$h_{\mathrm{Fal}}(X) = -\sum_{\tau\in\Phi} h(E,\Phi,\tau) - \frac{1}{2}\mu_{\mathrm{Art}}(a^0_{E,\Phi})$$

depends only on its CM type (E, Φ).

Proof. (idea) We use $\omega = \bigwedge_{\tau\in\Phi}\omega_\tau$ in the definition of $h_{\mathrm{Fal}}(X)$. For each $\sigma : K \to \overline{\mathbb{Q}}$, take a fractional ideal $\mathfrak{a}_\sigma \leq E$ such that $H_1(X^\sigma(\mathbb{C}), \mathbb{Z}) = \mathfrak{a}_\sigma u_\sigma$. We have

$$\frac{1}{2}\log\left|\int_{X^\sigma(\mathbb{C})}\omega^\sigma\wedge\overline{\omega^\sigma}\right| = \sum_{\tau\in\Phi}\log|\langle\omega_\tau, u_\sigma\rangle|_\infty + \frac{1}{2}\log N\mathfrak{a}_\sigma + \frac{1}{4}\log|D_E|_\infty.$$

One can show that

$$\begin{aligned}&\sum_{\sigma\in\overline{\mathbb{Q}}}\left(\sum_{\tau\in\Phi}\log|\langle\omega_\tau,u_\sigma\rangle|_\infty + \frac{1}{2}\log N\mathfrak{a}_\sigma\right)\\ &= \sum_{\sigma:K\to\overline{\mathbb{Q}}}\sum_{\tau\in\Phi}\left(\log|\langle\omega^\sigma_\tau,\omega^\sigma_{c\tau},u_\sigma\rangle_\infty|_\infty + \frac{1}{2}\sum_{p<\infty}(v_p(\omega^\sigma_\tau)+v_p(\omega^\sigma_{c\tau}))\log p\right)\end{aligned}$$

On the other hand, we have

$$v_p(\omega^\sigma) = \sum_{\tau\in\Phi} v_p(\omega^\sigma_\tau) - \frac{1}{2}\mu_{\mathrm{Art},p}(a_{\sigma E,\sigma\Phi}) + \frac{1}{4}v_p(D_E)$$

by 9(2). Combining these, we obtain

$$\begin{aligned}h_{\mathrm{Fal}}(X) &= -\frac{1}{[K:\mathbb{Q}]}\sum_{\tau\in\Phi}\sum_{\sigma:K\to\overline{\mathbb{Q}}}\left(\log|\langle\omega^\sigma_\tau,\omega^\sigma_{c\tau},u_\sigma\rangle_\infty|_\infty - \frac{1}{2}\sum_{p<\infty}(v_p(\omega^\sigma_\tau)-v_p(\omega^\sigma_{c\tau}))\log p\right)\\ &\quad - \frac{1}{2}\sum_{p<\infty}\mu_{\mathrm{Art},p}(a^0_{E,\Phi})\log p + \frac{1}{4}\left(-\log|D_E|_\infty + \sum_{p<\infty}v_p(D_E)\log p\right)\\ &= -\sum_{\tau\in\Phi}h(E,\Phi,\tau) - \frac{1}{2}\mu_{\mathrm{Art}}(a^0_{E,\Phi})\end{aligned}$$

□

Based on 14, we obtain yet another reformulation of the conjecture.

Conjecture 15. Let X be a CM abelian variety that has a good reduction everywhere. Let (E, Φ) be its CM type. Then,

$$h_{\mathrm{Fal}}(X) = -Z((a^0_{E,\Phi})^*, 0) - \frac{1}{2}\mu_{\mathrm{Art}}(a^0_{E,\Phi}).$$

0.4 Colmez's proof for abelian CM fields

Definition 16. We define the $\mathbb{Q}$-vector space

$$\mathcal{C}^{\mathrm{ab}} := \{a \in \mathcal{C}^{\mathrm{CM}} \mid a \text{ factors through } \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})\} \subseteq \mathcal{C}^{\mathrm{CM},0}.$$

Note that $\mathcal{C}^{\mathrm{ab}}$ has a $\mathbb{Q}$-basis consisting of all odd Dirichlet characters.

In this subsection, we present Colmez's proof that $h(a)$ and $Z(a^*, 0)$ can only differ by a rational multiple of $\log 2$ when $a \in \mathcal{C}^{\mathrm{ab},0}$. The indeterminacy at the prime 2 was later removed by Andrew Obus, hence fully resolving the case for abelian CM fields.

We identify

$$\begin{aligned} \mathbb{Q}/\mathbb{Z} &\to \mu_\infty \\ r &\mapsto e^{2\pi i r} \end{aligned}$$

so that $G_{\mathbb{Q}}$ acts on $\mathbb{Q}/\mathbb{Z}$. For $r \in \mathbb{Q}/\mathbb{Z}$, we denote its lift in $\mathbb{Q} \cap [0,1)$ by $\langle r \rangle$. We also define an element $a_r \in \mathcal{C}^{\mathrm{ab}}$ by $a_r(g) := \langle gr \rangle - \frac{1}{2}$.

Write $r = \frac{c}{d}$ with $0 \le c < d$ and $(c,d) = 1$. Then, a_r factors through $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q}) \simeq (\mathbb{Z}/d)^\times$, so a_r admits a decomposition into Dirichlet characters.

Proposition 17. For $r \neq 0$, we have

$$a_r = \sum_{\substack{\chi \in \widehat{(\mathbb{Z}/d)^\times} \\ \chi \text{ odd}}} c_\chi \chi,$$

where

$$c_\chi = -\frac{1}{\varphi(d)} \chi^{-1}(c) L(\chi, 0).$$

Proof. We have

$$\begin{aligned} c_\chi &= \frac{1}{\varphi(d)} \sum_{g \in (\mathbb{Z}/d)^\times} \chi^{-1}(g) a_r^*(g) \\ &= \frac{1}{\varphi(d)} \sum_{g \in (\mathbb{Z}/d)^\times} \chi(g) a_r(g) \\ &= \frac{1}{\varphi(d)} \sum_{g \in (\mathbb{Z}/d)^\times} \chi(g) \left(\langle gr \rangle - \frac{1}{2} \right). \end{aligned}$$

We need a fact about Dirichlet L-functions. We can write

$$\begin{aligned} L(\chi, s) &= \sum_{j=0}^{d-1} \chi(j) \sum_{\substack{n \ge 1 \\ n \equiv j \ (d)}} n^{-s} \\ &= d^{-s} \sum_{j=0}^{d-1} \chi(j) \sum_{n \ge 0} \left(n + \frac{j}{d} \right)^{-s} \\ &= d^{-s} \sum_{g \in (\mathbb{Z}/d)^\times} \chi(d) \zeta\left(\left\langle \frac{x}{d} \right\rangle, s \right), \end{aligned}$$

where $\zeta(x,s)$ is the Hurwitz zeta function. The Laurent expansion of $\zeta(x,s)$ at $s = 0$ is given by

$$\zeta(x,s) = \left(\frac{1}{2} - x \right) + \left(\log \frac{\Gamma(x)}{\sqrt{2\pi}} \right) s + O(s^2).$$

Therefore, we obtain

$$L(\chi, 0) = \sum_{x \in (\mathbb{Z}/d)^\times} \chi(x) \left(\frac{1}{2} - \left\langle \frac{x}{d} \right\rangle \right),$$

Note that $L(\chi, 0) = 0$ when χ is even. This concludes the proof. □

Corollary 18. $\mathcal{C}^{\mathrm{ab}}$ is spanned by a_r for $r \in \mathbb{Q}/\mathbb{Z}$ subject to the relations

$$a_r + a_{-r} = 0 \text{ for } r \neq 0,$$

$$\textstyle\sum_{j=0}^{m-1} a_{r+j/m} = a_{mr}.$$

Proof. This follows from 17 and the character orthogonality. □

We now compute

$$Z(a_r^*, 0) = \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} c_\chi \frac{L'}{L}(\chi_{\mathrm{prim}}, 0).$$

Using

$$L(\chi, s) = L(\chi_{\mathrm{prim}}, s) \prod_{p \mid d} (1 - \chi_{\mathrm{prim}}(p) p^{-s}),$$

we have

$$\begin{aligned} Z(a_r^*, 0) &= -\frac{1}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi^{-1}(c) L(\chi, s) \left(\frac{L'}{L}(\chi, s) - \sum_{p \mid d} \frac{\chi_{\mathrm{prim}}(p) p^{-s} \log p}{1 - \chi_{\mathrm{prim}}(p) p^{-s}} \right) \Bigg|_{s=0} \\ &= -\frac{1}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \left(\chi^{-1}(c) L'(\chi, 0) - \sum_{p \mid d} \chi^{-1}(c) \chi_{\mathrm{prim}}(p) \frac{L(\chi, s)}{1 - \chi_{\mathrm{prim}}(p) p^{-s}} \Big|_{s=0} \log p \right). \end{aligned}$$

For each prime $p \mid d$, there is a bijection

$$\left\{ \begin{array}{c} \text{odd Dirichlet characters } \chi \\ \text{mod } d \text{ such that } \chi_{\mathrm{prim}}(p) \neq 0 \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{odd Dirichlet characters} \\ \text{mod } d_{(p)} := d/p^{v_p(d)} \end{array} \right\}$$

$$\chi \mapsto \chi_{(p)}.$$

When $\chi_{\mathrm{prim}}(p) = 0$, the term in the inner sum vanishes. When $\chi_{\mathrm{prim}}(p) \neq 0$, we have

$$\frac{L(\chi, s)}{1 - \chi_{\mathrm{prim}}(p) p^{-s}} = L(\chi_{(p)}, s).$$

Hence,

$$Z(a_r^*, 0) = -\frac{1}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi^{-1}(c) L'(\chi, 0) - \sum_{p \mid d} \sum_{\substack{\chi \bmod d_{(p)} \\ \chi \text{ odd}}} \chi^{-1}(c) \chi(p) L(\chi, 0) \log p.$$

Using the Hurwizt zeta function again, we obtain a formula

$$L'(\chi, 0) = \sum_{x \in (\mathbb{Z}/d)^\times} \chi(x) \left(\log \frac{\Gamma(\langle x/d \rangle)}{\sqrt{2\pi}} - \left(\frac{1}{2} - \left\langle \frac{x}{d} \right\rangle \right) \log d \right).$$

Therefore,

$$\begin{aligned} Z(a_r^*, 0) = &-\frac{1}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \sum_{x \in (\mathbb{Z}/d)^\times} \chi^{-1}(c) \chi(x) \left(\log \frac{\Gamma(\langle x/d \rangle)}{\sqrt{2\pi}} - \left(\frac{1}{2} - \left\langle \frac{x}{d} \right\rangle \right) \log d \right) \\ &+ \frac{1}{\varphi(d)} \sum_{p \mid d} \sum_{\substack{\chi \bmod d_{(p)} \\ \chi \text{ odd}}} \sum_{x \in (\mathbb{Z}/d_{(p)})^\times} \chi^{-1}(c) \chi(p) \chi(x) \left(\frac{1}{2} - \left\langle \frac{x}{d_{(p)}} \right\rangle \right) \log p. \end{aligned}$$

For $x \in (\mathbb{Z}/d)^\times$, we have

$$\sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(x) = \frac{\varphi(d)}{2} (\delta_{x,1} - \delta_{x,-1}).$$

This yields

$$Z(a_r^*, 0) = -\frac{1}{2} \left(\left(\log \frac{\Gamma(c/d)}{\sqrt{2\pi}} - \left(\frac{1}{2} - \frac{c}{d} \right) \log d \right) - \left(\log \frac{\Gamma(1 - c/d)}{\sqrt{2\pi}} - \left(-\frac{1}{2} + \frac{c}{d} \right) \log d \right) \right)$$

$$+\frac{1}{2}\sum_{p|d}\frac{\varphi(d_{(p)})}{\varphi(d)}\left(\left(\frac{1}{2}-\frac{c_{(p)}}{d_{(p)}}\right)-\left(-\frac{1}{2}+\frac{c_{(p)}}{d_{(p)}}\right)\right)\log p$$
$$=\frac{F(r)-F(-r)}{2},$$

where

$$F(r):=\log\frac{\Gamma(1-c/d)}{\sqrt{2\pi}}-\left(\frac{c}{d}-\frac{1}{2}\right)\log d-\sum_{p|d}\frac{1}{\varphi(p^{v_p(d)})}\left(\frac{c_{(p)}}{d_{(p)}}-\frac{1}{2}\right)\log p$$
$$=\log\frac{\Gamma(1-c/d)}{\sqrt{2\pi}}-\sum_{p|d}\left(v_p(d)\left(\frac{c}{d}-\frac{1}{2}\right)+\frac{1}{(p-1)p^{v_p(d)-1}}\left(\frac{c_{(p)}}{d_{(p)}}-\frac{1}{2}\right)\right)\log p.$$

Here, $c_{(p)}\in\mathbb{Z}$ is the unique integer such that $0\le c_{(p)}<d_{(p)}$ and $pc_{(p)}\equiv c\ (d_{(p)})$.

Proposition 19. For $r\neq 0$, we have

$$Z(a_r^*,0)=\frac{F(r)-F(-r)}{2},$$

where

$$v_p(r):=\min(\langle r\rangle,0)\in\mathbb{Z}_{\le 0},$$
$$r_{(p)}:=p^{-v_p(r)}r\in\mathbb{Z}_{(p)}/\mathbb{Z},$$
$$V_p(r):=v_p(r)a_r(1)+\frac{1}{(p-1)p^{-v_p(r)-1}}a^*_{r_{(p)}}(\mathrm{Fr}_p),$$
$$F(r):=\log\frac{\Gamma(1-\langle r\rangle)}{\sqrt{2\pi}}+\sum_{p<\infty}V_p(r)\log p.$$

We next study h on $\mathcal{C}^{\mathrm{ab}}$ by investigating Fermat curves.

Definition 20. Let $m\ge 3$. The degree m *Fermat curve* is a projective curve over $\mathbb{Q}$ defined as the projective closure of

$$F_m: x^m+y^m=1$$

which is smooth of genus $(m-1)(m-2)/2$. We denote its Jacobian by J_m.

We state some facts about F_m and J_m. Identify $H^1_{\mathrm{dR}}(F_m)$ with $H^1_{\mathrm{dR}}(J_m)$ and $H_1(F_m(\mathbb{C}),\mathbb{Z})$ with $H_1(J_m(\mathbb{C}),\mathbb{Z})$. There is an obvious action $\mu_m^2\curvearrowright(F_m)_{\overline{\mathbb{Q}}}$, and we obtain a map $\mathbb{Z}[\mu_m^2]\to\mathrm{End}(H_1(J_m(\mathbb{C}),\mathbb{Z}))$. Let A_m denote the quotient of $\mathbb{Z}[\mu_m^2]$ by the kernel of this map. Then, J_m has CM by $A_m\otimes\mathbb{Q}$. Moreover, $H_1(J_m(\mathbb{C}),\mathbb{Z})$ is a free rank 1 A_m-module generated by the *Pochhammer contour* P_m. In order to produce factors of J_m, we consider the index set

$$I:=\{(r,s,t)\in(\mathbb{Q}/\mathbb{Z})^3\mid r+s+t=0\text{ and }r,s,t\neq 0\}.$$

For $q\in I$, we set $d_q>0$ to be the smallest integer such that $d_qr,d_qs,d_qt=0$. For $m>0$ such that $d_q\mid m$, we set

$$\omega_{m,q}:=m\langle r+s\rangle^{\langle r\rangle+\langle s\rangle+\langle t\rangle-1}x^{m\langle r\rangle}y^{m\langle s\rangle}\frac{y}{x}d\left(\frac{y}{x}\right)\in H^1_{\mathrm{dR}}(J_m)$$

It turns out that $\omega_{m,q}$ is an A_m-eigenvector. In particular, we can produce an abelian variety

$$J_q=J_{m,q}:=J_m/\mathrm{ann}_{A_m}(\omega_{m,q})J_m$$

that has CM by $\mathcal{O}_{\mathbb{Q}(\mu_m)}$ and turns out to be independent of m. We set $P_q=P_{m,q}\in H_1(J_q(\mathbb{C}),\mathbb{Z})$ to be the pushforward of P_m in $J_{m,q}$ which is also independent of m. On the other hand, for $q'\in\mathrm{Orb}_{G_{\mathbb{Q}}}(q)$, $\omega_{m,q'}$ is the pullback of some $\omega_q\in H^1_{\mathrm{dR}}(J_q(\mathbb{C}))$ that is independent of m. Now, $\omega_{q'}$ is an $\mathcal{O}_{\mathbb{Q}(\mu_d)}$-eigenvector

associated with some embedding $\tau_{q'} : \mathcal{O}_{\mathbb{Q}(\mu_d)} \to \overline{\mathbb{Q}}$. The CM type Φ_q of J_q is completely described by

$$\begin{cases} \tau_{q'} \in \Phi_q & \Leftrightarrow \quad \langle r' \rangle + \langle s' \rangle + \langle t' \rangle = 1, \\ \tau_{q'} \notin \Phi_q & \Leftrightarrow \quad \langle r' \rangle + \langle s' \rangle + \langle t' \rangle = 2, \end{cases}$$

where $q' = (r', s', t')$. In particular,

$$a_{\mathbb{Q}(\mu_d),\Phi_q,\tau_q} = \frac{1}{2} - (a_r + a_s + a_t).$$

We define an element $b_q \in \mathcal{C}^{\mathrm{ab}}$ by $b_q := a_r + a_s + a_t$.

Proposition 21. $\mathcal{C}^{\mathrm{ab}}$ is spanned by 1 and b_q for $q \in I$.

Proof. Let $V \leq \mathcal{C}^{\mathrm{ab}}$ denote the subspace spanned by b_q for $q \in I$. If suffices to show that every a_r for $0 \neq r \in \mathbb{Q}/\mathbb{Z}$ is in V. Write $r = \frac{c}{d}$ with $d \geq 3$ and $0 < c < d$. For each $1 \leq j \leq d-2$, we have

$$a_{1/d} + a_{j/d} - a_{(j+1)/d} = a_{1/d} + a_{j/d} + a_{1-(j+1)/d} \in V.$$

Adding successive terms, we obtain for $2 \leq j \leq d-1$

$$ja_{1/d} - a_{j/d} \in V.$$

When $j = d-1$, this yields

$$da_{1/d} = (d-1)a_{1/d} - a_{(d-1)/d} \in V,$$

hence $a_{1/d} \in V$. This forces $a_r = a_{c/d} \in V$. □

We now compute

$$\begin{aligned} h(b_q) &= \frac{1}{2}h(1) - h(\mathbb{Q}(\mu_d), \Phi_q, \tau_q) \\ &= \frac{1}{2}\log 2\pi - \frac{1}{2}\left(\log 2\pi + \log\left|\frac{\langle \omega_q, P_q \rangle}{\langle \omega_{-q}, P_q \rangle}\right|\right) + \frac{1}{2}\sum_{p<\infty}(v_p(\omega_q) - v_p(\omega_{-q}))\log p \\ &= -\frac{1}{2}\log\left|\frac{\langle \omega_q, P_q \rangle}{\langle \omega_{-q}, P_q \rangle}\right| + \frac{1}{2}\sum_{p<\infty}(v_p(\omega_q) - v_p(\omega_{-q}))\log p. \end{aligned}$$

The integral pairing over $\mathbb{C}$ was computed by Rohrlich. It is given by

$$\langle \omega_q, P_q \rangle_\infty = \frac{(2\pi i)^2}{\Gamma(\langle -r \rangle)\Gamma(\langle -s \rangle)\Gamma(\langle -t \rangle)}.$$

Combining this with 19, we obtain

$$\frac{\langle \omega_q, P_q \rangle_\infty}{\langle \omega_{-q}, P_q \rangle_\infty} = \exp(-2Z(b_q^*, 0)) \prod_{p<\infty} p^{V_p(q) - V_p(-q)},$$

hence

$$h(b_q) = Z(b_q^*, 0) + \frac{1}{2}\sum_{p<\infty}(v_p(\omega_q) - V_p(q) - v_p(\omega_{-q}) + V_p(-q))\log p.$$

One can show that

$$v_p(\omega_q) + v_p(\omega_{-q}) = V_p(q) + V_p(-q),$$

hence

$$h(b_q) = Z(b_q^*, 0) + \sum_{p<\infty}(v_p(\omega_q) - V_p(q))\log p.$$

Proposition 22. There is a unique $\mathbb{Q}$-linear map $w_p : \mathcal{C}^{\mathrm{ab}} \to \mathbb{Q}$ that satisfies $w_p(b_q) = v_p(\omega_q) - V_p(q)$.

Proof. This follows from the linear independency of $\log p$ over $\mathbb{Q}$. □

The Colmez conjecture for abelian CM fields would be proved if we can show $w_p = 0$ for all primes p.

Let $G_{\mathbb{Q}}$ act $\mathbb{Q}$-linearly on $\mathcal{C}^{\mathrm{ab}}$ by $(ga)(h) := a(hg)$. This action factors through $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \mathbb{Z}^\times$. Now, $\mathbb{Z}_p^\times$ sits inside $\mathbb{Z}^\times$ as a subgroup. Consider the filtration of $\mathbb{Z}_p^\times$

$$G_p^{(n)} := \begin{cases} \mathbb{Z}_p^\times & \text{if } n = 0, \\ 1 + p^n\mathbb{Z}_p & \text{if } \mathbb{Z}_p^\times. \end{cases}$$

Definition 23. We define $\mathcal{C}^{\mathrm{ab}}_{(p^n)} \leq \mathcal{C}^{\mathrm{ab}}$ to be $\mathbb{Q}$-subspace fixed by $G_p^{(n)}$.

Lemma 24. $w_p(a) = 0$ for all primes p and $a \in \mathcal{C}^{\mathrm{ab}}_{(p)}$.

Proof. It suffices to show $w_p(b_q) = 0$ for all $q = (r, s, t) \in I$ such that $v_p(r) = v_p(s) = v_p(t) = 0$. In this case, we have $V_p(q) = 0$ by definition. We also have $v_p(\omega_q) = 0$ since J_q has a good reduction at p and $\omega_{q'}$ for $q' \in \mathrm{Orb}_{G_{\mathbb{Q}}}(q)$ form a $\mathbb{Z}_p$-basis for $H^1_{\mathrm{dR}}(\mathcal{J}_q)$. □

We quote a lemma by Coleman (or rather its consequence).

Lemma 25. (1) If $p \geq 3$, then $w_p(ga_r) = w_p(a_r)$ for all $g \in G_p^{(1)}$ and $r \in \mathbb{Q}/\mathbb{Z}$.
(2) $w_2(ga_r) - w_2(a_r) = w_2(ga_s) - w_2(a_s)$ for all $g \in G_2^{(1)}$ and $r, s \in \mathbb{Q}/\mathbb{Z}$ that satisfies $v_2(r - s) = 0$.

Proposition 26. $w_p = 0$ for all primes $p \geq 3$.

Proof. Let $a \in \mathcal{C}^{\mathrm{ab}}$. Then,

$$\sum_{g \in \mathrm{Stab}_{G_p^{(1)}}(a) \backslash G_p^{(1)}} ga \in \mathcal{C}^{\mathrm{ab}}_{(p)},$$

so we have

$$\sum_{g \in \mathrm{Stab}_{G_p^{(1)}}(a) \backslash G_p^{(1)}} \omega_p(ga) = 0$$

by 24. However, $\omega_p(ga) = \omega_p(a)$, so this forces $w_p(a) = 0$. □

Colmez verified that $w_2 = 0$ on $\mathcal{C}^{\mathrm{ab}}_{(16)}$ which reduces to a finite check for $b_{1/4,1/4,1/2}$ and $b_{1/8,3/8,1/2}$. Obus later proved that $w_2 = 0$ on all of $\mathcal{C}^{\mathrm{ab}}$ as well.

Theorem 27. The Colmez conjecture is true for abelian CM fields.

Remark 28. When E is an imaginary quadratic field, the Colmez conjecture reduces to the Chowla–Selberg formula.

References

[Col93] Pierre Colmez. Périodes des variétés abéliennes à multiplication complexe. *Annals of Mathematics, Vol. 138, No. 3, 625-683*, 1993.